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Information Sciences

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Distributed optimal consensus of second-order multi-agent systems

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In this section, there are two parts: Appendix A presents the detailed procedure of the parameterization of cost functional (3); Appendix B presents the specific process of particle swarm optimization to derive optimal gain parameters, and gives an simulation example.

Appendix A Parameterization of cost functional (3)

For the later development, the following denotations are (iteratively) defined by

$$\left\{\begin{array}{l}\left\{\begin{array}{l}A_{10}(\eta_{[3]})=\frac{\eta_3-\eta_1}{\eta_2-\eta_1},\ A_{11}(\eta_{[5]})=\frac{\eta_5(\eta_4-\eta_1)}{(\eta_2-\eta_1)(\eta_3-\eta_1)},\\ A_{1i}(\eta_{[3i+1]})=\frac{\prod_{j=2i+2}^{3i+1}\eta_j(\eta_{2i+1}-\eta_1)}{\prod_{j=1}^{i}(\eta_{2j}-\eta_{2j-1})(\eta_3-\eta_1)},\\ A_{20}(\eta_{[3]})=\frac{\eta_3}{\eta_2-\eta_1},\ A_{21}(\eta_{[5]})=\frac{-\eta_4\eta_5}{(\eta_2-\eta_1)(\eta_3-\eta_1)},\\ A_{2i}(\eta_{[3i+1]})=\frac{-\prod_{j=2i+1}^{3i+1}\eta_j}{\prod_{j=1}^{i}(\eta_{2j}-\eta_{2j-1})(\eta_3-\eta_1)},\\ A_{30}(\eta_{[2]})=\frac{1}{\eta_2-\eta_1},\ A_{31}(\eta_{[4]})=\frac{\eta_4}{(\eta_2-\eta_1)(\eta_3-\eta_1)},\\ A_{3i}(\eta_{[3i]})=\frac{\prod_{j=2i+1}^{3i}\eta_j}{\prod_{j=1}^{i}(\eta_{2j}-\eta_{2j-1})(\eta_3-\eta_1)},\\ A_{40}(\eta_{[2]})=\frac{-\eta_1}{\eta_2-\eta_1},\ A_{41}(\eta_{[4]})=\frac{-\eta_1\eta_4}{(\eta_2-\eta_1)(\eta_3-\eta_1)},\\ A_{4i}(\eta_{[3i]})=\frac{-\prod_{j=2i+1}^{3i}\eta_j\eta_1}{\prod_{j=1}^{i}(\eta_{2j}-\eta_{2j-1})(\eta_3-\eta_1)},\ i=1,\ldots,N-1,\end{array}\right.\\ \left\{\begin{array}{l}\Phi_0(t,\eta_1)=e^{-\eta_1 t},\Phi_1(t,\eta_{[2]})=e^{-\eta_1 t}-e^{-\eta_2 t},\\ \Phi_i(t,\eta_{[i+1]})=\frac{\Phi_{i-1}(t,[\eta_1,\eta_{[3,i+1]}^{\mathrm{T}}]^{\mathrm{T}})}{\eta_3-\eta_1}-\frac{\Phi_{i-1}(t,[\eta_2,\eta_{[3,i+1]}^{\mathrm{T}}]^{\mathrm{T}})}{\eta_3-\eta_2},\ i=2,\ldots,N-1,\end{array}\right.\end{array}\right. \tag{A1}$$

where $\eta_{[i]}=[\eta_1,\ldots,\eta_i]^{\mathrm{T}}\in\mathbf{R}^i$ and $\eta_{[3,i+1]}=[\eta_3,\ldots,\eta_{i+1}]$.

We next specially introduce the following crucial lemma about $\Phi_i(t,\eta_{[i+1]})$ (when η_i's are positive real numbers), which is the same as Lemma 1 of [1] (noting that $\Phi_i(\cdot)$'s in (A1) can be recursively derived from those in (1) of [1]).

Lemma 1. For $\{\Phi_i(\cdot)\}$ in (A1), each pairwise product is absolutely integral on $[0,+\infty)$, that is,

$$\int_0^{+\infty}\left|\Phi_i(t,\eta_{[i+1]})\Phi_j(t,\bar{\eta}_{[j+1]})\right|\,\mathrm{d}t\ <\ +\infty,\ i,j=0,1,\ldots,N-1.$$

Particularly, defining that for $i,j=0,1,\ldots,N-1$,

$$B_{ij}(\eta_{[i+1]},\,\bar{\eta}_{[j+1]})\triangleq\int_0^{+\infty}\Phi_i(t,\eta_{[i+1]})\Phi_j(t,\bar{\eta}_{[j+1]})\ \mathrm{d}t, \tag{A2}$$

