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Two dimensional sparse signal reconstruction via 2D inverse-free sparse Bayesian learning

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Dear editor,

With the ability to reconstruct signals represented using sparse bases in a certain transform domain from an incomplete set of samples, sparse signal recovery (SSR) has been used in various applications. The reported SSR algorithms include basis pursuit (BP) [1], orthogonal matching pursuit (OMP) [2], and sparse Bayesian learning (SBL) [3,4]. These algorithms are effective for reconstructing one-dimensional (1D) signals from incomplete data. However, for two-dimensional (2D) signals, they generally experience heavy computational and memory burden. In particular, the atoms used in 2D signal representation and reconstruction are usually separable, yielding

$$\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}\boldsymbol{B}^{\mathrm{T}} + \boldsymbol{N}, \tag{1}$$

where $\boldsymbol{Y} \in \mathbb{R}^{P\times Q}$, $\boldsymbol{X} \in \mathbb{R}^{M\times N}$, $\boldsymbol{A} \in \mathbb{R}^{P\times M}$, $\boldsymbol{B} \in \mathbb{R}^{Q\times N}$, and $\boldsymbol{N} \in \mathbb{R}^{P\times Q}$ represent the observation, sparse signal, left dictionary, right dictionary, and noise, respectively. Noting that $\boldsymbol{Y}$ is incomplete, i.e., $P < M$ and $Q < N$, reconstructing $\boldsymbol{X}$ from $\boldsymbol{Y}$ is nonunique. The goal of SSR is to determine $\boldsymbol{X}$ with maximal possible zero elements. For traditional SSR algorithms, a trivial approach of solving (1) is to convert the 2D signals into 1D signals by column stacking, which can then be solved using the traditional SSR algorithms. However, the resulting dictionary is excessively large to be practically computed.

In this letter, a computationally efficient statistical 2D SSR algorithm is proposed. In particular, inverse-free sparse Bayesian learning (IFSBL) [5] is modified to directly work on 2D signals; therefore, we name the proposed algorithm 2D-IFSBL. All procedures of IFSBL, including the derivation of posteriors and parameter learning, are adjusted to suit 2D signal processing.

Proposed algorithm. Similar to SBL, the proposed 2D-IFSBL formulates the problem in (1) in the statistical framework. The signal $\boldsymbol{X}$ is Gaussian distributed, with the reciprocal of variance further Gamma disturbed as

$$\begin{aligned} p\left(\boldsymbol{X}|\boldsymbol{\Gamma}\right) &= \prod_{m=1}^{M}\prod_{n=1}^{N}\mathcal{N}\left(\boldsymbol{X}_{mn}|0,\boldsymbol{\Gamma}_{mn}^{-1}\right),\\ p\left(\boldsymbol{\Gamma},a,b\right) &= \prod_{m=1}^{M}\prod_{n=1}^{N}\mathcal{G}\left(\boldsymbol{\Gamma}_{mn};a,b\right). \end{aligned} \tag{2}$$

Moreover, noise $\boldsymbol{N}$ in (1) is zero-mean Gaussian distributed, with the reciprocal of noise variance Gamma distributed, which yields a Gaussian-distributed likelihood of the observation $\boldsymbol{Y}$ as

$$\begin{aligned} p(\boldsymbol{Y}|\boldsymbol{X},\alpha) &= \prod_{n=1}^{N}\mathcal{N}\left(\boldsymbol{Y}_{m\cdot}|\left(\boldsymbol{A}\boldsymbol{X}\boldsymbol{B}^{\mathrm{T}}\right)_{m\cdot},\alpha^{-1}\boldsymbol{I}\right),\\ p\left(\alpha;c,d\right) &= \mathcal{G}\left(\alpha;c,d\right). \end{aligned} \tag{3}$$

To achieve SSR, the expectations of posteriors of $\boldsymbol{X}$, $\boldsymbol{\Gamma}$, and α should be derived from the prior and likelihood provided by (2) and (3), respectively. IFSBL uses the expectation maximization to maximize the relaxed evidence lower bound to avoid the time-consuming matrix inverse involved in traditional SBL. The proposed 2D-IFSBL uses this strategy to derive the expectations of posteriors. The posterior of $\boldsymbol{X}$ is Gaussian disturbed, with the expectation as

$$\begin{aligned} \langle\boldsymbol{X}\rangle = \langle\alpha\rangle &\left(-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{Z}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{Y}\boldsymbol{B} + \frac{T}{2}\boldsymbol{Z}\right)\\ &\oslash\left(\frac{T\langle\alpha\rangle}{2}\boldsymbol{1}_{M\times N} + \langle\boldsymbol{\Gamma}\rangle\right), \end{aligned} \tag{4}$$

where $\boldsymbol{Z}$ represents an auxiliary variable introduced to relax the evidence lower bound [5], $\langle\cdot\rangle$ denotes the expectation operation, and T is a constant set to $T = 2\lambda_{\max}\left(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}\right)\cdot\lambda_{\max}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\right) + 10^{-10}$, which is slightly larger than the Lipschitz constant. $\oslash$ represents the element-wise division.

