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• **Supplementary File** •

Optimal Replacement of Degrading Components: A Control-Limit Policy

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Appendix A Proof of Theorem 1

Let $V^n(k,l_k)$ denote the value function in the nth iteration to during applying the value iteration method. The initial value is defined as $V^0(k,l_k)=0$ for $n=0$ and k. In such case, $V^0(k,l_k)$ is nondecreasing in k. Now, we assume that $V^n(k,l_k)$ is nondecreasing in l_k. Then, we have

$$V^{n+1}(k,l_k)=\begin{cases} c_2+V^n(0,l_0), & l_k>\xi' \\ \min\left\{c_1+V^n(0,l_0),\lambda(c_3+E\left[V^n(k+1,L)\right])\right\}, & l_k\leqslant\xi' \end{cases} \tag{A1}$$

From Proposition 2, it is known that the random variable L is stochastically increasing in l_k. By applying the induction principle, if $V^n(k,l_k)$ is nondecreasing in l_k, we can prove that $E\left[V^n(k+1,L)\right]$ is also nondecreasing by the result of stochastic order in [1] (Proposition 9.1.2, pp. 405, i.e. if two random variables X and Y such that $X\geqslant_{st}Y$, then $E\left[h(X)\right]\geqslant E\left[h(Y)\right]$ for all increasing functions $h(\cdot)$). Together with these results, it is observed that (A1) are nondecreasing in l_k, i.e. $V^{n+1}(k,l_k)$ is also nondecreasing. The proof is completed.

Appendix B Proof of Theorem 2

Similar to the proof of Theorem 1, we first define the initial value $V^0(k,l_k)=0$ for $n=0$ and k, where $V^0(k,l_k)$ is nonincreasing in k. Then, by applying the induction principle, we suppose $V^n(k,l_k)$ is nonincreasing in k. As such, we have

$$V^{n+1}(k,l_k)=\begin{cases} c_2+V^n(0,l_0), & l_k>\xi' \\ \min\left\{c_1+V^n(0,l_0),\lambda(c_3+E\left[V^n(k+1,L)\right])\right\}, & l_k\leqslant\xi' \end{cases} \tag{B1}$$

From (B1), the main task of the proof is to show that $E\left[V^n(k+1,L)\right]$ is also nonincreasing in k. First, we prove that $E\left[V^n(k,L)\right]$ is nonincreasing in k as follows. Suppose that there are two different observation epochs k^+ and k^-, with $k^+\geqslant k^-$. Let random variables L^+ and L^- denote the predicted future degradation signals made at k^+ and k^-, respectively, given the same degradation observation l_k at k^+ and k^-. To prove $E\left[V^n(k,L)\right]$ is nonincreasing in k, we need to show $E\left[V^n(k^+,L^+)\right]\leqslant E\left[V^n(k^-,L^-)\right]$ for any $k^+\geqslant k^-$. To do so, we first have

$$E\left[V^n(k^+,L^+)\right]-E\left[V^n(k^-,L^-)\right]=\left(E\left[V^n(k^+,L^+)\right]-E\left[V^n(k^+,L^-)\right]\right)+\left(E\left[V^n(k^+,L^-)\right]-E\left[V^n(k^-,L^-)\right]\right). \tag{B2}$$

Recall from Proposition 2 that random variable L is stochastically decreasing in k, i.e. $L^+\leqslant_{st}L^-$ in this case. Furthermore, we learn from Theorem 1 that $V^n(k,l_k)$ is nondecreasing in l_k for all k. Therefore, we have $E\left[V^n(k^+,L^+)\right]\leqslant E\left[V^n(k^+,L^-)\right]$ by the standard result of stochastic order in [1](Proposition 9.1.2, pp. 405, i.e. if two random variables X and Y such that $X\geqslant_{st}Y$, then $E\left[h(X)\right]\geqslant E\left[h(Y)\right]$ for all increasing functions $h(\cdot)$).

For the second term of the right-hand side of (B2), we have

$$E\left[V^n(k^+,L^-)\right]-E\left[V^n(k^-,L^-)\right]=E\left[V^n(k^+,L^-)-V^n(k^-,L^-)\right]=\int_{x\in\Re}\left(V^n(k^+,x)-V^n(k^-,x)\right)f_{L^-}(x)dx, \tag{B3}$$

where $f_{L^-}(x)$ is the PDF of L^-.

