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Nonlinear output-feedback tracking in multiagent systems with an unknown leader and directed communication

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Abstract This paper addresses cooperative global robust output regulation for heterogeneous and uncertain multiagent nonlinear systems in output-feedback normal form. Specifically, we develop a Lyapunov-based dynamic output-feedback law using a nonlinear internal model approach. We show that an effective control law can be constructed under general (static) directed communication topologies even when the leader is unknown. Hence, the present study offers a more general investigation of the problem in comparison with the developments in the recent literature.

Keywords multiagent systems, internal model, unknown leaders, output regulation, output-feedback

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1 Introduction

Cooperative tracking or synchronization control of leader-follower multiagent systems has been extensively studied. The problem, however, is that not every follower can access the reference for feedback, and followers merely communicate with their neighbors. This research contributes toward the development for various cooperative tracking control problems from different dynamics to communication topologies. For example, Refs. [1–7] noted interesting results for linear and [8–12] nonlinear scenarios.

For cooperative tracking, the leader system usually generates the desired reference trajectory. For instance, Ref. [1] studied constant references for linear networks. Ref. [13] further addressed the problem associated with nonlinear followers. Concerning the problem with active leaders, there have been many interesting studies, boosted in [2]. More recently, the internal model principle in output regulation theory has been shown to be a useful tool for handling the active leaders as well as the much more interesting robustness issues such as parametric uncertainties and external disturbances; see [14–17] and references therein. Such a problem is also referred to as cooperative robust output regulation in literature. Specifically, when the leader dynamics is known or without any uncertainty, the standard linear internal model principle-based approach can be used to some extent. In this direction, several output-feedback-based results have been well developed. For example, the semi-global output-feedback designs for nonlinear networks in strict-feedback normal form can be found in [8, 16], and the global output-feedback design for nonlinear networks in output-feedback normal form can be found in [9].

In practice, the active leader's dynamics may be uncertain or unknown, covering the case that the references and disturbances can be fundamental sinusoidal signals with unknown frequencies, unknown amplitudes, and unknown phases. However, the standard linear internal model fails when such frequency information is lacking or uncertain. An attempt to tackle such unknown active leaders can be found in [18] using an adaptive canonical internal model method.

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Surmounting common hurdles such as system uncertainties or special system relative degree, or bi-directed communication topologies encountered in the aforementioned literature served motivation for the present study to search for new techniques serving cooperative global robust output regulation (CGOR). Specifically, we focus on a broader class of heterogeneous and uncertain multiagent nonlinear systems in output-feedback normal form with unknown active leaders. The problem is significantly extended in this study because the follower agents may have arbitrary non identical relative degrees, and the communication topology can be directed. It is worth noting that the follower agents are general enough to model physical systems such as mass-spring-damper system, van der Pol oscillator, and single-link robotic manipulator and have served as a typical class of multiagent nonlinear systems for distributed output-feedback design (see [9, 13, 19]). The main contribution of the present study is two folds. On the one hand, in sharp contrast to our previous work (see [20]) on a state-feedback design, we further develop a Lyapunov-based dynamic output-feedback control law. In particular, an interesting integral input-to-state stable (iISS)-based nonlinear output-feedback design method (see [21]) can be established as an extended iISS-based tracking control investigation in a generalized distributed control fashion (see [22, 23]). On the other hand, the present study allows a general uncertain nonlinear network with non identical higher relative degrees and directed communication topologies without adding any extra observer network, and thus extends some results in the literature. To show the present study's practical potential, a heterogeneous network of mass-spring-damper systems with hardening springs is also given.

The rest of this paper is organized as follows. Section 2 introduces the formulation of the CGOR. Section 3 elaborates the design procedure and Section 4 presents the cooperative global stabilizer, giving rise to a solution to the CGOR. Section 5 gives an illustrative example and Section 6 concludes this paper.

Notation & definition. $\|\cdot\|$ is the Euclidean norm. $\mathbb{R}_+$ denotes the set of nonnegative real numbers. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if, it is continuous, strictly increasing, and $\alpha(0) = 0$; it is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and moreover unbounded. Id denotes the identical $\mathcal{K}_\infty$ function. The set of bounded $\mathcal{K}$ functions is denoted by $\mathcal{K}^o$, i.e., $\mathcal{K}^o = \mathcal{K}\backslash\mathcal{K}_\infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if, for each fixed $s \in \mathbb{R}_+$, $\beta(s, t)$ is continuous and decreases to zero as $t \to +\infty$, and for each fixed $t \in \mathbb{R}_+$, $\beta(\cdot, t) \in \mathcal{K}$. Given $\alpha \in \mathcal{K}$, let $\widehat{\mathcal{O}}(\alpha)$ be the set of functions:

$$\widehat{\mathcal{O}}(\alpha) \triangleq \left\{\gamma \in \mathcal{K} : \limsup_{s\to 0^+} \gamma(s)/\alpha(s) < \infty, \text{ and } \limsup_{s\to\infty} \gamma(s)/\alpha(s) < \infty \text{ if } \alpha \in \mathcal{K}^o\right\}.$$

We use $f_1 \circ f_2(x)$ as the function composition $f_1(f_2(x))$ for functions f_1, f_2 of compatible dimensions. For a vector $\xi_i \in \mathbb{R}^{n_i}$ for $1 \leqslant i \leqslant N$ and an integer $n_i \geqslant 1$, we use $\xi_{i,k} \in \mathbb{R}$ for $1 \leqslant k \leqslant n_i$ to denote the k-th element of the component ξ_i.

2 Problem formulation

Consider a class of heterogeneous and uncertain multiagent nonlinear systems transformable into the following form:

$$\dot{x}_i = A_i x_i + g_i(y_i, v, w) + B_i u_i, \quad y_i = C_i x_i, \quad 1 \leqslant i \leqslant N, \tag{1}$$

where, for $1 \leqslant i \leqslant N$, $x_i \in \mathbb{R}^{r_i}$ is the state, $y_i \in \mathbb{R}$ is the output, $u_i \in \mathbb{R}$ is the control input, $w \in \mathbb{W} \subseteq \mathbb{R}^{n_w}$ is the constant system parametric uncertainty, and $v \in \mathbb{R}^{n_v}$ is the commanded reference and encountered disturbances generated by an active leader system

$$\dot{v} = S(\sigma)v, \quad y_0 = q(v, w). \tag{2}$$

Here, the leader system (2) can be unknown owing to the constant parametric uncertainty $\sigma \in \mathbb{S} \subseteq \mathbb{R}^{n_\sigma}$.

For systems (1) and (2), some technical specifications are used throughout this paper. Both the sets $\mathbb{S}$ and $\mathbb{W}$ are assumed to be compact and known. For $1 \leqslant i \leqslant N$, we assume that the relative degree $r_i \geqslant 2$ and the matrix triplet (A_i, B_i, C_i) takes the r_i-dimensional Brunovsky normal form, i.e.,

$$A_i = \left[\begin{array}{c|c} 0 & I_{r_i-1} \\ \hline 0 & 0 \end{array}\right], \quad B_i = [0, \ldots, 0, 1]^{\mathrm{T}}, \quad C_i = [1, 0, \ldots, 0]. \tag{3}$$

For each $\sigma \in \mathbb{S}$, all eigenvalues of $S(\sigma)$ are distinct with zero real parts and the leader (2) is invariant in a known compact set $\mathbb{V}$, i.e., $v(t) \in \mathbb{V}$, $\forall t \geqslant 0$. Functions $g_i(y_i, v, w)$ and $q(v, w)$ are polynomials in their arguments y_i, v with coefficients continuously depending on w.

Remark 1. System (1) is in output-feedback normal form with higher relative degree, that has been widely studied in areas of nonlinear stabilization and consensus of multiagent nonlinear systems; see [19, 24, 25] and references therein. By doing an equivalent transformation, many physical systems, including van der Pol oscillator, mass-spring-damper system with hardening springs, single link robotic manipulator with a flexible joint, and one-degree-of-freedom Euler-Lagrange systems, can be transformed into (1); see [24, 26, 27] for more physical systems. The conditions under which a nonlinear system can be transformed into the output-feedback form were elaborated in [26].

The regulated output (to be unavailable in general) and local measurement for feedback design are given by

$$e_i = y_i - q(v, w), \quad e_{mi} = \sum_{j=0}^{N} a_{ij}(y_i - y_j), \quad 1 \leqslant i \leqslant N,$$

respectively, with the weighted adjacency matrix $[a_{ij}]_{0\leqslant i,j\leqslant N}$ corresponding to a communication digraph $\mathcal{G}$[1)]. Let

$$H = [h_{ij}]_{1\leqslant i,j\leqslant N}, \quad h_{ij} \triangleq \begin{cases} \sum_{k=0}^{N} a_{ik}, & \text{if } i = j, \\ -a_{ij}, & \text{if } i \neq j, \end{cases}$$

and it can be seen that

$$e_m = He, \quad e_m \triangleq [e_{m1}, \ldots, e_{mN}]^{\mathrm{T}}, \quad e \triangleq [e_1, \ldots, e_N]^{\mathrm{T}}. \tag{4}$$

The problem undertaken in this paper is formulated as follows.

Problem 1 (CGOR). For systems (1) and (2) with a communication digraph $\mathcal{G}$, seek a controller of the form:

$$\dot{\chi}_i = f_{ci}(\chi_i, e_{mi}), \quad u_i = k_{ci}(\chi_i, e_{mi}), \quad 1 \leqslant i \leqslant N, \tag{5}$$

such that for any $[v(0)^{\mathrm{T}}, \sigma^{\mathrm{T}}, w^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{D} \triangleq \mathbb{V} \times \mathbb{S} \times \mathbb{W}$ and any initial conditions $x_i(0)$ and $\chi_i(0)$ in their entire spaces, the solution of the closed-loop system composed of (1) and (5) exists and is bounded over the time interval $[0, \infty)$, and moreover, the regulated output $e(t)$ satisfies $\lim_{t\to\infty} e(t) = 0$.

