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• **Supplementary File** •

A Correlation-Breaking Interleaving of Polar Codes in Concatenated Systems

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Appendix A Background of Polar Codes

The relevant theories on non-systematic polar codes [1] and systematic polar codes [2] are presented.

Appendix A.1 Preliminaries of Non-Systematic Polar Codes

The generator matrix for polar codes is $G_N = BF^{\otimes n}$ where B is a bit-reversal matrix, $F = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$, $n = \log_2 N$, and $F^{\otimes n}$ is the nth Kronecker power of the matrix F over the binary field $\mathbb{F}_2$. For compactness, the subscript of G_N is sometimes omitted as G without causing confusion of the block length N.

Let $W : \mathcal{X} \to \mathcal{Y}$ denote a B-DMC where $\mathcal{X} = \{0, 1\}$ is the input and $\mathcal{Y}$ is the output alphabet of the channel. The transition probability is denoted by $W(y|x)$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$.

The channel polarization process is performed as follows. The $N = 2^n (n \geqslant 1)$ independent copies of W are first combined and then split into N bit channels $\{W_N^{(i)}\}_{i=1}^N$ with:

$$W_N^{(i)}(y_1^N, u_1^{i-1}|u_i) = \sum_{u_{i+1}^N \in \mathcal{X}^{N-i}} \frac{1}{2^{N-1}} W_N(y_1^N|u_1), \tag{A1}$$

where

$$W_N(y_1^N|u_1^N) = W^N(y_1^N|u_1^N G_N) = \prod_{i=1}^{N} W(y_i|x_i). \tag{A2}$$

Mathematically, the encoding is a process to obtain the codeword x_1^N through $x_1^N = u_1^N G$ for given source bits u_1^N. The source bits u_1^N consists of the information bits and the frozen bits, denoted by $u_{\mathcal{A}}$ and $u_{\bar{\mathcal{A}}}$, respectively. Frozen bits refer to the fixed transmission bits which are known to both the transmitter and the receiver. The set $\mathcal{A}$ includes the indices for the information bits and $\bar{\mathcal{A}}$ is the complementary set, which can be constructed as in [1,3–6]. Both sets $\mathcal{A}$ and $\bar{\mathcal{A}}$ are in $\{1, 2, ..., N\}$ for polar codes of length N. The source bits u_1^N can be split as $u_1^N = (u_{\mathcal{A}}, u_{\bar{\mathcal{A}}})$. The codeword can then be expressed as

$$x_1^N = u_{\mathcal{A}} G_{\mathcal{A}} + u_{\bar{\mathcal{A}}} G_{\bar{\mathcal{A}}}, \tag{A3}$$

where $G_{\mathcal{A}}$ is the submatrix of G_N with rows specified by the set $\mathcal{A}$.

An encoding diagram is shown in Fig. A1. Each node adds the signals on all incoming edges from the left and sends the result out on all edges to the right. The operations are done in the binary field $\mathbb{F}_2$. One such encoding process is highlighted in Fig. A1 for $x_2 = u_5 \oplus u_6 \oplus u_7 \oplus u_8$. If the nodes in Fig. A1 are viewed as memory elements, the encoding process is to calculate the corresponding binary values to fill all the memory elements from the left to the right. This view is helpful when it comes to systematic polar codes in the following section.

Appendix A.2 Systematic Polar Codes

The systematic polar code is constructed by specifying a set of indices of the codeword x_1^N as the indices to convey the information bits. Denote this set as $\mathcal{B}$ ($|\mathcal{B}| = K$) and the complementary set as $\bar{\mathcal{B}}$. The codeword x_1^N is thus split as $(x_{\mathcal{B}}, x_{\bar{\mathcal{B}}})$. Define a matrix $G_{\mathcal{AB}}$ that is a submatrix of the generator matrix with elements $\{G_{i,j}\}_{i\in\mathcal{A}, j\in\mathcal{B}}$. Splitting x_1^N in (A3) into $(x_{\mathcal{B}},\ x_{\bar{\mathcal{B}}})$ requires splitting the matrices $G_{\mathcal{A}}$ and $G_{\bar{\mathcal{A}}}$ as:

$$G_{\mathcal{A}} = (G_{\mathcal{AB}},\ G_{\mathcal{A}\bar{\mathcal{B}}}), \tag{A4}$$
$$G_{\bar{\mathcal{A}}} = (G_{\bar{\mathcal{A}}\mathcal{B}},\ G_{\bar{\mathcal{A}}\bar{\mathcal{B}}}). \tag{A5}$$