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with there hold the following relations (from which the explicit expressions of $B_{ij}(\cdot)$'s can be recursively derived):

$$\begin{cases} B_{ij}(\eta_{[i+1]},\,\bar{\eta}_{[j+1]}) = B_{ji}(\bar{\eta}_{[j+1]},\,\eta_{[i+1]}),\ i,j=0,1,\ldots,N-1, \\ \begin{cases} B_{00}(\eta_1,\,\bar{\eta}_1) = \frac{1}{\eta_1+\bar{\eta}_1}, \\ B_{0j}(\eta_1,\,\bar{\eta}_{[j+1]}) = \frac{\bar{\eta}_2-\bar{\eta}_1}{\prod_{l=1}^{j+1}(\bar{\eta}_l+\eta_1)}, \quad j=1,\ldots,N-1, \\ B_{1j}(\eta_{[2]},\bar{\eta}_{[j+1]}) = \frac{\bar{\eta}_2-\bar{\eta}_1}{\prod_{l=1}^{j+1}(\bar{\eta}_l+\eta_1)} - \frac{\bar{\eta}_2-\bar{\eta}_1}{\prod_{l=1}^{j+1}(\bar{\eta}_l+\eta_2)}, \\ \qquad j=1,\ldots,N-1, \\ B_{ij}(\eta_{[i+1]},\,\bar{\eta}_{[j+1]}) = \frac{B_{i,j-1}(\eta_{[i+1]},\,[\bar{\eta}_1,\bar{\eta}_{[3,j+1]}^{\mathrm{T}}]^{\mathrm{T}})}{\bar{\eta}_3-\bar{\eta}_1} - \frac{B_{i,j-1}(\eta_{[i+1]},\,[\bar{\eta}_2,\bar{\eta}_{[3,j+1]}^{\mathrm{T}}]^{\mathrm{T}})}{\bar{\eta}_3-\bar{\eta}_2}, \\ \quad i=2,\ldots,N-1,\ j=i,\ldots,N-1. \end{cases} \end{cases} \tag{A3}$$

Lemma 2. By substituting (2) into (1), the dynamics of $\varepsilon_{i,i-1}$'s and $\delta_{i,i-1}$'s can be derived as follows:

$$\begin{aligned} \varepsilon_{i,i-1}(t) \;=\; & \sum_{j=1}^{2} A_{10}(s_{ij}, s_{i\bar{j}}, l_i)\Phi_0(t, s_{ij})\varepsilon_{i,i-1}(0) + \sum_{j=1}^{2} A_{30}(s_{ij}, s_{i\bar{j}})\Phi_0(t, s_{ij})\delta_{i,i-1}(0) \\ & + \sum_{q=1}^{i-1}\sum_{j=1}^{2}\sum_{j_1=1}^{2}\cdots\sum_{j_q=1}^{2} A_{30}(s_{ij}, s_{i\bar{j}})\Phi_q(t, s_{[i-q,i]})\Big((A_{1q}(\cdot)+A_{2q}(\cdot))\varepsilon_{i-q,i-q-1}(0) \\ & + (A_{3q}(\cdot)+A_{4q}(\cdot))\delta_{i-q,i-q-1}(0)\Big), \end{aligned} \tag{A4}$$

and

$$\begin{aligned} \delta_{i,i-1}(t) \;=\; & \sum_{j=1}^{2} A_{20}(s_{ij}, s_{i\bar{j}}, k_i)\Phi_0(t, s_{ij})\varepsilon_{i,i-1}(0) + \sum_{j=1}^{2} A_{40}(s_{ij}, s_{i\bar{j}})\Phi_0(t, s_{ij})\delta_{i,i-1}(0) \\ & + \sum_{q=1}^{i-1}\sum_{j=1}^{2}\sum_{j_1=1}^{2}\cdots\sum_{j_q=1}^{2} A_{40}(s_{ij}, s_{i\bar{j}})\Phi_q(t, s_{[i-q,i]})\Big((A_{1q}(\cdot)+A_{2q}(\cdot))\varepsilon_{i-q,i-q-1}(0) \\ & + (A_{3q}(\cdot)+A_{4q}(\cdot))\delta_{i-q,i-q-1}(0)\Big), \end{aligned} \tag{A5}$$

where $s_{[i-q,i]} = [s_{i-q,j_q},\ldots,s_{i-1,j_1},s_{ij}]^{\mathrm{T}}$, $\bar{s}_{[i-q,i-1]} = [s_{i-q,j_q},s_{i-q,\bar{j}_q},\ldots,s_{i-1,j_1},s_{i-1,\bar{j}_1}]^{\mathrm{T}}$, $\bar{j} = \frac{2}{j}$ and $\bar{j}_q = \frac{2}{j_q}$; $A_{p1}(\cdot) = A_{p1}(s_{i-1,j_1}, s_{i-1,\bar{j}_1}, s_{ij}, l_{i-1}, k_{i-1})$, $A_{pq}(\cdot) = A_{pq}(\bar{s}_{[i-q,i-1]}, l_{i-q}, k_{[i-q,i-1]})$, $p=1,2$; $A_{31}(\cdot) = A_{31}(s_{i-1,j_1}, s_{i-1,\bar{j}_1}, s_{ij}, k_{i-1})$, $A_{3q}(\cdot) = A_{3q}(\bar{s}_{[i-q,i-1]}, k_{[i-q,i-1]})$; $A_{41}(\cdot) = A_{41}(s_{i-1,j_1}, s_{i-1,\bar{j}_1}, s_{ij}, l_{i-1})$, $A_{4q}(\cdot) = A_{4q}(\bar{s}_{[i-q,i-1]}, l_{i-q}, k_{[i-q+1,i-1]})$.