Before the end of this section, we list two standing assumptions.

Assumption 1. The communication digraph $\mathcal{G}$ contains a directed spanning tree with the leader as the root.

Assumption 2. For $1 \leqslant i \leqslant N$, the number of oscillation modes in the function $u_i^\star(\mu)$ is fixed uniformly in $(v(0), \sigma, w)$, where the function $u_i^\star(\mu)$ is given by

$$\frac{\partial x_i^\star}{\partial v}(\mu) S(\sigma) v = A_i x_i^\star(\mu) + g_i(q(v, w), v, w) + B_i u_i^\star(\mu), \tag{6}$$

for some function $x_i^\star(\mu)$ with $\mu \triangleq [v^{\mathrm{T}}, \sigma^{\mathrm{T}}, w^{\mathrm{T}}]^{\mathrm{T}}$.

Assumption 1 imposes a necessary condition in multiagent systems with fixed communication topologies. It implies existence of a diagonal matrix $Q = \mathrm{diag}(q_1, \ldots, q_N)$ with $q_i > 0$ such that

$$I_N - QH - H^{\mathrm{T}}Q \tag{7}$$

is negative definite; see [29, Theorem 2.3, pp. 134]. The diagonal structure of matrix Q will be used in the construction of certain non-quadratic Lyapunov function (see (36) later) for handling the asymmetric Laplacian matrix $\mathcal{L}$ of general directed communication graph $\mathcal{G}$.

Since functions $g_i(y_i, v, w)$ and $q(v, w)$ are polynomials in arguments y_i, v, Eq. (6) always admits polynomial functions $x_i^\star(\mu)$, $u_i^\star(\mu)$ in v. By [30], Assumption 2 is equivalent to the fact that for $1 \leqslant i \leqslant N$,

1) A communication digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ is defined with node set $\mathcal{V} \triangleq \{0, 1, 2, \ldots, N\}$, edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and weighted adjacency matrix $\mathcal{A} \triangleq [a_{ij}]_{0\leqslant i,j\leqslant N}$, where $a_{ii} = 0$, and $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$; see [28] for details.

there exist functions $C_{i,l}(v(0),w,\sigma)\neq 0$ for $1\leqslant l\leqslant s_i$ with an integer $s_i\geqslant 1$, and distinct constants $\hat{\omega}_{i,l}$, $1\leqslant l\leqslant s_i$ such that

$$u_i^\star(\mu)=\sum_{l=1}^{s_i}C_{i,l}(v(0),w,\sigma)\mathrm{e}^{\jmath\hat{\omega}_{i,l}t},\quad \jmath^2=-1. \tag{8}$$

This fact assures certain (incremental) iISS property of the designed nonlinear internal model.

To facilitate the subsequent analysis, we briefly introduce the notions of input-to-state stable (ISS) and iISS for the nonlinear system (see [21])

$$\dot{x}=f(x,u,\mu(t)), \tag{9}$$

where $x\in\mathbb{R}^{n_x}$ is the state, $u\in\mathbb{R}^{n_u}$ the control input, $\mu(t)\in\mathbb{D}$ for all $t\geqslant 0$ as the locally essentially bounded signal ranging in the compact set $\mathbb{D}$, and f as a smooth function with $f(0,0,\mu(t))=0$ for all $\mu(t)$. For the system (9), a smooth function $V:\mathbb{R}_+\times\mathbb{R}^{n_x}\rightarrow\mathbb{R}_+$ is called an iISS-Lyapunov function (with respect to state x and input u, robustly on μ) if, it satisfies, along trajectories of (9),

$$\underline{\alpha}(\|x\|)\leqslant V(t,x)\leqslant\bar{\alpha}(\|x\|),\ \dot{V}\leqslant\chi(\|u\|)-\alpha(\|x\|), \tag{10}$$

where $\underline{\alpha},\bar{\alpha}\in\mathcal{K}_\infty$, $\alpha\in\mathcal{K}$, and $\chi\in\mathcal{K}$. The system (9) is called iISS if it has an iISS-Lyapunov function satisfying (10). Moreover, if α in (10) can be chosen to be of $\mathcal{K}_\infty$, it establishes an ISS-Lyapunov function. Note that the iISS-Lyapunov function requires the function α to be either bounded $\mathcal{K}^o$ or unbounded $\mathcal{K}_\infty$ while the ISS one requires α to be a merely $\mathcal{K}_\infty$ function. Thus, the iISS-Lyapunov function is more general than the ISS one.

3 Filtering, internal models and network augmentation

It is well known that the main idea for approaching cooperative global robust output regulation is an effective problem conversion to an axillary cooperative global robust stabilization problem. The latter one can be made more tractable than the original CGOR. Specifically, this idea leads to a two-step design. In the first step, we address the problem conversion by constructing a reduced-order input-driven filter and a nonlinear internal model, and then performing a set of coordinate and input transformations for the so-called augmented system composed of the original plant, the proposed filters, and internal models. In the second step, we successfully construct a distributed stabilizer for the latter problem, eventually giving rise to a distributed controller for the original CGOR. In this section, we show the detailed construction of the reduced-order input-driven filter and the nonlinear internal model.

3.1 Local input-driven filters

Inspired by [23, 31], for $1\leqslant i\leqslant N$, we introduce a reduced-order input-driven filter taking the form:

$$\dot{\xi}_i=A_i^o\xi_i+B_i^ou_i,\quad \xi_i\in\mathbb{R}^{r_i-1}, \tag{11}$$

where

$$A_i^o=-\lambda_iI_{r_i-1}+\left[\begin{array}{c|c}0 & I_{r_i-2}\\ \hline 0 & 0\end{array}\right],\quad B_i^o=[0,\ldots,0,1]^{\mathrm{T}} \tag{12}$$

with $\lambda_i>0$. Thus, we can define the observation error as

$$\theta_i=\bar{L}_i\big(x_i-\bar{B}_i\xi_i\big),\quad 1\leqslant i\leqslant N, \tag{13}$$

where $\bar{B}_i\triangleq\big[[A_i+\lambda_iI_{r_i}]^{r_i-2}B_i,\ldots,[A_i+\lambda_iI_{r_i}]B_i,B_i\big]$, $\bar{L}_i=[-d_i,I_{r_i-1}]$ and $d_i=[d_{i,1},\ldots,d_{i,r_i-1}]^{\mathrm{T}}\in\mathbb{R}^{r_i-1}$ with $d_{i,1}=(r_i-1)\lambda_i$, $d_{i,2}=\frac{1}{2}(r_i-1)(r_i-2)\lambda_i^2,\ldots,d_{i,r_i-1}=\lambda_i^{r_i-1}$. It can be shown that

$$\dot{\theta}_i=F_i\theta_i+f_i^\theta(y_i,v,w),\quad 1\leqslant i\leqslant N, \tag{14}$$

where

$$f_i^\theta(y_i, v, w) = \bar{L}_i g_i(y_i, v, w) + \big[[d_{i,2}, \ldots, d_{i,r_i-1}, 0]^{\mathrm{T}} - d_{i,1} d_i\big] y_i,$$
$$F_i = \left[\begin{array}{c|c} -[d_{i,1}, \ldots, d_{i,r_i-2}]^{\mathrm{T}} & I_{r_i-2} \\ \hline -d_{i,r_i-1} & 0 \end{array}\right].$$

Note that instead of directly estimating the agent state x_i, by (13), the signal output $\bar{L}_i \bar{B}_i \xi_i$ of the filter (11) is to estimate $\bar{L}_i x_i = [x_{i,2} - d_{i,1} x_{i,1}, \ldots, x_{i,r_i} - d_{i,r_i-1} x_{i,1}]^{\mathrm{T}}$. Hence, the filter (11) can be understood as a local individual observer of reduced-order that achieves certain biased estimation of the partial state $[x_{i,2}, \ldots, x_{i,r_i}]$.

As a consequence, we obtain the filter extended system, comprised of (1) and (11), by the following equations:

$$\begin{aligned} \dot{\theta}_i &= F_i \theta_i + f_i^\theta(y_i, v, w), \\ \dot{y}_i &= \xi_{i,1} + g_i^y(\theta_i, y_i, v, w), \\ \dot{\xi}_i &= A_i^o \xi_i + B_i^o u_i, \quad 1 \leqslant i \leqslant N, \end{aligned} \tag{15}$$

where $g_i^y(\theta_i, y_i, v, w) = g_{i,1}(y_i, v, w) + \theta_{i,1} + d_{i,1} y_i$. Obviously, the CGOR for system (1) is equivalent to that for the filter extended system (15).

3.2 Internal models

Notice that functions $g_i(y_i, v, w)$ and $q(v, w)$ are polynomials in arguments y_i, v. Then, by [32, Lemmas 4.12 & 4.13], there exist smooth functions $\theta_i^\star(\mu)$, $y_i^\star(\mu)$ and $\xi_i^\star(\mu)$, which are polynomials in v, solving regulator equations:

$$\begin{aligned} \frac{\partial \theta_i^\star}{\partial v}(\mu) S(\sigma) v &= F_i \theta_i^\star(\mu) + f_i^\theta(y_i^\star(\mu), v, w), \\ \frac{\partial y_i^\star}{\partial v}(\mu) S(\sigma) v &= \xi_{i,1}^\star(\mu) + g_i^y(\theta_i^\star(\mu), y_i^\star(\mu), v, w), \\ \frac{\partial \xi_i^\star}{\partial v}(\mu) S(\sigma) v &= A_i^o \xi_i^\star(\mu) + B_i^o u_i^\star(\mu), \\ 0 &= y_i^\star(\mu) - q(v, w), \quad 1 \leqslant i \leqslant N. \end{aligned} \tag{16}$$

Note that as in [8,16], uncertain functions $\xi_i^\star(\mu)$ and $u_i^\star(\mu)$ are necessary steady-state information to be compensated. They have to be asymptotically reproduced by the so-called internal model.