Then x_1^N can be split as the following:

$$\begin{cases} x_{\mathcal{B}} = u_{\mathcal{A}} G_{\mathcal{AB}} + u_{\bar{\mathcal{A}}} G_{\bar{\mathcal{A}}\mathcal{B}}, \\ x_{\bar{\mathcal{B}}} = u_{\mathcal{A}} G_{\mathcal{A}\bar{\mathcal{B}}} + u_{\bar{\mathcal{A}}} G_{\bar{\mathcal{A}}\bar{\mathcal{B}}}. \end{cases} \tag{A6}$$

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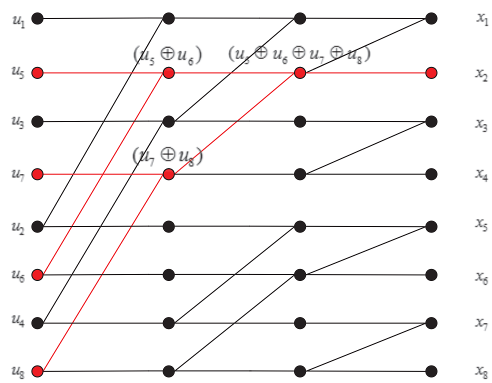

Figure A1 An encoding circuit of the non-systematic polar codes with $N = 8$. Signals flow from the left to the right. Each edge carries a signal of 0 or 1.

We can see from (A6) that, in systematic polar codes, $x_{\mathcal{B}}$ plays the role that $u_{\mathcal{A}}$ plays in non-systematic polar codes. Given a non-systematic encoder $(\mathcal{A}, u_{\bar{\mathcal{A}}})$, there exists a systematic encoder $(\mathcal{B}, u_{\bar{\mathcal{A}}})$ if $\mathcal{A}$ and $\mathcal{B}$ have the same number of elements and the matrix $G_{\mathcal{AB}}$ is invertible [2]. Then a systematic encoder can perform the mapping $x_{\mathcal{B}} \mapsto x_1^N = (x_{\mathcal{B}}, x_{\bar{\mathcal{B}}})$. To realize this systematic mapping, $x_{\bar{\mathcal{B}}}$ needs to be computed for any given information bits $x_{\mathcal{B}}$. To this end, we see from (A6) that $x_{\bar{\mathcal{B}}}$ can be computed if $u_{\mathcal{A}}$ is known. The vector $u_{\mathcal{A}}$ can be obtained as the following

$$u_{\mathcal{A}} = (x_{\mathcal{B}} - u_{\bar{\mathcal{A}}} G_{\bar{\mathcal{A}}\mathcal{B}})(G_{\mathcal{AB}})^{-1}. \tag{A7}$$

In [2], it is shown that $\mathcal{B} = \mathcal{A}$ satisfies all these conditions in order to establish the one-to-one mapping $x_{\mathcal{B}} \mapsto u_{\mathcal{A}}$. Therefore we can rewrite (A6) as

$$\begin{cases} x_{\mathcal{A}} = u_{\mathcal{A}} G_{\mathcal{AA}} + u_{\bar{\mathcal{A}}} G_{\bar{\mathcal{A}}\mathcal{A}}, \\ x_{\bar{\mathcal{A}}} = u_{\mathcal{A}} G_{\mathcal{A}\bar{\mathcal{A}}} + u_{\bar{\mathcal{A}}} G_{\bar{\mathcal{A}}\bar{\mathcal{A}}}. \end{cases} \tag{A8}$$

Note that the submatrix $G_{\mathcal{AA}}$ is a lower triangular matrix with ones at the diagonal. The entries above the diagonal are all zeros.

Let us go back to the diagram in Fig. A1. For systematic polar codes, the information bits are now conveyed in the right-hand side in $x_{\mathcal{A}}$. To calculate $x_{\bar{\mathcal{A}}}$, $u_{\mathcal{A}}$ in the left-hand side needs to be calculated first. Once $u_{\mathcal{A}}$ is obtained, systematic encoding can be performed in the same way as the non-systematic encoding: performing binary additions from the left to the right. Therefore, compared with non-systematic encoding, systematic encoding has an additional round of binary additions from the right to the left. The detailed analysis of systematic encoding can be found in [7, 8].