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Due to the induction hypothesis, we have that $V^n(k, l_k)$ is nonincreasing in k so that $V^n(k^+, x) \leqslant V^n(k^-, x)$. Therefore, we get

$$E\left[V^n(k^+, L^-)\right] - E\left[V^n(k^-, L^-)\right] = \int_{x\in\Re} \underbrace{\left(V^n(k^+, x) - V^n(k^-, x)\right)}_{\leqslant 0} \underbrace{f_{L^-}(x)}_{\geqslant 0} dx \leqslant 0. \tag{B4}$$

Together with these analyses, we have

$$E\left[V^n(k^+, L^+)\right] - E\left[V^n(k^-, L^-)\right] = \underbrace{E\left[V^n(k^+, L^+)\right] - E\left[V^n(k^+, L^-)\right]}_{\leqslant 0} + \underbrace{E\left[V^n(k^+, L^-)\right] - E\left[V^n(k^-, L^-)\right]}_{\leqslant 0} \leqslant 0. \tag{B5}$$

By the principle of induction and (B5), we can conclude that $E\left[V^n(k, L)\right]$ is nonincreasing in k. This result implies that $E\left[V^n(k+1, L)\right]$ is nonincreasing in k. As such, (B1) are nonincreasing in k, i.e. $V^{n+1}(k, l_k)$ is nonincreasing in k.

Appendix C Proof of Theorem 3

We first suppose that the results in Theorem 3 holds. Then, we have

$$c_1 + V(0, l_0) \leqslant \lambda(c_3 + E[V(k+1, L)]). \tag{C1}$$

Clearly, $c_1 + V(0, l_0)$ in (C1) is free of l_k. Then, by Theorem 1 and Proposition 3, $\lambda(c_3 + E[V(k+1, L)])$ in (C1) is nondecreasing in l_k. Therefore, (C1) is true for any pair $(k, l_k) \in \mathbf{W}$ such that $l_k \geqslant l_k^*$. As a result, under the condition that the optimal condition-based decision in (k, l_k^*) is to replace the system, the optimal condition-based decision for any $(k, l_k) \in \mathbf{W}$ with $l_k \geqslant l_k^*$ is to replace the system as well. In other words, the optimal condition-based replacement policy is a control-limit policy with the determined control limit l_k^*. Further, it is noted that $\lambda(c_3 + E[V(k+1, L)])$ in (C1) is nonincreasing in k according to previous results. Therefore, for each degradation state $l \in \Re$, a threshold for the age k_l^* exists and the optimal condition-based decision for any $(k, l_k) \in \mathbf{W}$ at $k \leqslant k_l^*$ is to preventively replace the system. Together with these results, by the existence of such a condition-based control limit l_k^* for k and a threshold for k_l^* of each l, l_k^* is nondecreasing in k. This completes the proof.

Appendix D Value iteration algorithm

This paper utilizes the following value iteration algorithm to find the optimal solution to (1). With the help of the above structural properties, the computational loads will be hopefully reduced.