Proof . Noting from the proof of Theorem 1 that $e(t) = e^{At}e(0)$, where $e = [\varepsilon_{10},\ldots,\varepsilon_{N,N-1},\delta_{10},\ldots,\delta_{N,N-1}]^{\mathrm{T}}$, the explicit formula of e (i.e, the explicit formulas of $\varepsilon_{i,i-1}$'s and $\delta_{i,i-1}$'s) can be derived by calculating $e^{At} = [e^{-s_{11}t}\upsilon_{11}, e^{-s_{12}t}\upsilon_{12}, \ldots, e^{-s_{N2}t}\upsilon_{N2}][\upsilon_{11},\upsilon_{12},\ldots,\upsilon_{N2}]^{-1}$ with $-s_{pj}$ and υ_{pj} ($p=1,\ldots,i$, $j=1,2$) respectively being the eigenvalues and corresponding eigenvectors of Hurwitz matrix A. From this, it can be seen that whether s_{pj}'s are complex numbers or positive real numbers, each entry of e^{At} can be represented as linear combinations of $e^{-s_{pj}t}$, whose coefficients are rational functions of s_{pj}'s with $l_p = s_{p1}+s_{p2}$ and $k_p = s_{p1}s_{p2}$. This implies that for complex or real numbers s_{pj}'s, the explicit formula of $\varepsilon_{i,i-1}$ represented as linear combinations of $e^{-s_{pj}t}$ is always the same, so is $\delta_{i,i-1}$. Thus, we only need to prove (A4) and (A5) in the scenario that s_{pj}'s are real numbers.

To begin with, substituting protocol (2) into system (1) yields

$$\begin{bmatrix} \dot{x}_1-\dot{x}_0 \\ \dot{v}_1-\dot{v}_0 \end{bmatrix} = \begin{bmatrix} v_1-v_0 \\ -k_1(x_1-x_0)-l_1(v_1-v_0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_1 & -l_1 \end{bmatrix}\begin{bmatrix} v_1-v_0 \\ x_1-x_0 \end{bmatrix}. \tag{A6}$$

From this, and noting the definitions $\varepsilon_{i,i-1} = x_i - x_{i-1}$ and $\delta_{i,i-1} = v_i - v_{i-1}$, we have

$$\begin{aligned} \begin{bmatrix} \varepsilon_{10}(t) \\ \delta_{10}(t) \end{bmatrix} &= \begin{bmatrix} \frac{l_1-s_{11}}{s_{12}-s_{11}}e^{-s_{11}t} + \frac{l_1-s_{12}}{s_{11}-s_{12}}e^{-s_{12}t} & \frac{1}{s_{12}-s_{11}}e^{-s_{11}t} + \frac{1}{s_{11}-s_{12}}e^{-s_{12}t} \\ \frac{-k_1}{s_{12}-s_{11}}e^{-s_{11}t} + \frac{-k_1}{s_{11}-s_{12}}e^{-s_{12}t} & \frac{-s_{11}}{s_{12}-s_{11}}e^{-s_{11}t} + \frac{-s_{12}}{s_{11}-s_{12}}e^{-s_{12}t} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_{10}(0) \\ \delta_{10}(0) \end{bmatrix} \\ &= \begin{bmatrix} \sum_{j=1}^{2} A_{10}(s_{1j}, s_{1\bar{j}}, l_1)\Phi_0(t, s_{1j})\varepsilon_{10}(0) + \sum_{j=1}^{2} A_{30}(s_{1j}, s_{1\bar{j}})\Phi_0(t, s_{1j})\delta_{10}(0) \\ \sum_{j=1}^{2} A_{20}(s_{1j}, s_{1\bar{j}}, k_1)\Phi_0(t, s_{1j})\varepsilon_{10}(0) + \sum_{j=1}^{2} A_{40}(s_{1j}, s_{1\bar{j}})\Phi_0(t, s_{1j})\delta_{10}(0) \end{bmatrix}, \end{aligned} \tag{A7}$$

where $-s_{i1}$ and $-s_{i2}$ are two solutions of $s^2 + l_i s + k_i = 0$, and $j\bar{j} = 2$. It is obvious that (A4) and (A5) hold for $i = 1$.

Assume for induction that for $i = 1,\ldots,w$, (A4) and (A5) always hold. Then, by (1) and (2), we have that for $i = 1,\ldots,j$,

$$\begin{aligned} \begin{bmatrix} \dot{\varepsilon}_{i+1,i} \\ \dot{\delta}_{i+1,i} \end{bmatrix} &= \begin{bmatrix} \delta_{i+1,i} \\ -k_{i+1}\varepsilon_{i+1,i} - l_{i+1}\delta_{i+1,i} + k_i\varepsilon_{i,i-1} + l_i\delta_{i,i-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -k_{i+1} & -l_{i+1} \end{bmatrix}\begin{bmatrix} \varepsilon_{i+1,i} \\ \delta_{i+1,i} \end{bmatrix} + \begin{bmatrix} 0 \\ k_i\varepsilon_{i,i-1} + l_i\delta_{i,i-1} \end{bmatrix}. \end{aligned}$$