Appendix A.3 SC Decoding

The SC decoding of polar codes follows the same graph as shown in Fig. A1. The likelihood ratio (LR) of bit channel i is defined as:

$$L_N^{(i)} = \frac{W_N^{(i)}(y_1^N, u_1^{i-1}|0)}{W_N^{(i)}(y_1^N, u_1^{i-1}|1)}. \tag{A9}$$

From [1], it is shown that the transition probability of bit channel i can be recursively calculated, which results in a recursive calculation of the LRs as:

$$L_N^{(2i-1)}(y_1^N, \hat{u}_1^{2i-2}) = \frac{L_{N/2}^{(i)}(y_1^{N/2}, \hat{u}_{1,o}^{2i-2} \oplus \hat{u}_{1,e}^{2i-2}) L_{N/2}^{(i)}(y_{N/2+1}^N, \hat{u}_{1,e}^{2i-2}) + 1}{L_{N/2}^{(i)}(y_1^{N/2}, \hat{u}_{1,o}^{2i-2} \oplus \hat{u}_{1,e}^{2i-2}) + L_{N/2}^{(i)}(y_{N/2+1}^N, \hat{u}_{1,e}^{2i-2})}, \tag{A10}$$

$$L_N^{(2i)}(y_1^N, \hat{u}_1^{2i-1}) = [L_{N/2}^{(i)}(y_1^{N/2}, \hat{u}_{1,o}^{2i-2} \oplus \hat{u}_{1,e}^{2i-2})]^{(1-2\hat{u}_{2i-1})} \cdot L_{N/2}^{(i)}(y_{N/2+1}^N, \hat{u}_{1,e}^{2i-2}). \tag{A11}$$

Appendix B Proof of Proposition 1

Proof. Assume the entries of the set $\mathcal{A}_i$ are arranged in the ascending order. Define $\mathcal{A}_i(a:b)$ as a vector containing elements of $\mathcal{A}_i$ obtained from elements a to b of column i of G. Suppose $i \leqslant N/2$. When $i > N/2$, the proof can be easily followed. Let N_i be the first number of power of two which is larger than (or equal to) i. Let $j = i + N_i$. Then $\mathcal{A}_i(N_i+1 : 2N_i) = \mathcal{A}_j(1 : N_i)$. The proof of this proposition is directly connected with the fact that the LR of u_j ($j \in \mathcal{A}_j$) involves the LR of $\sum_{l \in \mathcal{A}_i(1:N_i)} u_l$ (from the nature of the polar encoding and decoding graph). With $\mathcal{A}_i(N_i+1 : 2N_i) = \mathcal{A}_j(1 : N_i)$, the decision of bit u_i ($N_i + 1 \leqslant i \leqslant 2N_i$) is therefore affected by the sum of the already decoded bits of $\sum_{l \in \mathcal{A}_i(1:N/2)} u_l$. In other words, any error of the bit u_l ($l \in \mathcal{A}_i(1 : N_i)$) affects the decision of the subsequent bit u_i with $i \in \mathcal{A}_i(N_i + 1 : 2N_i)$. Therefore the errors in $\hat{u}_{\mathcal{A}_i}$ are correlated.

We provide an example to explain the meaning of this proposition. The notation $v_{\mathcal{A}}$ is a subvector of v_1^N with elements specified by the set $\mathcal{A}$. Here is an example to show what exactly $\hat{u}_{\mathcal{A}_i}$ is. Let the block length be $N = 16$, the code rate of the polar code be $R = 0.5$, and the underlying channel is the BEC channel with an erasure probability 0.2. The set $\mathcal{A}$ is calculated to be $\mathcal{A} = \{8, 10, 11, 12, 13, 14, 15, 16\}$. Let $i = 10$, then we take the indices of non-zero entries of column 10 of G as $\mathcal{A}_{10}$, which is a collecting set of indices 10, 12, 14, 16. Therefore, $\hat{u}_{\mathcal{A}_{10}}$ is a subvector of $\hat{u}_1^N$ which contains elements of $\hat{u}_{10}$, $\hat{u}_{12}$, $\hat{u}_{14}$, $\hat{u}_{16}$.

To show the coupling, an experiment is performed as below. The number of times the errors of $\hat{u}_{\mathcal{A}_i}$ ($i \in \mathcal{A}$) happening simultaneously (denoted by N_s) over the number of times any of the bits $\hat{u}_{\mathcal{A}_i}$ in error (denoted by N_e) is called the coupling coefficient, which is equal to N_s/N_e. The coupling coefficients (similar to the correlation coefficient) of bits indicated by non-zero positions of column 10, 11, and 13 is shown in Table B1. It can be seen from Table I that if there are errors in $\hat{u}_{\mathcal{A}_{10}} = \{\hat{u}_{10}, \hat{u}_{12}, \hat{u}_{14}, \hat{u}_{16}\}$, then 76% of times these bits errors happen simultaneously, resulting in a coupling coefficient 0.76 for errors in $\hat{u}_{\mathcal{A}_{10}}$. The coupling coefficients for $\hat{u}_{\mathcal{A}_{11}}$ and $\hat{u}_{\mathcal{A}_{13}}$ are 0.74 in Table B1.

