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Distributed optimal consensus of second-order multi-agent systems

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Dear editor,

For multi-agent systems, consensus is the most fundamental problem, for which, vast studies have been reported, but a few achieved optimal protocols [1–9]. As well known, optimal consensus is nontrivial due to the mismatching between the whole performance optimization (which usually leads to a centralized protocol) and the given distributed communication topology [1]. This makes that for a pre-given global cost functional, a distributed optimal protocol usually cannot be derived based on certain Riccati equation [1–3], for which suboptimal protocol could be attempted [4, 5]. Despite proposing distributed optimal protocols, Refs. [6, 7] merely focus on local cost functionals (rather than global), and Refs. [3, 8, 9] only consider post-given or man-made (to some extent) global cost functionals, which lack practical significance. Notably, Ref. [1] developed a novel design strategy of distributed optimal protocol for pre-given global cost functional and topology, but the strategy is only applicable to single-integrator agents.

This study is the first attempt of a distributed optimal protocol for second-order multi-agent systems. Essentially different from the existing studies, the global quadratic cost functional and the topology are both pre-given, and the optimal protocol to be sought is distributed. Recognizing the inapplicability of Riccati-based strategy, we develop an effective strategy of distributed optimal consensus. Motivated by [1], we first affirm the feasibility of distributed optimal protocol, i.e., the existence of optimal gain parameters. Then, by recursively deriving the explicit formula of the consensus error, an online implementable algorithm is developed to achieve the parameterization of the cost functional. Namely, the completely explicit formula of the cost functional depending on gain parameters of all agents is derived. Furthermore, the optimal gain parameters are obtained by minimizing the explicit formula.

Problem statement. We consider the following leader-following multi-agent system:

$$\begin{cases} \dot{x}_i = v_i,\ \dot{v}_i = u_i, \quad i = 1, \dots, N, \\ \dot{x}_0 = v_0,\ \dot{v}_0 = 0, \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ are the positions of the i-th follower and the leader, respectively; $v_i \in \mathbb{R}$ and $v_0 \in \mathbb{R}$ are the velocities, and $u_i \in \mathbb{R}$ is the control input of the i-th follower. In what follows, suppose that the topology associated with system (1) is the following digraph with single chain.

The objective is to find a distributed optimal protocol in the form:

$$\begin{aligned} u_i &= -\bar{k}_i a_{i,i-1}(x_i - x_{i-1}) - \bar{l}_i w_{i,i-1}(v_i - v_{i-1}) \\ &=: -k_i \varepsilon_{i,i-1} - l_i \delta_{i,i-1}, \quad i = 1, \dots, N, \end{aligned} \tag{2}$$

with $k_i = \bar{k}_i a_{i,i-1}$, $l_i = \bar{l}_i w_{i,i-1}$, $\varepsilon_{i,i-1} = x_i - x_{i-1}$ and $\delta_i = v_i - v_{i-1}$, such that all agents achieve consensus, i.e., $\lim_{t\to+\infty}(|x_i(t)-x_0(t)|+|v_i(t)-v_0(t)|) = 0$, while minimizing the following quadratic cost functional:

$$\begin{aligned} J &= \int_0^{+\infty} \sum_{i=1}^{N} \big(a_{i,i-1}\big(x_i(t) - x_{i-1}(t)\big)^2 \\ &\quad + w_{i,i-1}\big(v_i(t) - v_{i-1}(t)\big)^2 + r_i u_i^2(t)\big)\mathrm{d}t \\ &= \int_0^{+\infty} \sum_{i=1}^{N} \big(a_{i,i-1}\varepsilon_{i,i-1}^2(t) + w_{i,i-1}\delta_{i,i-1}^2(t) \\ &\quad + r_i u_i^2(t)\big)\ \mathrm{d}t, \end{aligned} \tag{3}$$

where a_{ij} and w_{ij}, all positive, are respectively the weights of exchanging the position and velocity information associated with the topology digraph (i.e., Figure 1); r_i are the positive weighted coefficients of the protocol.

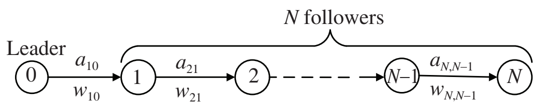

Figure 1 Digraph of one chain.

Remark 1. Although the optimal protocol problem can be viewed as an LQR problem, it cannot be solved by the means of certain Riccati equation. Otherwise, the deduced optimal protocol, which is usually centralized rather than distributed, violates the digraph in Figure 1 and is not the desired distributed protocol.