From this, we deduce

$$\begin{bmatrix} \varepsilon_{i+1,i}(t) \\ \delta_{i+1,i}(t) \end{bmatrix} = \begin{bmatrix} \frac{l_{i+1}-s_{i+1,1}}{s_{i+1,2}-s_{i+1,1}}e^{-s_{i+1,1}t} + \frac{l_{i+1}-s_{i+1,2}}{s_{i+1,1}-s_{i+1,2}}e^{-s_{i+1,2}t} & \frac{1}{s_{i+1,2}-s_{i+1,1}}e^{-s_{i+1,1}t} + \frac{1}{s_{i+1,1}-s_{i+1,2}}e^{-s_{i+1,2}t} \\ \frac{-k_{i+1}}{s_{i+1,2}-s_{i+1,1}}e^{-s_{i+1,1}t} + \frac{-k_{i+1}}{s_{i+1,1}-s_{i+1,2}}e^{-s_{i+1,2}t} & \frac{-s_{i+1,1}}{s_{i+1,2}-s_{i+1,1}}e^{-s_{i+1,1}t} + \frac{-s_{i+1,2}}{s_{i+1,1}-s_{i+1,2}}e^{-s_{i+1,2}t} \end{bmatrix}$$

$$\cdot\begin{bmatrix}\varepsilon_{i+1,i}(0)\\ \delta_{i+1,i}(0)\end{bmatrix}+\begin{bmatrix}\int_0^t\Big(\frac{1}{s_{i+1,2}-s_{i+1,1}}e^{-s_{i+1,1}(t-\tau)}+\frac{1}{s_{i+1,1}-s_{i+1,2}}e^{-s_{i+1,2}(t-\tau)}\Big)\big(k_i\varepsilon_{i,i-1}(\tau)+l_i\delta_{i,i-1}(\tau)\big)\,\mathrm{d}\tau\\ \int_0^t\Big(\frac{-s_{i+1,1}}{s_{i+1,2}-s_{i+1,1}}e^{-s_{i+1,1}(t-\tau)}+\frac{-s_{i+1,2}}{s_{i+1,1}-s_{i+1,2}}e^{-s_{i+1,2}(t-\tau)}\Big)\big(k_i\varepsilon_{i,i-1}(\tau)+l_i\delta_{i,i-1}(\tau)\big)\,\mathrm{d}\tau\end{bmatrix}. \quad \text{(A8)}$$

To complete the proof, based on the assumption and noting that $\int_0^t e^{-\bar{\eta}_1(t-\tau)}\Phi_i(\tau,\eta_{[i+1]})\,\mathrm{d}\tau=\Phi_{i+1}(t,[\eta_{[i+1]}^{\mathrm{T}},\bar{\eta}_1]^{\mathrm{T}})$ (which has been verified in the proof of theorem 1 in [1]), it suffices to calculate the following :

$$\begin{aligned}
&\int_0^{+\infty}e^{-s_{i+1,p}(t-\tau)}k_i\varepsilon_{i,i-1}(\tau)\,\mathrm{d}\tau\\
&=\int_0^{+\infty}e^{-s_{i+1,p}(t-\tau)}k_i\Bigg(\sum_{j=1}^{2}A_{10}(s_{ij},s_{i\bar{j}},l_i)\Phi_0(\tau,s_{ij})\varepsilon_{i,i-1}(0)+\sum_{j=1}^{2}A_{30}(s_{ij},s_{i\bar{j}})\Phi_0(\tau,s_{ij})\delta_{i,i-1}(0)\\
&\quad+\sum_{q=1}^{i-1}A_{30}(s_{ij},s_{i\bar{j}})\Bigg(\sum_{j_1=1}^{2}\cdots\sum_{j_{q+1}=1}^{2}\big(A_{1q}(\cdot)+A_{2q}(\cdot)\big)\Phi_q(\tau,s_{[i-q,i]})\varepsilon_{i-q,i-q-1}(0)\\
&\quad+\sum_{j_1=1}^{2}\cdots\sum_{j_{q+1}=1}^{2}(A_{3q}(\cdot)+A_{4q}(\cdot))\Phi_q(\tau,s_{[i-q,i]})\delta_{i-q,i-q-1}(0)\Bigg)\Bigg)\,\mathrm{d}\tau\\
&=\sum_{j=1}^{2}A_{11}(s_{ij},s_{i\bar{j}},s_{i+1,p},l_i,k_i)\Phi_1(t,[s_{ij},s_{i+1,p}]^{\mathrm{T}})\varepsilon_{i,i-1}(0)\\
&\quad+\sum_{j=1}^{2}A_{31}(s_{ij},s_{i\bar{j}},s_{i+1,p},k_i)\Phi_1(t,[s_{ij},s_{i+1,p}]^{\mathrm{T}})\delta_{i,i-1}(0)\\
&\quad+\sum_{q=1}^{i-1}\Bigg(\sum_{j_1=1}^{2}\cdots\sum_{j_{q+1}=1}^{2}\big(A_{1,q+1}(\cdot)+A_{2,q+1}(\cdot)\big)\Phi_{q+1}(t,[s_{[i-q,i]}^{\mathrm{T}},s_{i+1,p}]^{\mathrm{T}})\varepsilon_{i-q,i-q-1}(0)\\
&\quad+\sum_{j_1=1}^{2}\cdots\sum_{j_{q+1}=1}^{2}(A_{3,q+1}(\cdot)+A_{4,q+1}(\cdot))\Phi_{q+1}(t,[s_{[i-q,i]}^{\mathrm{T}},s_{i+1,p}]^{\mathrm{T}})\delta_{i-q,i-q-1}(0)\Bigg),
\end{aligned}\quad \text{(A9)}$$