The posterior of $\boldsymbol{\Gamma}$ is Gamma distributed, with the expectation as

$$\langle\boldsymbol{\Gamma}_{mn}\rangle = \frac{a + 0.5}{b + \langle\boldsymbol{X}_{mn}\rangle^{2} + \langle\boldsymbol{\Gamma}_{mn}\rangle}. \tag{5}$$

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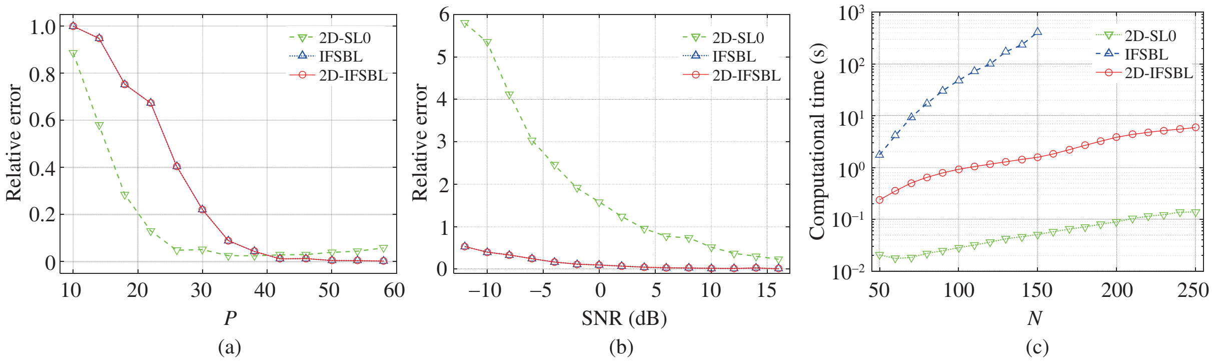

Figure 1 (Color online) Monte-Carlo experimental results. (a) Relative error versus P, (b) relative error versus SNR, and (c) computational time versus N of three algorithms.

The posterior of α is Gamma distributed, with the expectation as

$$\langle\alpha\rangle = \frac{c + \frac{1}{2}PQ}{d + \frac{1}{2}\langle g(\boldsymbol{X},\boldsymbol{Z})\rangle}, \tag{6}$$

where $\langle g(\boldsymbol{X},\boldsymbol{Z})\rangle$ is an auxiliary function for building the relaxed evidence lower bound, which is derived as

$$\begin{aligned}\langle g(\boldsymbol{X},\boldsymbol{Z})\rangle = &\left\|\boldsymbol{Y} - \boldsymbol{A}\boldsymbol{Z}\boldsymbol{B}^{\mathrm{T}}\right\|_2^2 + 2\cdot\mathrm{sum}\Big\{\Big(\langle\boldsymbol{X}_{mn}\rangle \\ &- \boldsymbol{Z}^{\mathrm{T}}\Big)\odot\Big[\boldsymbol{A}^{\mathrm{T}}\left(\boldsymbol{A}\boldsymbol{Z}\boldsymbol{B}^{\mathrm{T}} - \boldsymbol{Y}\right)\boldsymbol{B}\Big]\Big\} \\ &+ \frac{T}{2}\Big\{\|\langle\boldsymbol{X}_{mn}\rangle - \boldsymbol{Z}\|_2^2 + \mathrm{sum}\Big[\boldsymbol{1}_{M\times N} \\ &\oslash\left(\frac{T\langle\alpha\rangle}{2}\boldsymbol{1}_{M\times N} + \boldsymbol{\Gamma}\right)\Big]\Big\},\end{aligned} \tag{7}$$

where $\odot$ and $\mathrm{sum}(\cdot)$ denote the Hadamard product and summation of matrix elements, respectively.

Then, the proposed 2D-IFSBL proceeds by iteratively updating (4)–(6) until convergence, where the auxiliary variable $\boldsymbol{Z}$ is updated using the expectation of $\boldsymbol{X}$, i.e., $\boldsymbol{Z} = \langle\boldsymbol{X}\rangle$.

Experimental results. Some experimental results are presented to compare the performance of the proposed 2D-IFSBL with that of 2D smoothed l_0 norm (2D-SL0) [6] and IFSBL [5], in which the parameters of 2D-SL0 are carefully selected to achieve best performance [7].

In our experiments, we generate an $M \times N$ 2D sparse signal with K nonzero entries. For dictionaries $\boldsymbol{A}$ and $\boldsymbol{B}$, their entries are normally distributed. Moreover, the Gaussian distributed noise is added to observation $\boldsymbol{Y}$ to simulate noisy conditions. First, the parameters of the size of the sparse signal are set as $M = N = 80$ and $K = 10$, SNR is set to 20 dB, and P and Q are equal, ranging from 10 to 60 in steps of 10. Figure 1(a) shows the relative error ($\|\boldsymbol{X} - \hat{\boldsymbol{X}}\|_2/\|\boldsymbol{X}\|_2$) as a function of P obtained by averaging the relative errors of over 100 runs of three algorithms for different independently and randomly generated $\boldsymbol{A}$, $\boldsymbol{X}$, $\boldsymbol{B}$, and $\boldsymbol{N}$. 2D-SL0 obtains lower relative errors than IFSBL and 2D-IFSBL when P is smaller than 40. However, for large P, IFSBL and 2D-IFSBL achieve better performance than 2D-SL0. Similarly, Figure 1(b) shows the relative error with respect to SNR, with other parameters fixed at $M = N = 80$, $P = Q = 55$, and $K = 10$. The results indicate that IFSBL and 2D-IFSBL are more robust than 2D-SL0. Finally, we compare the computational times of the three algorithms under a computational platform of Intel(R) Core i7-8550U @ 1.8 GHz. The parameters are set as follows: $M = N$, $P = Q = N/2$, $K = N/10$, and SNR = 20 dB. Figure 1(c) shows the average computational time of 100 runs of the three algorithms under different N. Clearly, 2D-IFSBL is slightly slower than 2D-SL0 but is ten to hundred times faster than IFSBL.

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