SCIENCE CHINA
Information Sciences

• LETTER •

October 2021, Vol. 64 209204:1–209204:3
https://doi.org/10.1007/s11432-019-9857-5

pth moment exponential stability of general nonlinear discrete-time stochastic systems

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Received 3 February 2019/Accepted 2 April 2019/Published online 20 July 2020

Citation Jiang X S, Tian S P, Zhang W H. pth moment exponential stability of general nonlinear discrete-time stochastic systems. Sci China Inf Sci, 2021, 64(10): 209204, https://doi.org/10.1007/s11432-019-9857-5

Dear editor,

This study researches the pth moment exponential stability of general discrete-time nonlinear stochastic system

$$x_{k+1} = F_k(x_k, \omega_k),\ F_k(0, \cdot) \equiv 0. \tag{1}$$

Since Lyapunov initiated his stability theory, stability analysis has been one of the most important research topics in mathematics and modern control theory. Up to now, apart from the study of the stability of ordinary differential equations (ODEs), stochastic stability has also been extensively investigated [1–5]. The monographs [1, 2] were about stochastic stability of continuous-time Itô systems; however, there are few studies on the stability of stochastic difference equation (1). In particular, it seems that there has been no systematic monograph on stability of stochastic difference equations corresponding to continuous-time Itô systems. Difference systems become more and more important at present. Firstly, because it is almost impossible to obtain the analytical solution of continuous-time stochastic Itô equations, one has to search for numerical solutions, which is in fact a discretization process. Secondly, discrete-time systems are ideal models in engineering modeling [6]. Hence, it is very valuable to study the stability of stochastic difference equation (1). Below, we review the newly development in stochastic stability of difference systems.

In [7], we presented a discrete stochastic LaSalle invariance principle for stochastic system (1), which generalized the classic LaSalle invariance principle to discrete stochastic systems. Ref. [8] investigated the pth moment exponential input-to-state stability (ISS) of affine discrete-time impulsive stochastic delay systems with state-dependent noise, and some pth moment exponential ISS criteria were provided. In [9], asymptotic stability in probability of the following stochastic system

$$x_{k+1} = f_k(x_k) + g_k(x_k)\omega_k$$

was discussed. Different from stochastic Itô systems [1–3], we can find that most criteria on discrete stochastic stability are given via stochastic inequalities rather than deterministic inequalities, where the mathematical expectation of the system state [8] or conditional mathematical expectation [9] was involved, which makes the judging criteria more difficult to be verified.

This study presents a pth moment exponential stability criterion for the general nonlinear stochastic system (1). By introducing an efficient difference operator $\Delta V_k(x)$, which only contains the mathematical expectation of the white noise process $\{\omega_k\}_{k\in\mathcal{N}}$, a sufficient condition for pth moment exponential stability of the system (1) is given via some deterministic function inequalities that can be easily tested. Our main result can be viewed as a discrete version of pth moment exponential stability of stochastic Itô systems [1, 2].

Notations. $\mathbb{R}^n$: n-dimensional real vector space; $\mathcal{N} := \{0, 1, 2, \ldots\}$; $\|x\|$: 2-norm of a vector $x \in \mathcal{R}^n$; A': the transpose of the matrix or vector A; $\mathcal{E}$: the mathematical expectation operator.

Problem statements. Consider the following discrete-time time-varying nonlinear stochastic difference system

$$\begin{cases} x_{k+1} = F_k(x_k, \omega_k), & F_k(0, \cdot) \equiv 0, \\ k \in \mathcal{N} := \{0, 1, 2, \ldots\}, & x_0 \in \mathbb{R}^n, \end{cases} \tag{2}$$

where x_0 is, without loss of generality, assumed to be the deterministic initial state, x_k, $k \in \mathcal{N}$, are the $\mathbb{R}^n$-valued state variables, $\{\omega_k\}_{k\in\mathcal{N}}$ is an independent $\mathbb{R}^l$-valued random variable sequence defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_k\}_{k\in\mathcal{N}}, \mathscr{P})$ with $\mathcal{F}_k := \sigma(\omega_s, s = 0, 1, \ldots, k-1)$, $\mathcal{F}_0 = \{\phi, \Omega\}$. $F_k : \mathbb{R}^n \times \mathbb{R}^l \mapsto \mathbb{R}^n$ is a continuous function for each $k \in \mathcal{N}$. To introduce our main result, we first define pth moment exponential stability for system (2).

Assumption 1. $\mathcal{E}|F_k(x, \omega_k)|^p < +\infty$, $p > 0$.