and according to the same rule,

$$\begin{aligned}
&\int_0^{+\infty}e^{-s_{i+1,p}(t-\tau)}l_i\delta_{i,i-1}(\tau)\,\mathrm{d}\tau\\
&=\int_0^{+\infty}e^{-s_{i+1,p}(t-\tau)}l_i\Bigg(\sum_{j=1}^{2}A_{20}(s_{ij},s_{i\bar{j}},k_i)\Phi_0(\tau,s_{ij})\varepsilon_{i,i-1}(0)+\sum_{j=1}^{2}A_{40}(s_{ij},s_{i\bar{j}})\Phi_0(\tau,s_{ij})\delta_{i,i-1}(0)\\
&\quad+\sum_{q=1}^{i-1}\sum_{j=1}^{2}A_{40}(s_{ij},s_{i\bar{j}})\Bigg(\sum_{j_1=1}^{2}\cdots\sum_{j_q=1}^{2}\big(A_{1q}(\cdot)+A_{2q}(\cdot)\big)\Phi_q(\tau,s_{[i-q,i]})\varepsilon_{i-q,i-q-1}(0)\\
&\quad+\sum_{j_1=1}^{2}\cdots\sum_{j_q=1}^{2}(A_{3q}(\cdot)+A_{4q}(\cdot))\Phi_q(\tau,s_{[i-q,i]})\delta_{i-q,i-q-1}(0)\Bigg)\Bigg)\,\mathrm{d}\tau\\
&=\sum_{j=1}^{2}A_{21}(s_{ij},s_{i\bar{j}},s_{i+1,p},l_i,k_i)\Phi_1(t,[s_{ij},s_{i+1,p}]^{\mathrm{T}})\varepsilon_{i,i-1}(0)\\
&\quad+\sum_{j=1}^{2}A_{41}(s_{ij},s_{i\bar{j}},s_{i+1,p},l_i)\Phi_1(t,[s_{ij},s_{i+1,p}]^{\mathrm{T}})\delta_{i,i-1}(0)\\
&\quad+\sum_{q=1}^{i-1}\sum_{j=1}^{2}\Bigg(\sum_{j_1=1}^{2}\cdots\sum_{j_q=1}^{2}\big(A_{1,q+1}(\cdot)+A_{2,q+1}(\cdot)\big)\Phi_{q+1}(\tau,[s_{[i-q,i]}^{\mathrm{T}},s_{i+1,p}]^{\mathrm{T}})\varepsilon_{i-q,i-q-1}(0)\\
&\quad+\sum_{j_1=1}^{2}\cdots\sum_{j_q=1}^{2}(A_{3,q+1}(\cdot)+A_{4,q+1}(\cdot))\Phi_{q+1}(\tau,[s_{[i-q,i]}^{\mathrm{T}},s_{i+1,p}]^{\mathrm{T}})\delta_{i-q,i-q-1}(0)\Bigg).
\end{aligned}\quad \text{(A10)}$$

Substituting (A9) and (A10) into (A8) yields

$$\begin{aligned}
\varepsilon_{i+1,i}(t)&=\Bigg(\frac{l_{i+1}-s_{i+1,1}}{s_{i+1,2}-s_{i+1,1}}e^{-s_{i+1,1}t}+\frac{l_{i+1}-s_{i+1,2}}{s_{i+1,1}-s_{i+1,2}}e^{-s_{i+1,2}t}\Bigg)\varepsilon_{i+1,i}(0)\\
&\quad+\Bigg(\frac{1}{s_{i+1,2}-s_{i+1,1}}e^{-s_{i+1,1}t}+\frac{1}{s_{i+1,1}-s_{i+1,2}}e^{-s_{i+1,2}t}\Bigg)\delta_{i+1,i}(0)\\
&\quad+\sum_{j=1}^{2}\frac{1}{s_{i+1,\bar{j}}-s_{i+1,j}}\Bigg(\sum_{j_1=1}^{2}A_{11}(s_{ij_1},s_{i\bar{j}_1},s_{i+1,j},l_i,k_i)\Phi_1(t,[s_{ij_1},s_{i+1,j}]^{\mathrm{T}})\\
&\quad+\sum_{j_1=1}^{2}A_{21}(s_{ij_1},s_{i\bar{j}_1},s_{i+1,j},l_i,k_i)\Phi_1(t,[s_{ij_1},s_{i+1,j}]^{\mathrm{T}})\Bigg)\varepsilon_{i,i-1}(0)\\
&\quad+\sum_{j=1}^{2}\frac{1}{s_{i+1,\bar{j}}-s_{i+1,j}}\Bigg(\sum_{j_1=1}^{2}A_{31}(s_{ij_1},s_{i\bar{j}_1},s_{i+1,j},k_i)\Phi_1(t,[s_{ij_1},s_{i+1,j}]^{\mathrm{T}})
\end{aligned}$$

$$+\sum_{j_1=1}^{2} A_{41}(s_{ij_1}, s_{i\bar{j}_1}, s_{i+1,j}, l_i)\Phi_1(t, [s_{ij_1}, s_{i+1,j}]^{\mathrm{T}})\Big)\delta_{i,i-1}(0)$$
$$+\sum_{j=1}^{2}\frac{1}{s_{i+1,\bar{j}}-s_{i+1,j}}\sum_{q=1}^{i-1}\Big(\sum_{j_1=1}^{2}\cdots\sum_{j_{q+1}=1}^{2}(A_{1,q+1}(\cdot)+A_{2,q+1}(\cdot))\Phi_{q+1}(t,[s_{[i-q,i]},s_{i+1,j}]^{\mathrm{T}})\Big)\varepsilon_{i-q,i-q-1}(0)$$
$$+\sum_{j=1}^{2}\frac{1}{s_{i+1,\bar{j}}-s_{i+1,j}}\sum_{q=1}^{i-1}\Big(\sum_{j_1=1}^{2}\cdots\sum_{j_{q+1}=1}^{2}(A_{3,q+1}(\cdot)+A_{4,q+1}(\cdot))\Phi_{q+1}(t,[s_{[i-q,i]},s_{i+1,j}]^{\mathrm{T}})\Big)\delta_{i-q,i-q-1}(0),$$