By (8), the function $\xi_{i,1}(\mu)$ can be further expressed by

$$\xi_{i,1}^\star(\mu) = \sum_{l=1}^{s_i} \frac{1}{(\lambda_i + \jmath \hat{\omega}_{i,l})^{r_i-1}} C_{i,l}(v(0), w, \sigma) \mathrm{e}^{\jmath \hat{\omega}_{i,l} t} \tag{17}$$

implying that the number of oscillation modes in the function $\xi_{i,1}^\star(\mu)$ is fixed uniformly in $(v(0), \sigma, w)$. Hence, by [32, Chapter 6] and [33, Chapter 2], with $\tau_i(\mu) = [\xi_{i,1}^\star(\mu), \frac{d\xi_{i,1}^\star(\mu)}{dt}, \ldots, \frac{d^{s_i-1}\xi_{i,1}^\star(\mu)}{dt^{s_i-1}}]^{\mathrm{T}}$, we have

$$\dot{\tau}_i(\mu) = \Phi_i(p_i)\tau_i(\mu), \quad \xi_{i,1}^\star(\mu) = \Psi_i \tau_i(\mu) \tag{18}$$

with

$$\Phi_i(p_i) = \left[\begin{array}{c|c} 0 & I_{s_i-1} \\ \hline 0 & 0 \end{array}\right] - L_i p_i^{\mathrm{T}}, \quad \Psi_i = [1, 0, \ldots, 0]$$

for an uncertain vector $p_i \triangleq p_i(\sigma)$ and $L_i = [0, \ldots, 0, 1]^{\mathrm{T}}$. Inspired by [20,34] and for each $1 \leqslant i \leqslant N$, we further define

$$\eta_i^\star(\mu) = [\eta_i^{\star a}(\mu), \eta_i^{\star b}(\mu)^{\mathrm{T}}, \eta_i^{\star c}(\mu)^{\mathrm{T}}]^{\mathrm{T}}, \tag{19}$$

where $\eta_i^{\star a}(\mu) = [m_i - p_i]^{\mathrm{T}} T_i^{-1} \tau_i(\mu)$, $\eta_i^{\star b}(\mu) = T_i^{-1}\tau_i(\mu)$, and $\eta_i^{\star c}(\mu) = m_i - p_i$ with $T_i = [[[m_i - p_i]^{\mathrm{T}}[\Phi_i + I_{s_i}]]^{\mathrm{T}}, [[m_i - p_i]^{\mathrm{T}}[\Phi_i + I_{s_i}]\Phi_i]^{\mathrm{T}}, \ldots, [[m_i - p_i]^{\mathrm{T}}[\Phi_i + I_{s_i}]\Phi_i^{s_i-1}]^{\mathrm{T}}]^{\mathrm{T}}$ for a vector m_i such that the matrix

$$M_i = \left[\begin{array}{c|c} 0 & I_{s_i-1} \\ \hline 0 & 0 \end{array}\right] - L_i m_i^{\mathrm{T}}$$

is Hurwitz[2]. Then, by (18), we obtain

$$\dot{\eta}_i^{\star}(\mu) = \varphi_i(\eta_i^{\star}(\mu)) + G_i \xi_{i,1}^{\star}(\mu), \tag{20}$$

where

$$\varphi_i(\eta_i^{\star}) = \begin{bmatrix} -\eta_i^{\star a} \\ M_i\eta_i^{\star b} + L_i\eta_i^{\star a} \\ -\Lambda_i\eta_i^{\star b}(\eta_i^{\star b\mathrm{T}}\eta_i^{\star c} - \eta_i^{\star a}) \end{bmatrix}, \quad G_i = \begin{bmatrix} 1, 0, \ldots, 0 \end{bmatrix}^{\mathrm{T}} \tag{21}$$

for a positive definite matrix Λ_i. As an immediate consequence, it leads to an internal model candidate (see [32]):

$$\dot{\eta}_i = \varphi_i(\eta_i) + G_i\xi_{i,1} \tag{22}$$

with $\eta_i \in \mathbb{R}^{2s_i+1}$.

To establish a tractable augmented system, we define functions $\Gamma_i^{\xi} \triangleq [\Gamma_{i,1}, \ldots, \Gamma_{i,r_i-1}]^{\mathrm{T}}$ and $\Gamma_i^{u} \triangleq \Gamma_{i,r_i}$ by

$$\Gamma_{i,k}(\eta_i) = \eta_i^{c\mathrm{T}}[M_i + L_i\eta_i^{c\mathrm{T}} + I_{s_i}][M_i + L_i\eta_i^{c\mathrm{T}} + \lambda_i I_{s_i}]^{k-1}\eta_i^{b}\Omega(\delta_i + 1 - \|\eta_i\|^2), \quad 1 \leqslant k \leqslant r_i, \tag{23}$$

where $\delta_i \geqslant \max_{\eta_i \in \{\eta_i^{\star}(\mu):\mu\in\mathbb{D}\}} \|\eta_i\|^2$ and $\Omega(\zeta) = \frac{\nu(\zeta)}{\nu(\zeta)+\nu(1-\zeta)}$ with $\nu(\zeta) = \mathrm{e}^{-\frac{1}{\zeta}}$ when $\zeta > 0$; $\nu(\zeta) = 0$ when $\zeta \leqslant 0$. Then, it can be verified that

$$\xi_i^{\star}(\mu) = \Gamma_i^{\xi}(\eta_i^{\star}(\mu)), \quad u_i^{\star}(\mu) = \Gamma_i^{u}(\eta_i^{\star}(\mu)).$$

With the aid of the internal model (22), the CGOR for the filter extended system (15) is then equivalent to the cooperative global robust stabilization for the augmented system composed of the filter extended system (15) and the internal model (22), with respect to the global output zeroing manifold $\{(\theta_i, \eta_i, e_i, \xi_i, u_i) : \theta_i = \theta_i^{\star}(\mu),\ \eta_i = \eta_i^{\star}(\mu),\ y_i = y_i^{\star}(\mu),\ \xi_i = \Gamma_i^{\xi}(\eta_i),\ u_i = \Gamma_i^{u}(\eta_i)\}$.

3.3 Coordinate/input transformations

For making the augmented system more tractable, we define the new coordinates $(\bar{\theta}_i, \bar{\eta}_i, e_i, \bar{\xi}_i, \bar{u}_i)$ with

$$\begin{aligned} &\bar{\theta}_i = \theta_i - \theta_i^{\star}(\mu), \quad \bar{\eta}_i = \eta_i - \eta_i^{\star}(\mu) - G_i e_i, \\ &\bar{\xi}_i = \xi_i - \Gamma_i^{\xi}(\eta_i), \quad \bar{u}_i = u_i - \Gamma_i^{u}(\eta_i). \end{aligned} \tag{24}$$

Then, we can establish the following equivalent augmented system:

$$\begin{aligned} &\dot{\bar{\theta}}_i = F_i\bar{\theta}_i + \bar{f}_i^{\theta}(e_i, \mu), \\ &\dot{\bar{\eta}}_i = \bar{f}_i^{\eta}(\bar{\eta}_i, \bar{\theta}_i, e_i, \mu), \\ &\dot{e}_i = \bar{\xi}_{i,1} + \bar{g}_i^{e}(\bar{\theta}_i, \bar{\eta}_i, e_i, \mu), \\ &\dot{\bar{\xi}}_i = A_i^{o}\bar{\xi}_i + B_i^{o}\bar{u}_i + \bar{g}_i^{\xi}(\bar{\eta}_i, e_i, \bar{\xi}_{i,1}, \mu), \quad 1 \leqslant i \leqslant N, \end{aligned} \tag{25}$$

where

$$\begin{aligned} &\bar{f}_i^{\theta}(e_i, \mu) = f_i^{\theta}(e_i + q(v, w), v, w) - f_i^{\theta}(q(v, w), v, w), \\ &\bar{f}_i^{\eta}(\bar{\eta}_i, \bar{\theta}_i, e_i, \mu) = \varphi_i(\bar{\eta}_i + \eta_i^{\star}(\mu) + G_i e_i) - \varphi_i(\eta_i^{\star}(\mu)) - G_i\bar{g}_i^{y}(\bar{\theta}_i, e_i, \mu), \end{aligned}$$

2) Since the pair $([m_i - p_i]^{\mathrm{T}}[\Phi_i + I_{s_i}], \Phi_i)$ is observable, the matrix T_i is well-defined and nonsingular.

$$
\begin{aligned}
&\bar{g}_i^e(\bar{\theta}_i,\bar{\eta}_i,e_i,\mu) = \bar{g}_i^y(\bar{\theta}_i,e_i,\mu) + \Gamma_{i,1}^{\xi}(\bar{\eta}_i + \eta_i^\star(\mu) + G_i e_i) - \Gamma_{i,1}^{\xi}(\eta_i^\star(\mu)),\\
&\bar{g}_i^y(\bar{\theta}_i,e_i,\mu) = g_{i,1}(e_i + q(v,w),v,w) - g_{i,1}(q(v,w),v,w) + \bar{\theta}_{i,1} + d_{i,1}e_i,\\
&\bar{g}_i^{\xi}(\bar{\eta}_i,e_i,\bar{\xi}_{i,1},\mu) = A_i^o[\Gamma_i^{\xi}(\bar{\eta}_i + \eta_i^\star(\mu) + G_i e_i) - \Gamma_i^{\xi}(\eta_i^\star(\mu))],\\
&\quad + B_i^o[\Gamma_i^u(\bar{\eta}_i + \eta_i^\star(\mu) + G_i e_i) - \Gamma_i^u(\eta_i^\star(\mu))] - [\Gamma_i^{\xi'}(\eta_i)\dot{\eta}_i - \Gamma_i^{\xi'}(\eta_i^\star(\mu))\dot{\eta}_i^\star(\mu)].
\end{aligned} \tag{26}
$$