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Main results. We first affirm the feasibility of distributed optimal protocol (2), i.e., the existence of optimal gain parameters k_i and l_i (see Theorem 1 below). Then, an online implementable algorithm is presented to achieve the parameterization of cost functional (3), namely, to derive the completely explicit formula of the cost function- al with respect to gain parameters k_i and l_i. Moreover, the optimal gain parameters are found by minimizing the explicit formula.

Theorem 1. There is an optimal protocol in the form (2), which can achieve the consensus of multi-agent system (1), while minimizing cost functional (3).

Proof. Under protocol (2), system (1) and cost functional (3) become

$$\begin{cases} \dot{e} = Ae, \\ J_u(e(0)) = \int_0^{+\infty} e^{\mathrm{T}}(t)Qe(t)\ \mathrm{d}t, \end{cases}$$

where $e = [\varepsilon_{10}, \ldots, \varepsilon_{N,N-1}, \delta_{10}, \ldots, \delta_{N,N-1}]^{\mathrm{T}}$.

Noting the positiveness of k_i and l_i, we see that A is Hurwitz, and Q is symmetric and positive definite. Then, the protocol can guarantee the consensus of multi-agent system (1).

Let us next show the existence of optimal protocol. It is worth first pointing out that, if k_i and l_i are given (and of course positive), as well as A and Q, then there exists a unique symmetric positive definite matrix P satisfying the following Lyapunov equation:

$$A^{\mathrm{T}}P + PA = -Q, \tag{4}$$

and $J_u(e(0)) = e^{\mathrm{T}}(0)Pe(0)$ holds.

Because A and Q are linearly and quadratically depending on k_i and l_i, their entries can be viewed as (linear or quadratic) polynomials of k_i and l_i. Noting that the unique P satisfying (4) can be derived by solving linear equation (4) via the elimination method, we can see that each entry of P is a rational function of k_i and l_i (whose numerator and denominator are both polynomials).

Thus, to prove the existence of optimal protocol, it suffices to prove the continuity of P (as a rational matrix of k_i and l_i) on $\mathbb{R}_+^{2N}$. In fact, the discussion below for the case $N = 1$ (due to page limitation) shows that for nonzero $e(0)$, if any one of k_i and l_i approaches zero or infinity, then $J_u(e(0))$ would tend to infinity. This, together with the continuity of P, immediately means the existence of optimal protocol.

To prove the continuity of P, we only need to prove that no entry of P has denominator which vanishes for some positive k_i or l_i. Otherwise, one entry of P would be infinity for certain positive k_i and l_i, which contradicts the existence and uniqueness of P satisfying the above Lyapunov equation.

Let us check the case $N=1$. For the case, we have

$$J_u(e(0)) = \begin{bmatrix} \varepsilon_{10}(0) \\ \delta_{10}(0) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} \varepsilon_{10}(0) \\ \delta_{10}(0) \end{bmatrix},$$

where $p_1 = \frac{a_{10}(l_1^2+k_1)}{2k_1l_1} + \frac{w_{10}k_1}{2l_1} + \frac{r_1k_1^2}{2l_1}$, $p_2 = \frac{a_{10}+r_1k_1^2}{2k_1}$, and $p_3 = \frac{a_{10}+r_1k_1^2}{2k_1l_1} + \frac{w_{10}+r_1l_1^2}{2l_1}$. From this, it can be seen that $\lim_{k_1+l_1\to+\infty} J_u(e(0)) = +\infty$ and $\lim_{k_1l_1\to 0} J_u(e(0)) = +\infty$ provided that $\varepsilon_{10}(0)$ or $\delta_{10}(0)$ is nonzero.

From the above proof, we know that $J_u(e(0)) = e^{\mathrm{T}}(0)Pe(0)$, where P satisfies (4). Because P is a rational function of k_i and l_i, the optimal gain parameters k_i or l_i, which minimize $J_u(e(0))$, depend on the initial system conditions, and hence cannot be obtained by solving Lyapunov equation (4). This is completely different from the classical LQR problem which can be solved by merely solving a Riccati equation. Recognizing the particularity and complexity of our optimal problem, we next pursue the parameterization of $J_u(e(0))$, i.e., the explicit formula of $J_u(e(0))$ as a rational function of k_i and l_i, such that the existing optimization methods can be applied (e.g., gradient method and particle swarm optimization).

Substituting (2) into (3) yields

$$\begin{aligned} J_u(e(0)) = \sum_{i=1}^{N} \Big(&(a_{i,i-1} + r_ik_i^2)\int_0^{+\infty} \varepsilon_{i,i-1}^2(t)\ \mathrm{d}t \\ &+(w_{i,i-1} + r_il_i^2)\int_0^{+\infty} \delta_{i,i-1}^2(t)\ \mathrm{d}t \\ &+2r_ik_il_i\int_0^{+\infty} \varepsilon_{i,i-1}(t)\delta_{i,i-1}(t)\ \mathrm{d}t\Big). \end{aligned} \tag{5}$$

In the following, we need to compute the integrals of $\varepsilon_{i,i-1}^2$, $\delta_{i,i-1}^2$ and $\varepsilon_{i,i-1}\delta_{i,i-1}$ to finish the parameterization of $J_u(e(0))$.