which implies that

$$\begin{aligned}\varepsilon_{i+1,i}(t) \;=\; &\sum_{j=1}^{2} A_{10}(s_{i+1,j}, s_{i+1,\bar{j}}, l_{i+1})\Phi_0(t, s_{i+1,j})\varepsilon_{i+1,i}(0)\\ &+\sum_{j=1}^{2} A_{30}(s_{i+1,j}, s_{i+1,\bar{j}})\Phi_0(t, s_{i+1,j})\delta_{i+1,i}(0)\\ &+\sum_{q=1}^{i}\sum_{j=1}^{2}\sum_{j_1=1}^{2}\cdots\sum_{j_q=1}^{2} A_{30}(s_{i+1,j}, s_{i+1,\bar{j}})\Big((A_{1q}(\cdot)+A_{2q}(\cdot))\Phi_q(t, s_{[i-q+1,i+1]})\varepsilon_{i-q+1,i-q}(0)\\ &+(A_{3q}(\cdot)+A_{4q}(\cdot))\Phi_q(t, s_{[i-q+1,i+1]})\delta_{i-q+1,i-q}(0)\Big).\end{aligned}$$

This means that (A4) holds for $i = w + 1$. Similarly, it can be concluded that (A5) also holds for $i = w + 1$.

Consequently, by induction, (A4) and (A5) hold for any $i = 1, \ldots, N$. $\Diamond$

From the foregoing proof, we know that the explicit formulas of $\varepsilon_{i,i-1}$ and $\delta_{i,i-1}$, represented respectively by (A4) and (A5), are applicable to both the cases of complex and real numbers s_{pj}'s, which means that the explicit formula of $J_u(e(0)$ in (5) is the same for complex and real numbers s_{pj}'s. Therefore, without generality, by Lemma 1 and Lemma 2, we can recursively derive the explicit formula of $J_u(e(0))$ in the scenario that s_{pj}'s are positive real numbers.

Substituting (A4) and (A5) into (5), by a series of tedious but uncomplicated calculations, we can conclude that

$$\begin{aligned}J_u(e(0)) \;=\; \sum_{i=1}^{N}\Big(&\sum_{p=1}^{i}\sum_{q=p}^{i} E_{i1}^{pq}(k_{[p,i]}, l_{[p,i]})\varepsilon_{p,p-1}(0)\varepsilon_{q,q-1}(0)\\ &+\sum_{p=1}^{i}\sum_{q=p}^{i} E_{i2}^{pq}(k_{[p,i]}, l_{[p,i]})\delta_{p,p-1}(0)\delta_{q,q-1}(0)\\ &+\sum_{p=1}^{i}\sum_{q=p}^{i} E_{i3}^{pq}(k_{[p,i]}, l_{[p,i]})\varepsilon_{p,p-1}(0)\delta_{q,q-1}(0)\Big),\end{aligned}$$

with $E_{ij}^{pq}(\cdot) = (a_{i,i-1} + r_i k_i^2)E_{ij_1}^{pq}(\cdot) + (w_{i,i-1} + r_i l_i^2)E_{ij_2}^{pq}(\cdot) + 2r_i k_i l_i E_{ij_3}^{pq}(\cdot)$. In the following, define $\bar{\kappa} = \frac{2}{\kappa}$ and $\bar{w} = \frac{2}{w}$ for convenience. Here the formulas of $E_{ij_h}^{pq}(\cdot)$ $(j, h = 1, 2, 3)$ are represented as follows:

$$\begin{cases} E_{ij_h}^{ii}(\cdot) = \sum_{\kappa=1}^{2}\sum_{w=1}^{2} A_{2(j-1)+h,0}(s_{i\kappa}, s_{i\bar{\kappa}}, \cdot)A_{2(j-1)+h,0}(s_{iw}, s_{i\bar{w}}, \cdot)B_{00}(s_{i\kappa}, s_{iw}),\\ E_{ij_h}^{ip}(\cdot) = \sum_{\kappa=1}^{2}\sum_{w=1}^{2} A_{2(j-1)+h,0}(s_{i\kappa}, s_{i\bar{\kappa}}, \cdot)A_{j+2,0}(s_{iw}, s_{i\bar{w}})\\ \qquad\cdot\sum_{w_1=1}^{2}\cdots\sum_{w_{i-p}=1}^{2}(A_{2h-1,i-p}(\cdot) + A_{2h,i-p}(\cdot))B_{0,i-p}(s_{i\kappa}, s_{[p,i]}),\\ E_{ij_h}^{pq}(\cdot) = \sum_{\kappa=1}^{2} A_{j+2,0}(s_{i\kappa}, s_{i\bar{\kappa}})\sum_{\kappa_1=1}^{2}\cdots\sum_{\kappa_{i-p}=1}^{2}(A_{2h-1,i-p}(\cdot) + A_{2h,i-p}(\cdot))\\ \qquad\cdot\sum_{w=1}^{2} A_{j+2,0}(s_{iw}, s_{i\bar{w}})\sum_{w_1=1}^{2}\cdots\sum_{w_{i-q}=1}^{2}(A_{2h-1,i-q}(\cdot) + A_{2h,i-q}(\cdot))B_{i-p,i-q}(s_{[p,i]}, s_{[q,i]}),\\ \quad i = 1, \ldots, N,\ p = 1, \ldots, i-1,\ q = p, \ldots, i-1,\ j = 1, 2,\ h = 1, 2.\end{cases}$$