Table B1 Coupling effect for $N = 16$ and $R = 0.5$ in a BEC channel with an erasure probability of 0.2

Column Index	Coupling coefficient
10	76%
11	74%
13	74%

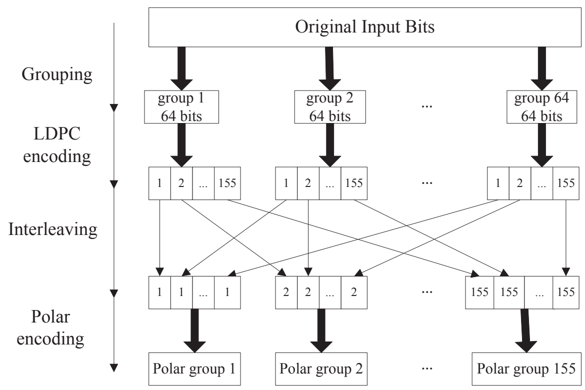

Figure C1 A blind interleaving scheme with direct product (BI-DP). The block length of the LDPC code is $N_l = 155$, and the code rate is 64/155. The block length of the polar code is $N = 256$, and the code rate is $R = 1/4$.

Appendix C The Blind Interleaving Schemes

In this section, the scheme of scattering all bits in a LDPC block into different polar code blocks is introduced. The N_l bits of one LDPC block are divided into N_l polar code blocks, which guarantees that the received error information bits in each LDPC block are independent as they come from different polar code blocks during de-interleaving.

Appendix C.1 Direct Product Blind Interleaving: BI-DP

Denote $c_i^{(j)}$ $(1 \leqslant i \leqslant N_l,\ 1 \leqslant j \leqslant K)$ as the ith coded bit of the jth LDPC block. Also denote $u_k^{(d)}$ as the kth information bit of the dth polar block. Bits i $(c_i^{(j)})$ of all LDPC code blocks form the input vector to the ith polar code encoder. The input bits of the ith polar block are arranged in the order of the LDPC blocks: $u_j^{(i)} = c_i^{(j)}$. For example, $c_1^{(j)}(1 \leqslant j \leqslant K)$ of all LDPC blocks produce the input for the first polar block, and $u_j^{(1)} = c_1^{(j)}$, meaning that bit one of the jth LDPC block is set as the jth input bit of polar block one. This interleaving is called the blind interleaving with direct product (BI-DP).

We give an example in Fig. C1 where $K_l = 64$ and $N_l = 155$. Polar code in this example has $N = 256$, $K = 64$ and a code rate $R = 1/4$. Fig. C1 is an exact illustration of the BI-DP scheme: bits one of all LDPC blocks serve as the input to polar block one, bits two of all LDPC blocks serve as the input to polar block two, and so on.

To compare with the subsequent improved blind interleaving, define a $N_l \times K$ matrix C, that contains the elements of the input of polar blocks. The entry of the ith row and jth column is $C_{i,j} = u_j^{(i)}$. For the BI-DP scheme, $C_{i,j} = u_j^{(i)} = c_i^{(j)}$.

Appendix C.2 Cyclic Direct Product Blind Interleaving: BI-CDP

One problem with the BI-DP scheme is that for LDPC block j, all the coded bits of it are placed as the jth input bits of all polar blocks. For example, all the bits $c_i^{(1)}$ of LDPC block one are the first information bits of all polar blocks in the receiver side. Given that information bits of polar codes are not equally protected, it can happen that LDPC block j is exposed to a large amount of errors if bit j of the polar code is a poorly protected bit in the decoding process. An improved BI, termed cyclic DP (BI-CDP), is thus introduced below to overcome this problem.

Denote $N_l = n_u K + k_l$, where n_u and k_l are the quotient and the reminder of N_l divided by K, respectively. Define a basic polynomial $p(x) = j' x^{j'}$ $(0 \leqslant j' \leqslant K-1)$. For the ith polar code block $(1 \leqslant i \leqslant n_u K)$, the assignments of the LDPC coded bits to this polar block can be obtained from the i'th $(i' = i-1)$ quasi cyclic shift $(0 \leqslant i' \leqslant n_u K - 1)$:

$$p^{(i')}(x) = ((j' + \lfloor i'/K \rfloor K) x^{(i'+j')})(\bmod K), \tag{C1}$$

where 'mod' is the modulo operator. Here the word 'quasi' means that it is not the traditional cyclic shift operation of $x^{i'} p(x)$ because of the jump of the coefficients every K shifts.