Equivalently, with $\bar{\theta} \triangleq [\bar{\theta}_1^{\rm T},\ldots,\bar{\theta}_N^{\rm T}]^{\rm T}$, $\bar{\eta} \triangleq [\bar{\eta}_1^{\rm T},\ldots,\bar{\eta}_N^{\rm T}]^{\rm T}$, $e \triangleq [e_1,\ldots,e_N]^{\rm T}$, $\bar{\xi} \triangleq [\bar{\xi}_1^{\rm T},\ldots,\bar{\xi}_N^{\rm T}]^{\rm T}$, and $\bar{u} \triangleq [\bar{u}_1,\ldots,\bar{u}_N]^{\rm T}$, we rewrite (25) in the following compact form:

$$
\begin{aligned}
\dot{\bar{\theta}} &= F\bar{\theta} + \bar{f}^{\theta}(\bar{\theta},e,\mu),\\
\dot{\bar{\eta}} &= \bar{f}^{\eta}(\bar{\eta},\bar{\theta},e,\mu),\\
\dot{e} &= C^o\bar{\xi} + \bar{g}^e(\bar{\theta},\bar{\eta},e,\mu),\\
\dot{\bar{\xi}} &= A^o\bar{\xi} + B^o\bar{u} + \bar{g}^{\xi}(\bar{\eta},e,C^o\bar{\xi},\mu),
\end{aligned} \tag{27}
$$

where, for short, $\bar{f}^{\theta} \triangleq [\bar{f}_1^{\theta{\rm T}},\ldots,\bar{f}_N^{\theta{\rm T}}]^{\rm T}$, $\bar{f}^{\eta} \triangleq [\bar{f}_1^{\eta{\rm T}},\ldots,\bar{f}_N^{\eta{\rm T}}]^{\rm T}$, $\bar{g}^e \triangleq [\bar{g}_1^e,\ldots,\bar{g}_N^e]^{\rm T}$, $\bar{g}^{\xi} \triangleq [\bar{g}_1^{\xi{\rm T}},\ldots,\bar{g}_N^{\xi{\rm T}}]^{\rm T}$, $F \triangleq {\rm diag}(F_1,\ldots,F_N)$, $A^o \triangleq {\rm diag}(A_1^o,\ldots,A_N^o)$, $B^o \triangleq {\rm diag}(B_1^o,\ldots,B_N^o)$, $C^o \triangleq {\rm diag}(C_1^o,\ldots,C_N^o)$ with $C_i^o = [1,0,\ldots,0]_{1\times(r_i-1)}$.

It can be shown that the augmented system (27) has an equilibrium point at $(\bar{\theta},\bar{\eta},e,\bar{\xi}) = (0,0,0,0)$, with respect to which, we can further define the cooperative global robust stabilization problem as follows.

Problem 2 (Cooperative global robust stabilization (CGS))**.** For the augmented system (25), seek a smooth controller of the form:

$$
\bar{u}_i = \rho_i(e_{mi},\bar{\xi}_i), \tag{28}
$$

such that the equilibrium point of the closed-loop system is globally uniformly asymptotically stable in the sense that

$$
\|x_c(t)\| \leqslant \beta(\|x_c(0)\|,t), \quad \forall x_c(0) \in \mathbb{R}^{n_c}, \quad \forall \mu \in \mathbb{D} \tag{29}
$$

for $\beta \in \mathcal{KL}$, where $x_c \triangleq [\bar{\theta}^{\rm T},\bar{\eta}^{\rm T},e^{\rm T},\bar{\xi}^{\rm T}]^{\rm T}$ and $n_c \triangleq \sum_{i=1}^N 2(r_i+s_i)$.

According to output regulation theory [32], it is concluded that CGOR can be solved by

$$
\begin{aligned}
\dot{\xi}_i &= A_i^o\xi_i + B_i^o u_i,\\
\dot{\eta}_i &= \varphi_i(\eta_i) + G_i\xi_{i,1},\\
u_i &= \rho_i\big(e_{mi},\xi_i - \Gamma_i^{\xi}(\eta_i)\big) + \Gamma_i^u(\eta_i), \quad 1 \leqslant i \leqslant N,
\end{aligned} \tag{30}
$$

as long as the auxiliary CGS for system (27) can be solved by a controller of the form (28). In this sense, we say the CGOR (Problem 1) has been converted into the CGS (Problem 2). The remained question is to find admissible nonlinear functions ρ_i for $1 \leqslant i \leqslant N$ in (28) so as to solve Problem 2.

4 Cooperative global stabilization for output regulation

In accordance with the preceding argument, it remains to solve the CGS for system (27) as described in Problem 2. The key technical challenge, compared with the ISS dynamic uncertainties as in [9, 35], lies in the fact that the augmented system (27) contains mixed ISS and iISS\ISS dynamic uncertainties $(\bar{\theta},\bar{\eta})$. Thus, as given in the forthcoming Lemma 1, we need to construct the corresponding ISS- and iISS-Lyapunov functions and a tight growth condition. A recursive design is then performed for a systematic design procedure of constructing a controller (28).

4.1 Mixed ISS and iISS\ISS dynamic uncertainties

To be more specific, we establish the following stabilizability result for the augmented system (27). For the sake of completeness, the proof of Lemma 1 is shown and put in Appendix A.

Lemma 1. Consider the augmented system (27) under Assumption 2. Then, both the following properties hold.

(i) There are an ISS-Lyapunov function $V_1(\bar{\theta})$ and an iISS-Lyapunov function $V_2(t,\bar{\eta})$ such that, along the trajectories of (27),

$$\underline{\alpha}_1\|\bar{\theta}\|^2 \leqslant V_1(\bar{\theta}) \leqslant \bar{\alpha}_1\|\bar{\theta}\|^2, \qquad \dot{V}_1 \leqslant -\alpha_1 V_1(\bar{\theta}) + \gamma_{1a}(\|e\|^2), \tag{31a}$$

$$\underline{\alpha}_2(\|\bar{\eta}\|) \leqslant V_2(t,\bar{\eta}) \leqslant \bar{\alpha}_2(\|\bar{\eta}\|), \quad \dot{V}_2 \leqslant -\alpha_2 \circ V_2(t,\bar{\eta}) + \gamma_{2a}(\|e\|^2) + \gamma_{21}V_1(\bar{\theta}), \tag{31b}$$

where $\underline{\alpha}_1, \bar{\alpha}_1, \alpha_1, \gamma_{21} > 0$, $\gamma_{1a}, \gamma_{2a} \in \widehat{\mathcal{O}}(\mathrm{Id})$, and

$$\underline{\alpha}_2(s) = \ln(1+\underline{\ell}_2 s^2), \quad \bar{\alpha}_2(s) = \bar{\ell}_2 s^2, \quad \alpha_2(s) = \frac{\ell_2 s}{1+s} \tag{32}$$

for some constants $\underline{\ell}_2, \bar{\ell}_2, \ell_2 > 0$.

(ii) The (global) growth conditions

$$\|\bar{g}^e(\bar{\theta},\bar{\eta},e,\mu)\|^2 \leqslant \phi_a(\|e\|^2) + \phi_{a1}V_1(\bar{\theta}) + \phi_{a2}\circ V_2(t,\bar{\eta}), \tag{33a}$$

$$\|\bar{g}^\xi(\bar{\eta},e,C^o\bar{\xi},\mu)\|^2 \leqslant \phi_{ba}(\|e\|^2) + \phi_{b2}\circ V_2(t,\bar{\eta}) + \phi_b\|C^o\bar{\xi}\|^2 \tag{33b}$$

hold for $\phi_a, \phi_{ba} \in \widehat{\mathcal{O}}(\mathrm{Id})$, $\phi_{a2}, \phi_{b2} \in \widehat{\mathcal{O}}(\alpha_2)$, and $\phi_{a1}, \phi_b > 0$.

Remark 2. Normally, functions in the right side of the growth condition (33) can be chosen all belonging to $\mathcal{K}$, and usually $\mathcal{K}_\infty$ for simplicity, provided that the dynamic uncertainties of the augmented system (27) are all ISS, see [9, 35]. In contrast, from (31b), the dynamic uncertainty $\bar{\eta}$ satisfies merely the iISS\ISS condition. Hence, the growth condition (33) has to be chosen much tighter in the sense that those nonlinear functions associated with $\bar{\eta}$, i.e., ϕ_{a2} and ϕ_{b2} have to be chosen belonging to $\mathcal{K}^o$ but not $\mathcal{K}_\infty$. Indeed, such growth condition is assured by the introduction of the function $\Omega(\cdot)$ in (23). The feature of this tighter growth condition is, as shown in Subsection 4.2, to ensure existence of the nonlinear feedback controller by a recursive design.

4.2 Recursive design

In what follows and for the sake of convenience, we define, for $1 \leqslant i \leqslant N$,

$$\begin{aligned}&\tilde{\xi} = [\tilde{\xi}_1^{\mathrm{T}}, \ldots, \tilde{\xi}_N^{\mathrm{T}}]^{\mathrm{T}}, \quad \tilde{\xi}_i = [\tilde{\xi}_{i,1}, \ldots, \tilde{\xi}_{i,r_i-1}]^{\mathrm{T}},\\ &\tilde{\xi}_{i,l} = \bar{\xi}_{i,l} - \varrho_{i,l}(e_{mi}, \bar{\xi}_{i,1}, \ldots, \bar{\xi}_{i,l-1}), \quad 1 \leqslant l \leqslant r_i - 1\end{aligned} \tag{34}$$

with

$$\begin{aligned}&\varrho_{i,1}(e_{mi}) = -\kappa_i(e_{mi}),\\ &\varrho_{i,2}(e_{mi},\bar{\xi}_{i,1}) = \lambda_i\bar{\xi}_{i,1} - (c+1)\tilde{\xi}_{i,1} - \frac{[\partial\varrho_{i,1}/\partial e_{mi}]^2}{2\epsilon}\tilde{\xi}_{i,1},\\ &\varrho_{i,l}(e_{mi},\bar{\xi}_{i,1},\ldots,\bar{\xi}_{i,l-1}) = \lambda_i\bar{\xi}_{i,l-1} - 2\tilde{\xi}_{i,l-1} - \frac{[\partial\varrho_{i,l-1}/\partial e_{mi}]^2}{2\epsilon}\tilde{\xi}_{i,l-1} - \sum_{k=1}^{l-2}\frac{[\partial\varrho_{i,l-1}/\partial\bar{\xi}_{i,k}]^2}{2\epsilon}\tilde{\xi}_{i,l-1}\\ &\qquad + \sum_{k=1}^{l-2}[\partial\varrho_{i,l-1}/\partial\bar{\xi}_{i,k}][-\lambda_i\bar{\xi}_{i,k} + \bar{\xi}_{i,k+1}], \quad 3 \leqslant l \leqslant r_i,\end{aligned} \tag{35}$$

where $c > 0$, $\epsilon > 0$, $\kappa_i(s) = \bar{\kappa}_i(s)s$ is a smooth function $\bar{\kappa}_i \geqslant 1$ that is even (i.e., $\bar{\kappa}_i(s) = \bar{\kappa}_i(-s)$) and increasing over $[0, +\infty)$.