Additionally, we have

$$\begin{cases} E_{ij_3}^{ii}(\cdot) = \sum_{\kappa=1}^{2}\sum_{w=1}^{2} A_{2j-1,0}(s_{i\kappa}, s_{i\bar{\kappa}}, \cdot)A_{2j,0}(s_{iw}, s_{i\bar{w}}, \cdot)B_{00}(s_{i\kappa}, s_{iw}),\\ E_{ij_3}^{ip}(\cdot) = \sum_{\kappa=1}^{2}\sum_{w=1}^{2} A_{2j-1,0}(s_{i\kappa}, s_{i\bar{\kappa}}, \cdot)A_{40}(s_{iw}, s_{i\bar{w}})\sum_{w_1=1}^{2}\cdots\sum_{w_{i-p}=1}^{2}(A_{2j-1,i-p}(\cdot) + A_{2j,i-p}(\cdot))\\ \qquad\cdot B_{0,i-p}(s_{i\kappa}, s_{[p,i]}) + \sum_{\kappa=1}^{2}\sum_{w=1}^{2} A_{2j,0}(s_{i\kappa}, s_{i\bar{\kappa}}, \cdot)A_{30}(s_{iw}, s_{i\bar{w}})\\ \qquad\cdot\sum_{w_1=1}^{2}\cdots\sum_{w_{i-p}=1}^{2}(A_{2j-1,i-p}(\cdot) + A_{2j,i-p}(\cdot))B_{0,i-p}(s_{i\kappa}, s_{[p,i]}),\\ E_{ij_3}^{pq}(\cdot) = \sum_{\kappa=1}^{2} A_{30}(s_{i\kappa}, s_{i\bar{\kappa}})\sum_{\kappa_1=1}^{2}\cdots\sum_{\kappa_{i-p}=1}^{2}(A_{2j-1,i-p}(\cdot) + A_{2j,i-p}(\cdot))\\ \qquad\cdot\sum_{w=1}^{2}\sum_{w_1=1}^{2}\cdots\sum_{w_{i-q}=1}^{2} A_{40}(s_{iw}, s_{i\bar{w}})(A_{2j-1,i-q}(\cdot) + A_{2j,i-q}(\cdot))B_{i-p,i-q}(s_{[p,i]}, s_{[q,i]}),\\ \quad i = 1, \ldots, N,\ p = 1, \ldots, i-1,\ q = p, \ldots, i-1,\ j = 1, 2,\end{cases}$$

$$\begin{cases} E^{ii}_{i3_h}(\cdot) = \sum_{\kappa=1}^{2}\sum_{w=1}^{2} A_{h0}(s_{i\kappa}, s_{i\bar{\kappa}}, \cdot) A_{h+2,0}(s_{iw}, s_{i\bar{w}}) B_{00}(s_{i\kappa}, s_{iw}), \\ E^{ip}_{i3_h}(\cdot) = \sum_{\kappa=1}^{2}\sum_{w=1}^{2} A_{h0}(s_{i\kappa}, s_{i\bar{\kappa}}, \cdot) A_{h+2,0}(s_{iw}, s_{i\bar{w}}) \\ \qquad \cdot \sum_{w_1=1}^{2} \cdots \sum_{w_{i-p}=1}^{2} (A_{3,i-p}(\cdot) + A_{4,i-p}(\cdot)) B_{0,i-p}(s_{i\kappa}, s_{[p,i]}), \\ E^{pi}_{i3_h}(\cdot) = \sum_{\kappa=1}^{2}\sum_{w=1}^{2} A_{h+2,0}(s_{i\kappa}, s_{i\bar{\kappa}}) A_{h+2,0}(s_{iw}, s_{i\bar{w}}) \\ \qquad \cdot \sum_{w_1=1}^{2} \cdots \sum_{w_{i-p}=1}^{2} (A_{1,i-p}(\cdot) + A_{2,i-p}(\cdot)) B_{0,i-p}(s_{i\kappa}, s_{[p,i]}), \\ E^{pq}_{i3_h}(\cdot) = \sum_{\kappa=1}^{2} A_{h+2,0}(s_{i\kappa}, s_{i\bar{\kappa}}) \sum_{\kappa_1=1}^{2} \cdots \sum_{\kappa_{i-p}=1}^{2} (A_{1,i-p}(\cdot) + A_{2,i-p}(\cdot)) \\ \qquad \cdot \sum_{w=1}^{2} A_{h+2,0}(s_{iw}, s_{i\bar{w}}) \sum_{w_1=1}^{2} \cdots \sum_{w_{i-q}=1}^{2} (A_{3,i-q}(\cdot) + A_{4,i-q}(\cdot)) B_{i-p,i-q}(s_{[p,i]}, s_{[q,i]}), \\ \quad i = 1, \ldots, N,\ p = 1, \ldots, i-1,\ q = p, \ldots, i-1,\ h = 1, 2, \end{cases}$$