Let $m = (j' + \lfloor i'/K \rfloor K) + 1$, $q = ((i' + j') \bmod K) + 1$ and $l = ((m-1) \bmod K) + 1$. Then the i'th cyclic shifted polynomial $p^{(i')}(x)$ carries the mth bit of the qth LDPC block $c_m^{(q)}$, which is applied to the lth bit of polar block $i = i'+1$, namely $u_l^{(i)} = c_m^{(q)}$.

This quasi-cyclic arrangement of LDPC coded bits to the corresponding input bits of polar blocks works for the first $n_u K$ polar blocks. However it does not work for the last k_l polar blocks because $m = (j' + n_u K) + 1 > N_l$ when $k_l \leqslant j' \leqslant K-1$.

There are many ways to arrange the input for the last k_l polar blocks. In the following, we propose one possible solution. Let $i' = i - 1 = n_u K + i_r$ $(n_u K < i \leqslant N_l$ and $0 \leqslant i_r \leqslant k_l - 1)$. For the original polynomial $p(x) = j' x^{j'}$, when $j' = j - 1 = i_r$, the i'th cyclic shift is defined as $p^{(i')}(x) = i' x^{j'}$. When $j' = j - 1 \neq i_r$, define a new parameter j'' $(0 \leqslant j'' \leqslant K-2)$ for the other

Polar \ LDPC	1	2	3	4
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3
4	4	4	4	4
5	5	5	5	5
6	6	6	6	6
7	7	7	7	7
8	8	8	8	8
9	9	9	9	9
10	10	10	10	10
11	11	11	11	11

(a) BI-DP

Polar \ LDPC	1	2	3	4
1	1	2	3	4
2	4	1	2	3
3	3	4	1	2
4	2	3	4	1
5	5	6	7	8
6	8	5	6	7
7	7	8	5	6
8	6	7	8	5
9	9	9	10	11
10	11	10	9	10
11	10	11	11	9

(b) BI-CDP

Figure C2 An example of the matrix C for $N_l = 11$ and $K = 4$. The row and column indices are the indices of polar and LDPC blocks. The entries of one column are the indices of LDPC coded bits of that specific LDPC block. The four colors of the background corresponding to the four positions of each polar block.

$K-1$ elements of the i'th shift of $p(x)$ ($i' = i - 1 = n_u K + i_r$):

$$p^{(i')}(x) = \begin{cases} i' x^{j'}, & \text{if } j' = i_r, \\ (j'' \bmod k_l + n_u K) x^{(i'+j''+1) \bmod K}, & \text{otherwise.} \end{cases} \tag{C2}$$

It can be verified that the proposed arrangements assign the remaining LDPC coded bits to the last k_l polar blocks.

This arrangement can be viewed from the matrix C defined in Section Appendix C.1. Fig. C2 shows the assignments of LDPC coded bits to polar blocks, stored by this matrix C. In this example, $N_l = 11$ and $K = 4$. For LDPC block j ($1 \leqslant j \leqslant 4$), the subscript of the coded bits $c_i^{(j)}$ ($1 \leqslant i \leqslant 11$) are stored in column j of the two tables. For polar block i, the input information bits $u_j^{(i)}$ are stored in the ith row of the tables. Since the entries of the tables in Fig. C2 are the subscripts of $c_i^{(j)}$, the subscripts of the information bits $u_j^{(i)}$ are represented by different colors: yellow is $j = 1$ ($u_1^{(i)}$), orange is $j = 2$, green is $j = 3$, and blue is $j = 4$.

For BI-DP, the assignments of each LDPC coded bits are designed according to Section Appendix C.1. Clearly it can be seen from the same color of columns of Fig. C2-(a) that the coded bits of LDPC block j are assigned to the same bits (the jth bits) of all polar blocks. The assignments of LDPC coded bits for BI-CDP are done according to equations (C1) and (C2). Take column 1 (LDPC block 1) of Fig. C2-(b) as an example. It is shown that three coded bits (three colored yellow of the column 1) of LDPC block one are put as the first information bits for three polar blocks (polar block 1, 5, and 9), two coded bits (two colored orange) are the second information bits of polar blocks 4 and 8, three coded bits (three colored green) are the third information bits of polar blocks 3, 7 and 11, and three coded bits (three colored blue) are the fourth information bits of polar blocks 2, 6 and 10. On the other hand, all eleven coded bits of LDPC block one are the first information bits of eleven polar blocks for the BI-DP scheme.