Definition 1 ([1, 2])**.** The trivial solution $x_k \equiv 0$ of the stochastic difference system (2) is said to be pth moment exponentially stable, if there exist two positive constants λ

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and C such that

$$\mathcal{E}\|x_k\|^p \leqslant C\|x_0\|^p \mathrm{e}^{-\lambda k},\ k \in \mathcal{N},\ p > 0. \tag{3}$$

When $p = 2$, the trivial solution $x_k \equiv 0$ of the stochastic difference system (2) is said to be exponentially stable in mean square.

For system (2), we define the difference operator as

$$\Delta V_k(x) := \mathcal{E}V_{k+1}(F_k(x, \omega_k)) - V_k(x). \tag{4}$$

Under Assumption 1, $\Delta V_k(x)$ is a deterministic function, which involves only the mathematical expectation of ω_k, and such an important feature differs from the previous literature, where $\Delta V_k(x_k)$ was used to test stochastic stability. It should be pointed out that it is easier to test $\Delta V_k(x) < 0 (\leqslant 0)$ than $\Delta V_k(x_k) < 0 (\leqslant 0)$, because $\Delta V_k(x_k)$ contains the mathematical expectation of the state x_k.

The following lemma is well known in stochastic processes, which is called the smoothness of the conditional mathematical expectation; see [7] and the reference therein.

Lemma 1. If $\mathbb{R}^n$-valued random variable ξ is independent of the σ-field $\mathcal{G}$, and $\mathbb{R}^d$-valued random variable η is $\mathcal{G}$-measurable, then for any nonnegative function f: $\mathbb{R}^n \times \mathbb{R}^d \mapsto \mathbb{R}^+$,

$$\mathcal{E}[f(\xi, \eta)|\mathcal{G}] = \mathcal{E}[f(x, \eta)]|_{x=\xi}, \quad \text{almost surely}.$$

Main result. Our main result is the following theorem.

Theorem 1. If there exist a positive definite and continuous Lyapunov function sequence $\{V_k(x)\}_{k\in\mathcal{N}}$ on $\mathcal{N} \times \mathbb{R}^n$, and three positive constants k_1, k_2 and k_3, such that

$$k_1\|x\|^p \leqslant V_k(x) \leqslant k_2\|x\|^p \tag{5}$$

and

$$\begin{aligned}\Delta V_k(x) &= \mathcal{E}V_{k+1}(F_k(x, \omega_k)) - V_k(x)\\ &\leqslant -k_3\|x\|^p,\end{aligned} \tag{6}$$

then system (2) under Assumption 1 is pth moment exponentially stable.

Proof. By (5), $-V_k(x) \geqslant -k_2\|x\|^p$. Hence,

$$-k_3V_k(x) \geqslant -k_3k_2\|x\|^p.$$

So

$$-k_3\|x\|^p \leqslant -\frac{k_3}{k_2}V_k(x) \leqslant -\frac{k_3}{k_2 + k_3}V_k(x).$$

In view of the condition (6), we further have

$$\Delta V_k(x) \leqslant -k_3\|x\|^p \leqslant -\frac{k_3}{k_2 + k_3}V_k(x).$$

On the other hand, because $V_k(x)$ is a continuous function, x_k is $\mathcal{F}_k$-adapted, and ω_k is independent of ω_k for all $k \in \mathcal{N}$, it follows that

$$\begin{aligned}&\mathcal{E}[V_{k+1}(F_k(x_k, \omega_k))|\mathcal{F}_k] - V_k(x_k)\\ &\quad= \mathcal{E}[V_{k+1}(F_k(x_k, \omega_k)) - V_k(x_k)|\mathcal{F}_k]\\ &\quad= \mathcal{E}[V_{k+1}(F_k(x, \omega_k)) - V_k(x)]|_{x=x_k}\\ &\quad= \Delta V_k(x)|_{x=x_k}\\ &\quad\leqslant -\frac{k_3}{k_2 + k_3}V_k(x)|_{x=x_k}\\ &\quad= -\frac{k_3}{k_2 + k_3}V_k(x_k), \quad \text{almost surely},\end{aligned} \tag{7}$$

where the second equality is due to Lemma 1. Taking the mathematical expectation on both sides of (7), it gives that

$$\begin{aligned}\mathcal{E}V_{k+1}(x_{k+1}) &= \mathcal{E}V_{k+1}(F_k(x_k, \omega_k))\\ &\leqslant \frac{k_2}{k_2 + k_3}\mathcal{E}V_k(x_k).\end{aligned} \tag{8}$$