Theorem 1. Under Assumptions 1 and 2, Problem 1 can be solved by a controller of the form (30) with $\rho_i(e_{mi},\bar{\xi}_i) = \varrho_{i,r_i}(e_{mi},\bar{\xi}_i)$ recursively given by (35).

Proof. According to the argument presented in Section 3, it remains to show that the controller (28) with $\rho_i(e_{mi},\bar{\xi}_i) = \varrho_{i,r_i}(e_{mi},\bar{\xi}_i)$ solves Problem 2.

First, consider the e subsystem. Define

$$V_a(e_m) = \sum_{i=1}^{N} q_i \int_0^{e_{mi}} \kappa_i(s)ds, \tag{36}$$

where q_i is as in (7), and κ_i is the same function as in (35). Since

$$\sum_{i=1}^{N} \frac{1}{2} q_i e_{mi}^2 \leqslant V_a(e_m) \leqslant \sum_{i=1}^{N} q_i \bar{\kappa}_i(e_{mi}) e_{mi}^2, \tag{37}$$

the function $V_a(e_m)$ is positive definite and radially unbounded. It can be further shown that, along the trajectories of the closed-loop system,

$$\begin{aligned} \dot{V}_a &= \kappa(e_m)^{\mathrm{T}} QH[C^o\bar{\xi} + \bar{g}^e(\bar{\theta}, \bar{\eta}, e, \mu)] \\ &= \kappa(e_m)^{\mathrm{T}} QH[-\kappa(e_m) + C^o\tilde{\xi} + \bar{g}^e(\bar{\theta}, \bar{\eta}, e, \mu)], \end{aligned} \tag{38}$$

where $\kappa(e_m) \triangleq [\kappa_1(e_{m1}), \ldots, \kappa_N(e_{mN})]^{\mathrm{T}}$. In (38), by (7), we have

$$\kappa(e_m)^{\mathrm{T}} QH\kappa(e_m) = \frac{1}{2}\kappa(e_m)^{\mathrm{T}}(QH + H^{\mathrm{T}}Q)\kappa(e_m) \geqslant \frac{1}{2}\|\kappa(e_m)\|^2.$$

Moreover, by completing the squares and using (33a), we have

$$\begin{aligned} &\kappa(e_m)^{\mathrm{T}} QH\big[C^o\tilde{\xi} + \bar{g}^e(\bar{\theta}, \bar{\eta}, e, \mu)\big] \\ &\quad\leqslant \frac{1}{6}\|\kappa(e_m)\|^2 + 2\|QH\|^2\|C^o\tilde{\xi}\|^2 + 2\|QH\|^2\big[\phi_a(\|e\|^2) + \phi_{a1}V_1(\bar{\theta}) + \phi_{a2}\circ V_2(t, \bar{\eta})\big]. \end{aligned}$$

Consequently, it implies that

$$\dot{V}_a \leqslant -\frac{1}{3}\|\kappa(e_m)\|^2 + 3\|QH\|^2\|C^o\tilde{\xi}\|^2 + 3\|QH\|^2\big[\phi_a(\|e\|^2) + \phi_{a1}V_1(\bar{\theta}) + \phi_{a2}\circ V_2(t, \bar{\eta})\big].$$

Second, consider the $\bar{\xi}$ subsystem under transformation (34). Let $V_b(\tilde{\xi}) = \frac{1}{2}\sum_{i=1}^{N}\tilde{\xi}_i^{\mathrm{T}}\tilde{\xi}_i$, which, by (33), satisfies

$$\begin{aligned} \dot{V}_b &\leqslant -c\|C^o\tilde{\xi}\|^2 - \|\tilde{\xi}\|^2 + \frac{1}{2}\bar{r}\epsilon\big[\|H\|^2\|C^o\bar{\xi} + \bar{g}^e(\bar{\theta}, \bar{\eta}, e, \mu)\|^2 + \|\bar{g}^{\xi}(\bar{\eta}, e, C^o\bar{\xi}, \mu)\|^2\big] \\ &\leqslant -\big[c - \bar{r}\epsilon(2\|H\|^2 + \phi_b)\big]\|C^o\tilde{\xi}\|^2 - \|\tilde{\xi}\|^2 + \bar{r}\epsilon(\|H\|^2 + \phi_b)\|\kappa(e_m)\|^2 \\ &\quad + \frac{1}{2}\bar{r}\epsilon\big[4\|H\|^2\phi_a(\|e\|^2) + \phi_{ba}(\|e\|^2)\big] + 2\bar{r}\epsilon\|H\|^2\phi_{a1}V_1(\bar{\theta}) \\ &\quad + \frac{1}{2}\bar{r}\epsilon\big[4\|H\|^2\phi_{a2}\circ V_2(t, \bar{\eta}) + \phi_{b2}\circ V_2(t, \bar{\eta})\big] \end{aligned}$$

with $\bar{r} = \max_{1\leqslant i\leqslant N}(r_i - 1)$.

Finally, let

$$V(t, \bar{\theta}, \bar{\eta}, e_m, \tilde{\xi}) = \nu_1 V_1(\bar{\theta}) + \nu_2 V_2(t, \bar{\eta}) + V_a(e_m) + V_b(\tilde{\xi}) \tag{39}$$

for some constants $\nu_1, \nu_2 > 0$ determined by (43) later. Then, by (31) and (37), it can be verified that

$$\underline{\alpha}(\|\bar{\theta}, \bar{\eta}, e_m, \tilde{\xi}\|) \leqslant V(t, \bar{\theta}, \bar{\eta}, e_m, \tilde{\xi}) \leqslant \bar{\alpha}(\|\bar{\theta}, \bar{\eta}, e_m, \tilde{\xi}\|) \tag{40}$$

for some functions $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$. Moreover, the function V satisfies, along the trajectories of the closed-loop system,

$$\begin{aligned} \dot{V} \leqslant& -\left[\frac{1}{3} - (\|H\|^2 + \phi_b)\bar{r}\epsilon\right]\|\kappa(e_m)\|^2 + \gamma(\|e\|^2) - \big[c - \bar{r}\epsilon(2\|H\|^2 + \phi_b) - 3\|QH\|^2\big]\|C^o\tilde{\xi}\|^2 - \|\tilde{\xi}\|^2 \\ &- \big[\nu_1\alpha_1 - \nu_2\gamma_{21} - 3\|QH\|^2\phi_{a1} - 2\|H\|^2\bar{r}\epsilon\phi_{a1}\big]V_1(\bar{\theta}) - \nu_2\alpha_2\circ V_2(t, \bar{\eta}) + \frac{1}{2}\bar{r}\epsilon\phi_{b2}\circ V_2(t, \bar{\eta}) \\ &+ \big(3\|QH\|^2 + 2\|H\|^2\bar{r}\epsilon\big)\phi_{a2}\circ V_2(t, \bar{\eta}) \end{aligned} \tag{41}$$

with

$$\gamma(\|e\|^2) = \nu_1\gamma_{1a}(\|e\|^2) + \nu_2\gamma_{2a}(\|e\|^2) + 3\|QH\|^2\phi_a(\|e\|^2) + 2\|H\|^2\bar{r}\epsilon\phi_a(\|e\|^2) + \frac{1}{2}\bar{r}\epsilon\phi_{ba}(\|e\|^2).$$

Since $\phi_{a2}, \phi_{b2} \in \widehat{\mathcal{O}}(\alpha_2)$, by [20, Lemma A.3], there exists a constant $\nu_2^\star > 0$ such that

$$\big(3\|QH\|^2 + 2\|H\|^2\bar{r}\epsilon\big)\phi_{a2} \circ V_2(t, \bar{\eta}) + \frac{1}{2}\bar{r}\epsilon\phi_{b2} \circ V_2(t, \bar{\eta}) \leqslant \nu_2^\star \alpha_2 \circ V_2(t, \bar{\eta}).$$

Moreover, since $\gamma_{1a}, \gamma_{2a}, \phi_a, \phi_{ba} \in \widehat{\mathcal{O}}(\mathrm{Id})$, by [32, Lemma 7.8], it is easy to obtain a function $\kappa_i^\star$, written by $\kappa_i^\star(s) = \bar{\kappa}_i^\star(s)s$ for some smooth even function $\bar{\kappa}_i^\star \geqslant 1$, such that

$$\gamma(\|e\|^2) = \gamma(\|H^{-1}e_m\|^2) \leqslant \sum_{i=1}^{N} \kappa_i^\star(e_{mi})^2. \tag{42}$$