Overall, the improved BI-CDP can scatter the LDPC coded bits evenly to the input of polar blocks to reduce the chance of simultaneous errors. It is expected that the BI-CDP scheme performing better than the BI-DP scheme.

Appendix D Proof of Proposition 2

Proof. First, let us bear in mind that the submatrix $G_{\mathcal{AA}}$ is a lower triangular matrix as discussed in Section Appendix A.2. This proposition is equivalent to the following assignment:

$$\begin{cases} i \in \bar{\mathcal{A}}_{cs}, & \text{if } \omega_i = 1, \\ i \in \mathcal{A}_{cs}, & \text{if } \omega_i > 1. \end{cases} \tag{D1}$$

For $\omega_i = 1$, there is only one non-zero entry $G_{i,i} = 1$ for row i. Let $K_c = |\mathcal{A}_{cs}|$ and $K_{uc} = |\bar{\mathcal{A}}_{cs}|$. Denote the submatrix formed by the rows of $G_{\mathcal{AA}}$ indicated by $\bar{\mathcal{A}}_{cs}$ as $G_{\mathcal{AA}}(\bar{\mathcal{A}}_{cs}, :)$. Then each row of the submatrix $G_{\mathcal{AA}}(\bar{\mathcal{A}}_{cs}, :)$ has Hamming weight one. Extract the columns specified by $\bar{\mathcal{A}}_{cs}$ of $G_{\mathcal{AA}}(\bar{\mathcal{A}}_{cs}, :)$ to obtain a matrix denoted as G_{uc}. Similar to the process of extracting $G_{\mathcal{AA}}$ from G, the extraction of rows and columns (indicated by $\bar{\mathcal{A}}_{cs}$) from $G_{\mathcal{AA}}$ results in a final $K_{uc} \times K_{uc}$ identity matrix $G_{uc} = I_{K_{uc}}$.

According to Proposition 1, the errors in $\hat{u}_{\mathcal{A}_i}$ ($\hat{u}_{\mathcal{A}_c}$) are coupled. Now that each column of $G_{uc} = I_{K_{uc}}$ has Hamming weight one, the errors contained in $\hat{u}_{\bar{\mathcal{A}}_c}$ are not coupled as indicated by Proposition 1. The proof is complete.

Table E1 The CBI scheme for LDPC (21,8) and polar (32,16). The top row contains indices of LDPC blocks and the first column is the indices of polar blocks. The entries of the table are bit indices of LDPC blocks.

Polar \ LDPC	1	2	3	4	5	6	7	8	9	10
1	1 : 7	8	9	10	11	12	13	14	15	16
2	16	1 : 7	8	9	10	11	12	13	14	15
3	15	16	1 : 7	8	9	10	11	12	13	14
4	14	15	16	1 : 7	8	9	10	11	12	13
5	13	14	15	16	1 : 7	8	9	10	11	12
6	12	13	14	15	16	1 : 7	8	9	10	11
7	11	12	13	14	15	16	1 : 7	8	9	10
8	10	11	12	13	14	15	16	1 : 7	8	9
9	9	10	11	12	13	14	15	16	1 : 7	8
10	8	9	10	11	12	13	14	15	16	1 : 7
11	17 : 21	17	18	19	20	21	17	18	19	20
12	0	18 : 21	17	18	19	20	21	17	18	19
13	0	0	19 : 21	17	18	19	20	21	17	18
14	0	0	0	20 : 21	17	18	19	20	21	17
15	0	0	0	0	21 : 21	17	18	19	20	21

We use the same example as before (the one after the proof of Proposition 1 in Section Appendix B) to show how to use Proposition 2 to find the sets $\mathcal{A}_c$ and $\bar{\mathcal{A}}_c$. With Proposition 1, we can easily find that $\mathcal{A}_{cs} = \{4, 6, 7, 8\}$ for the submatrix $G_{\mathcal{A}\mathcal{A}}$. Relative to the matrix G_{16}, this set is $\mathcal{A}_c = \{12, 14, 15, 16\}$. The uncorrelated set is thus $\bar{\mathcal{A}}_c = \{8, 10, 11, 13\}$.