Using (8) repeatedly, it follows that

$$\begin{aligned}\mathcal{E}V_k(x_k) &\leqslant \left(\frac{k_2}{k_2 + k_3}\right)^k V_0(x_0)\\ &= \varepsilon^k V_0(x_0),\end{aligned}$$

where $0 < \varepsilon = \frac{k_2}{k_2+k_3} < 1$. By condition (5),

$$V_0(x_0) \leqslant k_2\|x_0\|^p,$$

so

$$\mathcal{E}V_k(x_k) \leqslant \varepsilon^k k_2\|x_0\|^p, \quad p > 0. \tag{9}$$

Still by condition (5),

$$k_1\|x\|^p \leqslant V_k(x), \quad \forall x \in \mathbb{R}^n,$$

which implies that

$$k_1\|x_k\|^p \leqslant V_k(x_k),\ \text{almost surely},\ k \in \mathcal{N}. \tag{10}$$

Taking the mathematical expectation on both sides of (10) and considering (9), we have

$$k_1\mathcal{E}\|x_k\|^p \leqslant \mathcal{E}V_k(x_k) \leqslant \varepsilon^k k_2\|x_0\|^p,$$

which yields that

$$\mathcal{E}\|x_k\|^p \leqslant \varepsilon^k \frac{k_2}{k_1}\|x_0\|^p. \tag{11}$$

In (11), if we set $C = \frac{k_2}{k_1} > 0$, $\lambda = -\log \varepsilon > 0$ (due to $0 < \varepsilon < 1$), then

$$\mathcal{E}\|x_k\|^p \leqslant \varepsilon^k \frac{k_2}{k_1}\|x_0\|^p = C\|x_0\|^p \mathrm{e}^{-\lambda k}.$$

This theorem is hence proved.

From Theorem 1, we can find that, in the study of pth moment exponential stability, $\mathcal{K}$-function is not necessary to be used as in other stochastic stabilities such as Theorem 2.2 of [2] on page 111.

In particular, for the linear stochastic difference system

$$x_{k+1} = Ax_k + Bx_k\omega_k, \quad x_0 \in \mathbb{R}^n, \quad k \in \mathcal{N}, \tag{12}$$

where $\{\omega_k\}_{k\in\mathcal{N}}$ is a one-dimensional independent white noise sequence, $\mathcal{E}\omega_k = 0$, $\mathcal{E}\omega_k^2 = 1$, $\mathcal{E}\omega_i\omega_j = 0$ for $i \neq j$, Theorem 1 leads to the following corollary.

Corollary 1. For the linear stochastic difference system (12), if

$$A'PA + B'PB - P < 0,$$

then system (12) is exponentially stable in mean square.

Proof. Set $V_k(x) = x'Px$. This corollary can be directly derived from Theorem 1.

To illustrate the effectiveness of Theorem 1, we give the following example.

Example 1. We consider a one-dimensional stochastic time-varying difference system

$$x_{k+1} = x_k \sin(kx_k)\omega_k, \quad x_0 \in \mathbb{R}, \tag{13}$$

where $\{\omega_k\}_{k\in\mathcal{N}}$ is an independence random variable sequence with $\mathcal{E}\omega_k^4 = 0.5$. Set $V_k(x) = x^4$. Obviously, $V_k(x)$ satisfies (5), i.e.,

$$k_1 x^4 \leqslant V_k(x) \leqslant k_2 x^4$$

for any $0 < k_1 \leqslant 1$ and $k_2 \geqslant 1$. In addition,

$$\begin{aligned}\Delta V_k(x) &= \mathcal{E}V_{k+1}(F_k(x,\omega_k)) - V_k(x)\\ &= x^4[\sin(kx)]^4\mathcal{E}\omega_k^4 - x^4\\ &= 0.5x^4[\sin(kx)]^4 - x^4\\ &\leqslant -0.5x^4, \quad \forall x \in \mathbb{R}.\end{aligned}$$

So condition (6) also holds. According to Theorem 1, system (2) is the 4th moment exponentially stable.

Conclusion. This study has presented a theorem on pth moment exponential stability for system (2), which can be viewed as a discrete version of Theorem 4.4 of [2] and Theorem 5.11 of [1] about continuous-time Itô systems. Because conditions (5) and (6) are easily verified, Theorem 1 is more applicable in practical computations. We believe that, following the proof line of Theorem 1, some previous results such as Theorems 1 and 2 in [8] can be further improved, which remains to be our future work.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 61374104, 61573227, 61773170), Natural Science Foundation of Guangdong Province of China (Grant No. 2016A030313505), the SDUST Research Fund (Grant No. 2015TDJH105), and the Fund for the Taishan Scholar Project of Shandong Province of China.

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