Consequently, in (41), choose constants $\epsilon, \nu_1, \nu_2, c$ and the smooth function $\bar{\kappa}_i$ such that

$$\begin{aligned}
&\epsilon \leqslant \frac{1}{4(\|H\|^2 + \phi_b)\bar{r}}, \quad \nu_2 \geqslant \nu_2^\star + 1, \\
&c \geqslant \bar{r}\epsilon(2\|H\|^2 + \phi_b) + 3\|QH\|^2, \\
&\nu_1 \geqslant \frac{1}{\alpha_1}\big(\nu_2\gamma_{21} + 3\|QH\|^2\phi_{a1} + 2\|H\|^2\bar{r}\epsilon\phi_{a1} + 1\big), \\
&\frac{1}{2}\bar{\kappa}_i(e_{mi}) \geqslant \bar{\kappa}_i^\star(e_{mi}) + 1.
\end{aligned} \tag{43}$$

Then, by (37), we obtain the dissipation inequality

$$\dot{V} \leqslant -V_1(\bar{\theta}) - \alpha_2 \circ V_2(t, \bar{\eta}) - \alpha_a V_a(e_m) - V_b(\tilde{\xi}) \tag{44}$$

with $\alpha_a \triangleq \frac{1}{6\max_{1\leqslant i\leqslant N} q_i}$. By (32), (39) and using inequality $\frac{a}{1+a} + b \geqslant \frac{a+b}{1+a+b}$ for $a, b \in \mathbb{R}_+$, the above dissipation inequality gives

$$\dot{V} \leqslant -\alpha \circ V(t, \bar{\theta}, \bar{\eta}, e_m, \tilde{\xi}), \quad \alpha(s) = \frac{c_1 s}{1 + c_2 s}$$

with $c_1 \triangleq \frac{\min\{1, \ell_2, \alpha_a\}}{\max\{1, \nu_1, \nu_2\}}$ and $c_2 \triangleq \frac{\max\{1, \alpha_a\}}{\min\{1, \nu_1, \nu_2\}}$.

Thus, by [36, Lemma 3.4 & Lemma 4.4], it can be shown that, there exists a function $\bar{\beta}(s,t) = \underline{\alpha}^{-1} \circ \alpha_0^{-1} \circ (\alpha_0 \circ \bar{\alpha}(s) \cdot \mathrm{e}^{-c_1 t}) \in \mathcal{KL}$ with $\alpha_0(s) \triangleq s\mathrm{e}^{c_2 s}$ such that, by (40),

$$\|(\bar{\theta}(t), \bar{\eta}(t), e_m(t), \tilde{\xi}(t))\| \leqslant \bar{\beta}(\|(\bar{\theta}(0), \bar{\eta}(0), e_m(0), \tilde{\xi}(0))\|, t).$$

Moreover, by (4) and (34), we can find functions $\underline{\alpha}_c, \bar{\alpha}_c \in \mathcal{K}_\infty$ such that $\underline{\alpha}_c(\|x_c\|) \leqslant \|(\bar{\theta}, \bar{\eta}, e_m, \tilde{\xi})\| \leqslant \bar{\alpha}_c(\|x_c\|)$, which completes the proof with $\beta(s,t) = \underline{\alpha}_c^{-1} \circ \bar{\beta}(\bar{\alpha}_c(s), t)$.

Remark 3. Though the subsystems of system (1) are just in the representative output-feedback form, it is of interest to note that our developed result can be extended to the more general output-feedback form with extra various dynamic uncertainties; see [9] for ISS one and [22,37] for iISS ones. Such an extension can be addressed in almost the same way as the one in the present paper except for a precise redesign of κ_i by further taking the ISS or iISS properties of these extra dynamic uncertainties into account, and thus, is not the main objective of this paper and omitted here.

Remark 4. A relevant result can be found in [13] on an output-feedback controller for the multiagent system (1) with a special constant leader, i.e., the reference y_0 is constant. From this viewpoint, the present result extends that of [13] from constant leaders to unknown active leaders. Some efforts have also been made to establish adaptive distributed observers; see [10]. For a comparison, in [10], the dynamics of the leader is not required for each follower, and only the so-called informed followers need to access full information of the leader dynamics.

4.3 Case study: controlling linear networks by nonlinear output-feedback

We show an impact of preceding result for an unknown leader-follower linear network. Let us consider the linear single-input single-output network described by

$$\dot{z}_i = \boldsymbol{A}_i z_i + \boldsymbol{B}_i u_i, \quad y_i = \boldsymbol{C}_i z_i, \quad z_i \in \mathbb{R}^{r_i}, \quad 0 \leqslant i \leqslant N \tag{45}$$

with $u_0 \equiv 0$. Suppose that each matrix triplet $(\boldsymbol{A}_i, \boldsymbol{B}_i, \boldsymbol{C}_i)$ for $0 \leqslant i \leqslant N$ takes the typical controllable and observable form:

$$\boldsymbol{A}_i = \left[\begin{array}{c|c} 0 & I_{r_i-1} \\ \hline \bar{a}_{i,1} & \bar{a}_{i,2}, \ldots, \bar{a}_{i,r_i} \end{array}\right], \quad \boldsymbol{B}_i = [0, \ldots, 0, 1]^{\mathrm{T}}, \quad \boldsymbol{C}_i = [1, 0, \ldots, 0]. \tag{46}$$

Here system (45) is called unknown in the sense that all the parameters $\bar{a}_{i,1}, \ldots, \bar{a}_{i,r_i}$ are unknown bounded constants.

Similar to (35), for $1 \leqslant i \leqslant N$, let

$$\begin{aligned} &k_{i,1} = -c_i\left(c_0 + 1 + \frac{c_i^2}{2\epsilon}\right), \quad K_{i,1} = -\left(c_0 + 1 + \frac{c_i^2}{2\epsilon} - \lambda_i\right), \\ &k_{i,l} = k_{i,l-1} l_{i,l-1}, \quad l_{i,l-1} \triangleq \frac{1}{2\epsilon}(k_{i,l-1}^2 + K_{i,l-1}K_{i,l-1}^{\mathrm{T}}) + 2, \\ &K_{i,l} = (l_{i,l-1} - \lambda_i)[K_{i,l-1}, 1] + [0, K_{i,l-1}], \quad 2 \leqslant l \leqslant r_i - 1 \end{aligned} \tag{47}$$

for constants $c_i, \epsilon > 0$ for $0 \leqslant i \leqslant N$ to be determined. In particular, without loss of generality, we assume that the initial condition $z_0(0)$ excites all oscillation modes of the leader dynamics.

Corollary 1. Consider network (45) under Assumption 1. Suppose that, for $1 \leqslant i \leqslant N$, characterization polynomials of the matrix pair $(\boldsymbol{A}_0, \boldsymbol{A}_i)$ are coprime. Then, Problem 1 for the linear unknown network (45) can be solved by the controller

$$\begin{aligned} &\dot{\xi}_i = A_i^o \xi_i + B_i^o u_i, \quad \dot{\eta}_i = \varphi_i(\eta_i) + G_i \xi_{i,1}, \\ &u_i = k_{i,r_i-1} e_{mi} + K_{i,r_i-1}(\xi_i - \Gamma_i^{\xi}(\eta_i)) + \Gamma_i^u(\eta_i), \quad 1 \leqslant i \leqslant N \end{aligned} \tag{48}$$

with k_{i,r_i-1} and K_{i,r_i-1} given by (47).

Sketch of the Proof. The proof of Corollary 1 can be done by the following.

- First, performing the coordinate transformation $x_i = T_i z_i$ with $T_i \triangleq [\boldsymbol{A}_i^{r_i-1}\boldsymbol{B}_i, \ldots, \boldsymbol{A}_i\boldsymbol{B}_i, \boldsymbol{B}_i]^{-1}$ for $1 \leqslant i \leqslant N$, we obtain the linear case of the form (1), i.e.,

$$\dot{x}_i = A_i x_i + E_i y_i + B_i u_i, \quad y_i = C_i x_i, \quad 1 \leqslant i \leqslant N, \tag{49}$$

where $E_i = T_i \boldsymbol{A}_i^{r_i} \boldsymbol{B}_i$; see [38].

- Second, the rest of proof can be completed as a direct consequence of that of Theorem 1 and constants $c_i, \epsilon > 0$ for $0 \leqslant i \leqslant N$ in (47) can be specified. The coprime condition on the matrix pair $(\boldsymbol{A}_0, \boldsymbol{A}_i)$ is used to assure Assumption 2.

For the above result on linear unknown networks, we note that the parameters of system matrices $\boldsymbol{A}_i$ for $0 \leqslant i \leqslant N$ are all unknown. In others words, the developed control is independent of those parameters.

Remark 5. It is known that when the active leader is known, a linear controller is enough to achieve the control goal for linear networks. As shown in (48), the output-feedback design is essentially nonlinear if it is the case of unknown active leaders. Thus such nonlinearity is basically necessary to cope with this situation. In fact, by the internal model principle (see [32, 39]), to achieve the CGOR, the leader dynamics should be incorporated in any successful controller and thus leads to the nonlinearity under discussion.

Remark 6. Compared with [2], the present study is distinguished from the following aspects. First, the active leader is allowed to be unknown and thus includes more general reference and disturbance signals. Second, the linear network (45) is heterogeneous and uncertain, rather than the simply single-integrator or second-order multiagent systems as in [2]. Third, by introducing reduced-order input driven filers, an output-feedback controller has been established, while the controller developed in [2] relies on agent velocities besides the relative positions.

5 Illustration

In this section, we present an example to illustrate the developed method. Consider a heterogeneous network of five mass-spring-damper systems with hardening springs, adopted from [40], described by

$$\ddot{y}_i + w_{i,1}\dot{y}_i + f_i(y_i, w_{i,2}, w_{i,3}) = u_i, \quad 1 \leqslant i \leqslant 5, \tag{50}$$

where $y_i \in \mathbb{R}$ is the output, $u_i \in \mathbb{R}$ is the controller, $f_i(y_i, w_{i,2}, w_{i,3}) \triangleq w_{i,2}y_i + w_{i,3}y_i^3$ denotes the hardening spring, $w \triangleq [w_{1,1}, w_{1,2}, w_{1,3}, \ldots, w_{5,1}, w_{5,2}, w_{5,3}]^{\mathrm{T}}$ describes the parametric uncertainty with $w_{i,k} > 0$.