Appendix E Design of the CBI Interleaving

Let $N_l = n_u K + k_l$ and $K_n = K_c + 1$. For the general CBI scheme, the number of polar blocks, n_p, to transmit K_n LDPC blocks, is expressed as:

$$n_p = \begin{cases} (n_u + 1)K_n, & \text{if } k_l = 0 \text{ or } k_l > K_n, \\ n_u K_n + k_l, & \text{otherwise.} \end{cases} \tag{E1}$$

As in Section Appendix C.1 and Section Appendix C.2, the general rules to determine the elements of the matrix C of the CBI scheme are the following:

- Consider elements of C within the first $n_u K_n$ rows. When $j' = i' \bmod K_n$, $C_{i,j}$ contains K_{uc} bits from LDPC block $j = (j'+1)$. These bits are put into positions $\bar{\mathcal{A}}_c$ of polar block $i = i' + 1$. For the remaining K_c correlated information bits of polar block i, it takes coded bits from other different K_c LDPC blocks to put into correlated positions $\mathcal{A}_c$ in the same fashion as the BI-CDP scheme.
- Consider the rest of the rows (for the remaining $n_p - n_u K_n$ polar blocks). When $j' = i' \bmod K_n$, $C_{i,j}$ contains the remaining bits (smaller than K_{uc}) from LDPC block $j = (j' + 1)$. These bits are also put into positions $\bar{\mathcal{A}}_c$ of polar block $i = i' + 1$. Polar block i' takes coded bits from other LDPC blocks for its correlated information bits, similarly to the arrangement of the BI-CDP scheme.

Two examples are given in Table E1 and Table E2 to explain the assignments for the two cases of (2): Table E1 is an example of the second case of (2) and Table E2 is an example of the first case of (2).

A polar code (32,16) concatenated with a LDPC code (21,8) shown in Table E1 is the example when $k_l < K_n$. The correlated set $\mathcal{A}_c = \{16, 24, 26, 27, 28, 29, 30, 31, 32\}$. Therefore $K_c = |\mathcal{A}_c| = 9$, $K_n = K_c + 1 = 10$, $n_u = \lfloor N_l/K \rfloor = 1$, and $k_l = 5 < K_n$. To transmit $K_n = 10$ LDPC blocks, $n_p = n_u K_n + k_l = 15$ polar blocks are required. In Table E1, the top row contains indices of the LDPC blocks, the first column is the indices of the polar blocks, and the entries of this table represent the indices of encoded bits of LDPC blocks. From Table E1, for polar block one, bit 1 to bit 7 are taken from LDPC block one, and the other 9 bits are bits $8, 9, \ldots, 16$ from LDPC blocks two to ten, respectively. The 7 bits from LDPC block one are placed at the uncorrelated positions $\bar{\mathcal{A}}_c$ of polar block one, and the other 9 bits from nine LDPC blocks are arranged at the correlated positions $\mathcal{A}_c$ of polar block one. The other polar blocks (polar block two to polar block ten) follow the same fashion in collecting the input bits. These first $n_u K_n$ rows follow the same cyclic assignments of LDPC coded bits to the inputs of polar blocks as the BI-CDP scheme. The remaining polar blocks (from polar block eleven to polar block fifteen) collect the remaining bits of LDPC blocks. For example, although polar block eleven can take $K_{uc} = 7$ uncorrelated bits from LDPC block one, there are not enough bits left from LDPC block one: only bits c_{17} to c_{21} are left. The assignments of the correlated positions of polar block eleven follows exactly that of the BI-CDP scheme.

Table E2 shows another example when $k_l > K_n$. In this example, the polar code (32,8) has an $\mathcal{A}_c = \{28, 30, 31, 32\}$ with $K_c = 4$ and the LDPC is a (22,8) code. The parameters are $k_l = 6$ and $K_n = K_c + 1 = 5$. The total polar blocks $n_p = n_u \times K_n = 3 \times 5$ are used to transmit $K_n = 5$ LDPC blocks. For both examples, there are 0s at the left low corner, which means that there are polar positions which are not used. These positions are wasted which are the cost of the universal CBI design.

Table E2 The CBI cheme for LDPC (22,8) and polar (32,8). The top row contains indices of LDPC blocks and the first column is the indices of polar blocks. The entries of the table are bit indices of LDPC blocks.