Algorithm D1 Value iteration algorithm

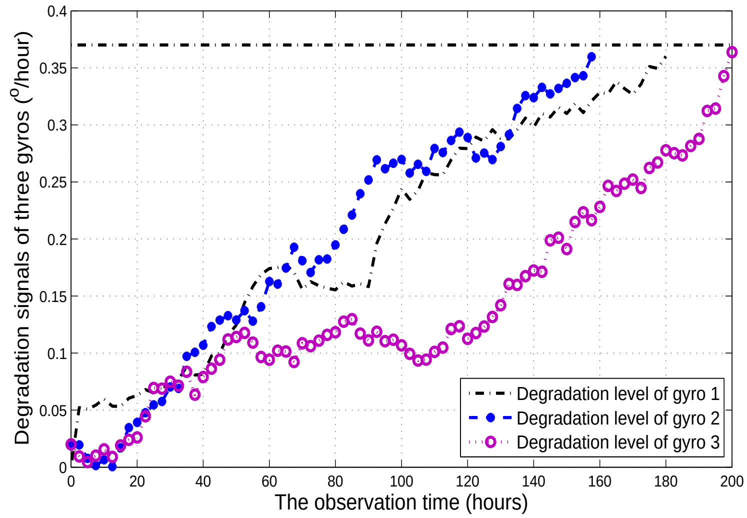
Require: Model settings for c_1, c_2, c_3;
Ensure: Minimized value function $V(k, l_k)$;
At current degradation state (k, l_k);
Set $V^0(k, l_k) = 0$;
Predict the distribution $F_L(x)$ of the future degradation state L using (6)-(8);
while The difference of the value functions $|V^n(k, l_k) - V^{n-1}(k, l_k)| < \delta$ **do**
  if $l_k > \xi'$ **then**
    set $V^{n+1}(k, l_k) = c_2 + V^n(0, l_0)$;
  else {The difference of the value functions $|V^n(k, l_k) - V^{n-1}(k, l_k)| \geqslant \delta$}
    set $V^{n+1}(k, l_k) = \lambda(c_3 + E\left[V^n(k+1, L)\right])$;
    if The value function $V^{n+1}(k, l_k) > c_1 + V^n(0, l_0)$ **then**
      set $V^{n+1}(k, l_k) = c_1 + V^n(0, l_0)$;
    end if
  end if
end while

Appendix E Data and main application results

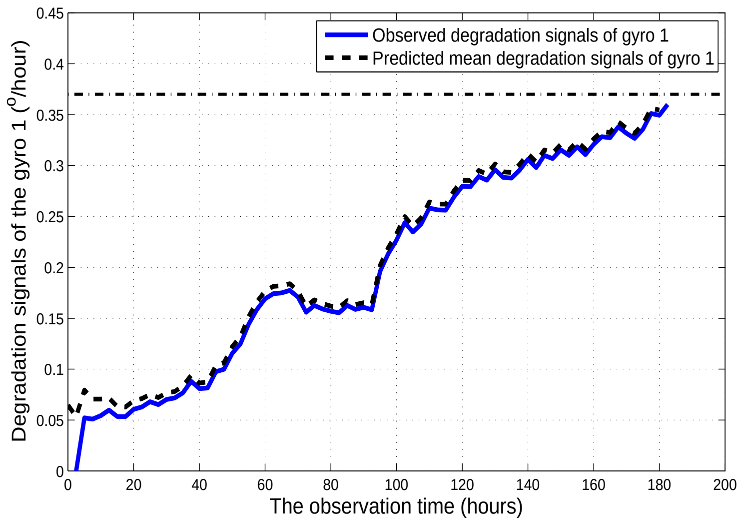
The data in the case study and main results by the proposed method are given in Figure E1.

References

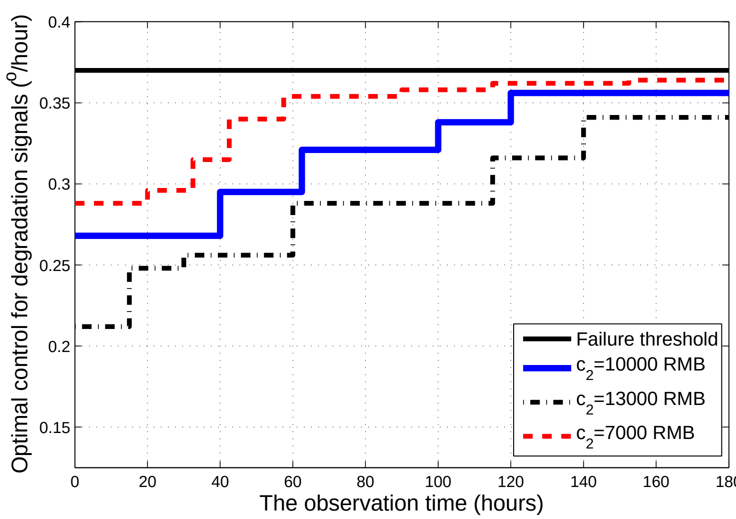
1 Ross S M. Stochastic Processes, 2nd ed. John Wiley & Sons, Inc., New York, 1996.



(a) Degradation signals of three gyros



(b) Predicted mean degradation signals



(c) Optimal control-limit solutions with different c_2

Figure E1 Data and main application results