We choose a reference signal $y_0 = A_m \cos(\sigma t)$ for simulation. It can be easily shown that y_0 can be exactly modeled by (2) with $n_v = 2$, $y_0 = v_1$, and $v(0) = [A_m, 0]^{\mathrm{T}}$, to be unknown. Choose the communication digraph specified by the matrix $H = [1, 0, 0, -1/3, -1/3; -1/2, 1, 0, 0, -1/2; 0, -1/3, 1, -1/3, 0; -1/2, -1/2, 0, 1, 0; 0, 0, -1/2, -1/2, 1]$, which verifies Assumption 1. Assume that

$$\mathbb{W} \triangleq \{w \in \mathbb{R}^{15} : 0.1 \leqslant w_k \leqslant 2,\ 1 \leqslant k \leqslant 15\},$$
$$\mathbb{S} \triangleq \{\sigma \in \mathbb{R} : 0.1 \leqslant \sigma \leqslant \pi/2\},\quad \mathbb{V} \triangleq \{v \in \mathbb{R}^2 : 0.1 \leqslant \|v\| \leqslant 1\}.$$

By letting $x_i = [y_i, \dot{y}_i + w_{i,1}y_i]^{\mathrm{T}}$, we rewrite (50) in the form (1) with $r_i = 2$. Note that using Euler's formula, we can express the function $u_i^\star(\mu)$ in (8) as

$$u_i^\star(\mu) = \varpi_{i,1}\mathrm{e}^{\jmath\sigma t} + \varpi_{i,2}\mathrm{e}^{-\jmath\sigma t} + \varpi_{i,3}\mathrm{e}^{\jmath(3\sigma)t} + \varpi_{i,4}\mathrm{e}^{-\jmath(3\sigma)t},$$

where $\varpi_{i,1} = \frac{1}{2}(-A_m\sigma^2 + w_{i,2}A_m + \frac{3}{4}A_m^3\sigma^3 + \jmath w_{i,1}A_m\sigma)$, $\varpi_{i,2} = \frac{1}{2}(-A_m\sigma^2 + w_{i,2}A_m + \frac{3}{4}A_m^3\sigma^3 - \jmath w_{i,1}A_m\sigma)$, $\varpi_{i,3} = \frac{1}{8}A_m^3$, $\varpi_{i,4} = \frac{1}{8}A_m^3$. Clearly, all $\varpi_{i,1}, \varpi_{i,2}, \varpi_{i,3}, \varpi_{i,4}$ are nonzero for $[v(0)^{\mathrm{T}}, w^{\mathrm{T}}, \sigma] \in \mathbb{V} \times \mathbb{W} \times \mathbb{S}$, verifying Assumption 2. Thus, by Theorem 1, the tracking problem can be solved by a two-step design.

In the first step, by introducing the single-order filter $\dot{\xi}_i = -\xi_i + u_i$, we obtain the filter extended system of the form (15) with

$$\xi_i^\star(\mu) = \frac{\varpi_{i,1}\mathrm{e}^{\jmath\sigma t}}{1 + \jmath\sigma} + \frac{\varpi_{i,2}\mathrm{e}^{-\jmath\sigma t}}{1 - \jmath\sigma} + \frac{\varpi_{i,3}\mathrm{e}^{3\jmath\sigma t}}{1 + 3\jmath\sigma} + \frac{\varpi_{i,4}\mathrm{e}^{-3\jmath\sigma t}}{1 - 3\jmath\sigma}.$$

Then, we can construct a nonlinear internal model $\dot{\eta}_i = \varphi_i(\eta_i) + G_i\xi_i$ with φ_i specified by (21) for $s_i = 4$, $m_i = [0.9999, 2.7000, 3.4000, 2.1000]^{\mathrm{T}}$, $L_i = [0, 0, 0, 1]^{\mathrm{T}}$, $\Lambda_i = \mathrm{diag}(5.21, 5.21, 5.21, 5.21)$, and $G_i = [1, 0, 0, 0, 0, 0, 0, 0, 0]^{\mathrm{T}}$. Also, by definition, we conclude that $\|\eta_i^\star(\mu)\| \leqslant 50$ for all $\mu \in \mathbb{V} \times \mathbb{W} \times \mathbb{S}$, leading to functions $\Gamma_i^\xi(\eta_i)$ and $\Gamma_i^u(\eta_i)$ in (23) with $\delta_i = 50$. Thus, by performing the transformation (24), we obtain the augmented system of the form (27). By a lengthy yet straightforward calculation as in the proof of Lemma 1, we can find functions $V_1(\bar{\theta})$ and $V_2(t, \bar{\eta})$ in (31) such that

$$\dot{V}_1 \leqslant -V_1 + 5\gamma_0(\|e\|^2),\quad \dot{V}_2 \leqslant -\alpha_2(V_2) + 5(\|e\|^2 + V_1)$$

with $\gamma_0(s) \triangleq 3s + s^3$, $\alpha_2(s) \triangleq \frac{s}{2(1+s)}$ and the growth condition (33) satisfies

$$\|\bar{g}^e(\bar{\theta}, \bar{\eta}, e, \mu)\|^2 \leqslant 10\|e\|^2 + 5V_1(\bar{\theta}) + 8\alpha_2 \circ V_2(t, \bar{\eta}),$$
$$\|\bar{g}^\xi(\bar{\eta}, e, \bar{\xi}, \mu)\|^2 \leqslant 15\|e\|^2 + 10\alpha_2 \circ V_2(t, \bar{\eta}) + 6\|\bar{\xi}\|^2.$$

In the second step, note that $\|H\| = 1.5224$, $Q = \mathrm{diag}(3.1515, 2.8048, 1.7648, 3.4036, 2.5002)$, $\|QH\| = 4.9652$. By (43), we can choose $\epsilon = 1/40$, $c = 75$. Furthermore, by (42), we can choose $\kappa_i(e_{mi}) = 30(5 + e_{mi}^2)e_{mi}$.

Summarized from the above two steps, we construct a distributed controller

$$\begin{aligned}
&\dot{\xi}_i = -\xi_i + u_i,\quad \dot{\eta}_i = \varphi_i(\eta_i) + G_i\xi_i, \\
&u_i = \Gamma_i^u(\eta_i) + \xi_i - \Gamma_i^\xi(\eta_i) - (c+1)[\xi_i - \Gamma_i^\xi(\eta_i) + \kappa_i(e_{mi})] - \frac{[\partial\kappa_i/\partial e_{mi}]^2}{2\epsilon}[\xi_i - \Gamma_i^\xi(\eta_i) + \kappa_i(e_{mi})].
\end{aligned} \tag{51}$$

A simulation result is shown in Figures 1 and 2 with parameters $(A_m, \sigma) = (0.8, 2\pi/5)$ and $(w_{i,1}, w_{i,2}, w_{i,3}) = (2 - 0.2i, 1 + 0.1i, 1 + 0.2i)$ and a randomly generated follower output initial condition $y_i(0)$, and $\dot{y}_i(0) = 0$, $\xi_i(0) = 0$, $\eta_i(0) = 0$ for $1 \leqslant i \leqslant 5$. Our designed distributed controller achieves the tracking control goal.

6 Conclusion

The CGOR for a class of heterogeneous and uncertain nonlinear multiagent systems was studied. For the general directed dynamic networks in the presence of an unknown active leader, we presented a nonlinear

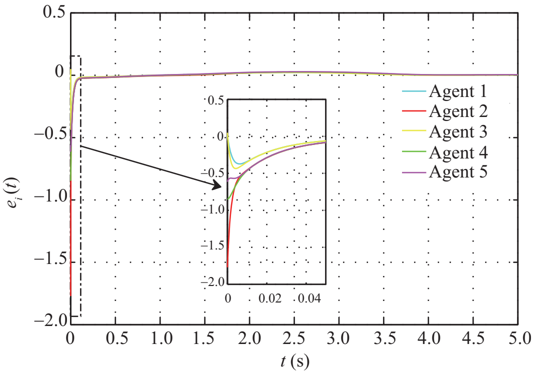

Figure 1 (Color online) Profiles of e_i for $1 \leqslant i \leqslant 5$.

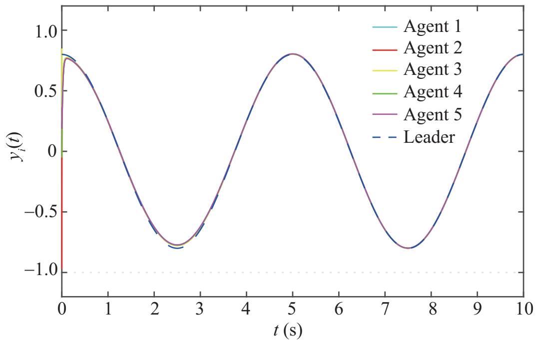

Figure 2 (Color online) Profiles of y_i for $1 \leqslant i \leqslant 5$ (dashed blue line is leader output y_0).

output-feedback design comprising three steps: filter design, internal model design, and stabilizer design. Particularly in the stabilization step, we presented successful output-feedback for the augmented system having mixed ISS and iISS\ISS dynamic uncertainties, which, eventually, gives rise to a solution to the CGOR. Note that our control gains are dependent on the information of the Laplacian matrix. One future direction is to further address distributed output-feedback design for the same problem with measurement uncertainty; see [41] for a novel adaptive output-feedback stabilization design for the single agent with measurement uncertainty.