Polar \ LDPC	1	2	3	4	5
1	1 : 4	5	6	7	8
2	8	1 : 4	5	6	7
3	7	8	1 : 4	5	6
4	6	7	8	1 : 4	5
5	5	6	7	8	1 : 4
6	9 : 12	13	14	15	16
7	16	9 : 12	13	14	15
8	15	16	9 : 12	13	14
9	14	15	16	9 : 12	13
10	13	14	15	16	9 : 12
11	17 : 20	17	18	19	20
12	21	18 : 21	17	18	19
13	22	22	19 : 22	17	18
14	0	0	0	20 : 22	17
15	0	0	0	0	21 : 22

Appendix F Complexity Analysis

For the CBI scheme, the interleaving requires a memory to store the decoded LR values from n_p polar blocks in order to do the de-interleaving. The memory size is therefore $[n_p, K]$. The decoder needs to wait n_p polar blocks to decode K_n LDPC blocks. The decoding delay of the outer code is therefore $n_p \times N \times T_s$, where T_s is the symbol duration in seconds. For the BI scheme, the memory size is $[N_l, K]$ and the decoding delay is $N_l \times N \times T_s$, which are both larger than the values of the CBI scheme.

Appendix G Additional Simulation Results

The submatrix $G_{\mathcal{A}\mathcal{A}}$ is formed from the generator matrix G. Based on the submatrix $G_{\mathcal{A}\mathcal{A}}$ and Proposition 2, for polar code (256,64), the correlated bits indices are calculated to be $\mathcal{A}_c$ ($K_c = 38$) and the uncorrelated bits indices $\bar{\mathcal{A}}_c$ ($K_{uc} = 26$) are also obtained. Similarly, for polar code (512,128), the correlated bits indices and the uncorrelated bits indices are calculated to be $\mathcal{A}_c$ ($K_c = 79$) and $\bar{\mathcal{A}}_c$ ($K_{uc} = 49$), respectively.

Fig. G1-(a) shows the BLER performance of different schemes, which follows the same trend as the BER performance.

The proposed CBI scheme can also work with other outer codes, such as BCH codes. Fig. G1-(b) shows the result of the polar code (256,64) with a BCH code (127,57) where the 127 and 57 are the code length and the number of information bits of BCH codes in one code block, respectively. It can be seen from Fig. G1-(b) that the CBI scheme employing BCH code as an outer code has a 0.7 dB gain over the direct concatenation scheme at a BER $= 10^{-4}$.

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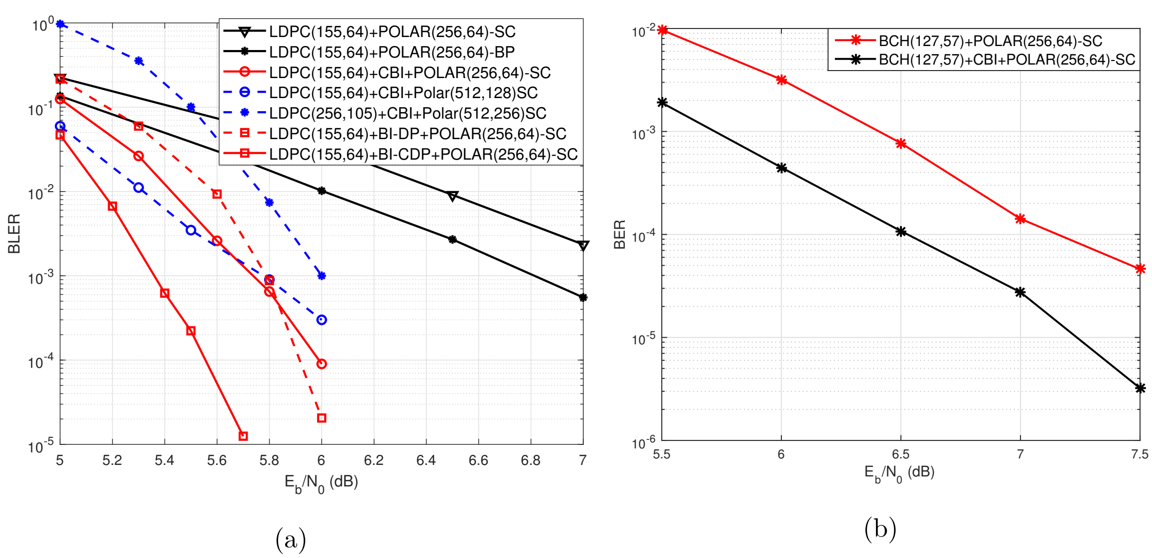

Figure G1 (a) The BLER performance of polar code concatenated with a LDPC code in AWGN channels. The LDPC codes are MacKay codes. (b)The BER performance of the concatenation scheme in AWGN channels. The BCH code is (127, 57) and the polar code is (256, 64).