By (1) and (2), we have

$$\begin{bmatrix} \dot{\varepsilon}_{i,i-1} \\ \dot{\delta}_{i,i-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_i & -l_i \end{bmatrix} \begin{bmatrix} \varepsilon_{i,i-1} \\ \delta_{i,i-1} \end{bmatrix} + \begin{bmatrix} 0 \\ k_{i-1}\varepsilon_{i-1,i-2} + l_{i-1}\delta_{i-1,i-2} \end{bmatrix},$$

where the system matrix is Hurwitz with its eigenvalues denoted by $-s_{i1}$ and $-s_{i2}$ for later development. From the above equation, it follows that the explicit formulas of the consensus errors $\varepsilon_{i,i-1}$ and $\delta_{i,i-1}$ can be recursively derived. Detailedly, the explicit formulas of $\varepsilon_{i,i-1}$ and $\delta_{i,i-1}$ can be represented as linear combinations of $e^{-s_{pj}t}$ ($p = 1, \ldots, i$, $j = 1, 2$), whose coefficients are rational functions of s_{pj} (see Lemma 2 in Appendix A).

Because $-s_{pj}$ are the eigenvalues of a Hurwitz matrix, the linear combination of $e^{-s_{pj}t}$ ($p = 1, \ldots, i$, $j = 1, 2$) must be square integrable, which implies the square integrability of $\varepsilon_{i,i-1}$ and $\delta_{i,i-1}$. Recalling that $J_u = e^{\mathrm{T}}(0)Pe(0)$ with P being a rational matrix depending on k_i and l_i, the integrals of $\varepsilon_{i,i-1}^2$ and $\delta_{i,i-1}^2$ in (5) are rational functions of k_i and l_i. Furthermore, the explicit formula of the cost functional with respect to gain parameters can be derived recursively as follows:

$$\begin{aligned} J_u(e(0)) = \sum_{i=1}^{N}\Big(&\sum_{p=1}^{i}\sum_{q=p}^{i} E_{i1}^{pq}(k_{[p,i]}, l_{[p,i]})\varepsilon_{p,p-1}(0)\varepsilon_{q,q-1}(0) \\ &+ \sum_{p=1}^{i}\sum_{q=p}^{i} E_{i2}^{pq}(k_{[p,i]}, l_{[p,i]})\delta_{p,p-1}(0)\delta_{q,q-1}(0) \\ &+ \sum_{p=1}^{i}\sum_{q=p}^{i} E_{i3}^{pq}(k_{[p,i]}, l_{[p,i]})\varepsilon_{p,p-1}(0)\delta_{q,q-1}(0)\Big), \end{aligned} \tag{6}$$

where the formulas of $E_{ij}^{pq}(\cdot)$ (which are continuous rational functions of $k_{[p,i]}$ and $l_{[p,i]}$) are postponed in Appendix A.

It is seen from the proof of Theorem 1 that if any one of k_i and l_i tends to zero or infinity, then $J_u(e(0))$ would tend to infinity. Thus, the minimum value of $J_u(e(0))$ must be achieved at extreme points. For clarity, Algorithm 1 is given to summarize the detailed procedure of deriving optimal gain parameters. Additionally, we employ the particle swarm optimization algorithm provided in [1] to seek the minimum value of a rational function (see Appendix B).

The simulation example is given in Appendix B.

Algorithm 1 Find optimal gain parameters

1: Implement the parameterization of cost functional by (6).
2: Calculate $\frac{\partial J_u(e(0))}{\partial k_i} = 0$ and $\frac{\partial J_u(e(0))}{\partial l_i} = 0$ to derive all extreme points.
3: Compute the corresponding value of $J_u(e(0))$ at each extreme point.
4: Derive the minimum $J_u(e(0))$ and the corresponding optimal gain parameters k_i and l_i.

Conclusion. This study is devoted to the distributed optimal consensus for leader-following second-order multi-agent systems. Remarkably, for pre-given global cost functional and topology, an effective strategy is developed to find distributed optimal protocol. Specifically, the explicit formula of the cost functional with respect to gain parameters is first derived, and then optimal gain parameters are obtained by minimizing the formula. Remark that in the leaderless case, the directed topology contains at least a cycle for consensus, which is essentially different from the leader-following case under investigation. So in the future, we will attempt to investigate the distributed optimal protocol for leaderless case.

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Supporting information Appendixes A and B. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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