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Appendix A Proof of Lemma 1

Appendix A.1 Proof of P1

First, consider $\bar{\theta}$ subsystem of (27). By the definition of the vector d_i, we know that the characteristic polynomial of F_i is $(s+\lambda_i)^{r_i-1}$ with $\lambda_i > 0$. So the matrix F_i is Hurwitz. Thus, we have a positive definite matrix P_i such that $P_iF_i + F_i^{\mathrm{T}}P_i = -2I_{r_i-1}$. Let $V_1(\bar{\theta}) = \sum_{i=1}^N \bar{\theta}_i^{\mathrm{T}} P_i \bar{\theta}_i$, which satisfies, along the trajectories of (27),

$$\underline{\alpha}_1 \|\bar{\theta}\|^2 \leqslant V_1(\bar{\theta}) \leqslant \bar{\alpha}_1 \|\bar{\theta}\|^2, \quad \dot{V}_1 \leqslant -\alpha_1 V_1(\bar{\theta}) + \sum_{i=1}^N \|P_i\|^2 \|\bar{f}_i^\theta(e_i,\mu)\|^2$$

for some constants $\underline{\alpha}_1 = \min_{1\leqslant i\leqslant N} \lambda_{\min}(P_i)$, $\bar{\alpha}_1 = \max_{1\leqslant i\leqslant N} \lambda_{\max}(P_i)$, $\alpha_1 = 1/\bar{\alpha}_1$ with $\lambda_{\min}(P_i), \lambda_{\max}(P_i)$ denoting the minimal and maximal eigenvalues of P_i. Further, since $\bar{f}_i^\theta(e_i,\mu)$ is a smooth function with $\bar{f}_i^\theta(0,\mu) = 0$ for $\mu \in \mathbb{D}$, by [20, Lemma A.1], we have a function $\gamma_{1a} \in \widehat{\mathcal{O}}(\mathrm{Id})$ such that

$$\sum_{i=1}^N \|P_i\|^2 \|\bar{f}_i^\theta(e_i,\mu)\|^2 \leqslant \gamma_{1a}(\|e\|^2)$$

which verifies (31a).

Second, consider the $\bar{\eta}$ subsystem of (27). Note that the $\bar{\eta}_i$ subsystem is in the form of (10) in [20]. By [20, Proposition 3.1], there exists a Lyapunov function $V_{2i}(t,\bar{\eta}_i)$ such that

$$\underline{\alpha}_{2i}(\|\bar{\eta}_i\|) \leqslant V_{2i}(t,\bar{\eta}_i) \leqslant \bar{\alpha}_{2i}(\|\bar{\eta}_i\|), \quad \dot{V}_{2i} \leqslant -\alpha_{2i} \circ V_{2i}(t,\bar{\eta}_i) + \bar{\gamma}_{2i}\|(\Delta_{i,1},\Delta_{i,2})\|^2 \tag{A1}$$

with a constant $\bar{\gamma}_{2i} > 0$, and

$$\underline{\alpha}_{2i}(s) = \ln(1 + \underline{\ell}_{2i}s^2), \quad \bar{\alpha}_{2i}(s) = \bar{\ell}_{2i}s^2, \quad \alpha_{2i}(s) = \frac{\ell_{2i}s}{1+s}$$

for some constants $\underline{\ell}_{2i}, \bar{\ell}_{2i}, \ell_{2i} > 0$ and $\Delta_{i,1} = e_i$, $\Delta_{i,2} = -\bar{g}_i^y(\bar{\theta}_i, e_i, \mu)$. Let $V_2(t,\bar{\eta}) = \sum_{i=1}^N V_{2i}(t,\bar{\eta}_i)$ and (32) with $\underline{\ell}_2 = \min_{1\leqslant i\leqslant N} \underline{\ell}_{2i}$, $\bar{\ell}_2 = \max_{1\leqslant i\leqslant N} \bar{\ell}_{2i}$, and $\ell_2 = \min_{1\leqslant i\leqslant N} \ell_{2i}$. By using the inequalities $\sum_{i=1}^N \ln(1+a_i) \geqslant \ln(1+\prod_{i=1}^N a_i)$ and $\sum_{i=1}^N \frac{a_i}{1+a_i} \geqslant \frac{\sum_{i=1}^N a_i}{1+\sum_{i=1}^N a_i}$ for $a_i \in \mathbb{R}_+$, it can be verified that

$$\underline{\alpha}_2(||\bar{\eta}||) \leqslant V_2(t,\bar{\eta}) \leqslant \bar{\alpha}_2(||\bar{\eta}||), \quad \sum_{i=1}^N \alpha_{2i} \circ V_{2i}(t,\bar{\eta}_i) \geqslant \alpha_2 \circ V_2(t,\bar{\eta}). \tag{A2}$$

Further, since $g_{i,1}(e_i + q(v,w), v, w) - g_{i,1}(q(v,w), v, w)$ is a smooth function vanishing at $e = 0$ for $\mu \in \mathbb{D}$, by (26) and [20, Lemma A.1], we have a constant $\hat{\gamma}_{21} > 0$ and a function $\bar{\gamma}_{2a} \in \widehat{\mathcal{O}}(\mathrm{Id})$ such that

$$\sum_{i=1}^{N} \|\bar{g}_i^y(\bar{\theta}_i, e_i, \mu)\|^2 \leqslant \hat{\gamma}_{21}\|\bar{\theta}\|^2 + \bar{\gamma}_{2a}(\|e\|^2) \tag{A3}$$

which implies (31b) with $\gamma_{21} = \hat{\gamma}_{21}\underline{\alpha}_1^{-1} \max_{1\leqslant i\leqslant N} \bar{\gamma}_{2i}$, $\gamma_{2a}(s) = \max_{1\leqslant i\leqslant N} \bar{\gamma}_{2i} \cdot [s + \bar{\gamma}_{2a}(s)]$.

Appendix A.2 Proof of P2

First, for $1 \leqslant i \leqslant N$, since both the functions Γ_i^ξ and Γ_i^u are smooth and compactly supported, the functions $\Gamma_i^\xi(\bar{\eta}_i + \eta_i^\star(\mu) + G_i e_i) - \Gamma_i^\xi(\eta_i^\star(\mu))$ and $\Gamma_i^u(\bar{\eta}_i + \eta_i^\star(\mu) + G_i e_i) - \Gamma_i^u(\eta_i^\star(\mu))$ are also smooth and compactly supported, vanishing at $(\bar{\eta}_i, e_i) = (0, 0)$. By [20, Lemma A.1], there exist functions $\bar{\phi}_2, \bar{\phi}_a \in \mathcal{K}^o \cap \widehat{\mathcal{O}}(\mathrm{Id})$ such that

$$\begin{aligned}&\sum_{i=1}^{N} \Big[\big\|\Gamma_i^\xi(\bar{\eta}_i + \eta_i^\star(\mu) + G_i e_i) - \Gamma_i^\xi(\eta_i^\star(\mu))\big\| + \big\|\Gamma_i^u(\bar{\eta}_i + \eta_i^\star(\mu) + G_i e_i) - \Gamma_i^u(\eta_i^\star(\mu))\big\|^2\Big] \\ &\quad \leqslant \bar{\phi}_a(\|e\|^2) + \bar{\phi}_2(\|\bar{\eta}\|^2).\end{aligned} \tag{A4}$$

Second, by the compactly supported property of $\Gamma_i^{\xi'}$, by [20, Lemma A.1] again, it can be shown that

$$\begin{aligned}&\sum_{i=1}^{N} \big\|\Gamma_i^{\xi'}(\eta_i)\dot{\eta}_i - \Gamma_i^{\xi'}(\eta_i^\star(\mu))\dot{\eta}_i^\star(\mu)\big\|^2 \\ &\quad\leqslant \sum_{i=1}^{N} 3\big\|\Gamma_i^{\xi'}(\bar{\eta}_i + \eta_i^\star(\mu) + G_i e_i) \cdot \varphi_i(\bar{\eta}_i + \eta_i^\star(\mu) + G_i e_i) - \Gamma_i^{\xi'}(\eta_i^\star(\mu)) \cdot \varphi_i(\eta_i^\star(\mu))\big\|^2 \\ &\qquad + \sum_{i=1}^{N} 3\big\|\Gamma_i^{\xi'}(\bar{\eta}_i + \eta_i^\star(\mu) + G_i e_i) \cdot G_i\Gamma_{i,1}(\bar{\eta}_i + \eta_i^\star(\mu) + G_i e_i) - \Gamma_i^{\xi'}(\eta_i^\star(\mu)) \cdot G_i\Gamma_{i,1}(\eta_i^\star(\mu))\big\|^2 \\ &\qquad + \sum_{i=1}^{N} 3\big\|\Gamma_i^{\xi'}(\bar{\eta}_i + \eta_i^\star(\mu) + G_i e_i) \cdot G_i\bar{\xi}_{i,1}\big\|^2 \\ &\quad\leqslant \hat{\phi}_a(\|e\|^2) + \hat{\phi}_2(\|\bar{\eta}\|^2) + \hat{\phi}_b \cdot \|C^o\bar{\xi}\|^2\end{aligned} \tag{A5}$$

for some functions $\hat{\phi}_2, \hat{\phi}_a \in \mathcal{K}^o \cap \widehat{\mathcal{O}}(\mathrm{Id})$ and a constant $\hat{\phi}_b > 0$.

Finally, let constants $\phi_{a1} = 2\hat{\gamma}_{21}\underline{\alpha}_1^{-1}$, $\phi_b = 2\hat{\phi}_b$, and functions $\phi_a(s) = 2\bar{\gamma}_{2a}(s) + 2\bar{\phi}_a(s)$, $\phi_{ba}(s) = 4(1 + \max_{1\leqslant i\leqslant N} \lambda_i)\bar{\phi}_a(s) + 2\hat{\phi}_a(s)$, $\phi_{a2}(s) = 2\bar{\phi}_2 \circ [\underline{\alpha}_2^{-1}(s)]^2$, $\phi_{b2}(s) = 4(1 + \max_{1\leqslant i\leqslant N} \lambda_i)\bar{\phi}_2(s) + 2\hat{\phi}_2(s)$ for $s \geqslant 0$. Note that using [20, Lemma A.2], we have $\phi_{a2}, \phi_{b2} \in \widehat{\mathcal{O}}(\alpha_2)$. Then by (A3) and (A4), we obtain (33a), and by (A4) and (A5), we can obtain (33b). The proof is complete.