and

$$\begin{cases} E^{ii}_{i3_3}(\cdot) = \sum_{\kappa=1}^{2}\sum_{w=1}^{2} A_{10}(s_{i\kappa}, s_{i\bar{\kappa}}, l_i) A_{40}(s_{iw}, s_{i\bar{w}}) B_{00}(s_{i\kappa}, s_{iw}) \\ \qquad + \sum_{\kappa=1}^{2}\sum_{w=1}^{2} A_{30}(s_{i\kappa}, s_{i\bar{\kappa}}) A_{20}(s_{iw}, s_{i\bar{w}}, k_i) B_{00}(s_{i\kappa}, s_{iw}), \\ E^{ip}_{i3_3}(\cdot) = \sum_{\kappa=1}^{2}\sum_{w=1}^{2} A_{10}(s_{i\kappa}, s_{i\bar{\kappa}}, l_i) A_{40}(s_{iw}, s_{i\bar{w}}) \sum_{w_1=1}^{2} \cdots \sum_{w_{i-p}=1}^{2} \big(A_{3,i-p}(\cdot) + A_{4,i-p}(\cdot)\big) \\ \qquad \cdot B_{0,i-p}(s_{i\kappa}, s_{[p,i]}) + \sum_{\kappa=1}^{2}\sum_{w=1}^{2} A_{20}(s_{i\kappa}, s_{i\bar{\kappa}}, k_i) A_{30}(s_{iw}, s_{i\bar{w}}) \\ \qquad \cdot \sum_{w_1=1}^{2} \cdots \sum_{w_{i-p}=1}^{2} \big(A_{3,i-p}(\cdot) + A_{4,i-p}(\cdot)\big) B_{0,i-p}(s_{i\kappa}, s_{[p,i]}), \\ E^{pi}_{i3_3}(\cdot) = 2\sum_{\kappa=1}^{2}\sum_{w=1}^{2} A_{30}(s_{i\kappa}, s_{i\bar{\kappa}}) A_{40}(s_{iw}, s_{i\bar{w}}) \\ \qquad \cdot \sum_{w_1=1}^{2} \cdots \sum_{w_{i-p}=1}^{2} \big(A_{1,i-q}(\cdot) + A_{2,i-p}(\cdot)\big) B_{0,i-p}(s_{i\kappa}, s_{[p,i]}), \\ E^{pq}_{i3_3}(\cdot) = \sum_{\kappa=1}^{2}\sum_{\kappa_1=1}^{2} \cdots \sum_{\kappa_{i-p}=1}^{2} A_{30}(s_{i\kappa}, s_{i\bar{\kappa}}) \big(A_{1,i-p}(\cdot) + A_{2,i-p}(\cdot)\big) \\ \qquad \cdot \sum_{w=1}^{2}\sum_{w_1=1}^{2} \cdots \sum_{w_{i-q}=1}^{2} A_{40}(s_{iw}, s_{i\bar{w}}) \big(A_{3,i-q}(\cdot) + A_{4,i-q}(\cdot)\big) B_{i-p,i-q}(s_{[p,i]}, s_{[q,i]}), \\ \quad i = 1, \ldots, N,\ p = 1, \ldots, i-1,\ q = p, \ldots, i-1. \end{cases}$$

Remark 2. It is worth pointing out that the idea/approach can be extended to the more general digraph, such as the case with directed tree. Actually, in the digraph of directed tree, the communication topology from the leader to any follower is a digraph with single chain. So according to Lemma 2, we can obtain the explicit dynamics of the consensus error of agents in each chain, and then achieve the parameterization of cost functional, which are the key and crucial procedures of our method. However, the digraph of directed tree is somewhat complex and may contain lots of chains, which requires a highly systematic and delicate strategy to represent the explicit dynamics of the consensus error of all agents. This will be deeply investigated in the future.

Appendix B Particle Swarm Optimization and A Simulation Example

Algorithm B1 Find optimal gain parameters by particle swarm optimization

.

Implement the parameterization of cost functional by (6).
Initialize the position $K_m = \{k_i, l_i | i = 1, \ldots, n\}$ and the velocity ν_m of particle m, $m = 1, \ldots, M$.
Calculate the cost functional $J_u(e(0))$ at K_m, denoted by $J(K_m)$, for particle m and set the local optimal gain parameters $pbest_m = K_m$.
Obtain the globally optimal gain parameters $gbest\_new = \min \{pbest_m\}$. Set the initial values of $gbest\_old$. Give a small threshold ϵ.
while $\|gbest\_new - gbest\_old\| > \epsilon$
    for $m = 1$ to M do
        $\nu_m = \omega\nu_m + c_1 rand()(pbest_m - K_m)$
            $+ c_2 rand()(gbest\_new - K_m)$.
        $K_m = K_m + \nu_m$.
        if $J(K_m) < J(pbest_m)$ **then**
            $pbest_m = K_m$.
        end if
        if $J(pbest_m) < J(gbest\_new)$ **then**
            $gbest\_old = gbest\_new$.
            $gbest\_new = pbest_m$.
        end if
    end for
end while
Use $gbest\_new$ as the final gain parameters.

Here $rand()$ is a random function on $[0, 1]$, and c_1 and c_2 are two positive constants.

We consider second-order leader-following multi-agent systems (1) with two followers to show the effectiveness of the above theoretical results. In the following, let $a_{10} = 2$, $a_{21} = 0.9$, $w_{10} = 1.6$, $w_{21} = 2.5$, $r_1 = 1$, $r_2 = 2$. The leader's initial values are set to $[x_0(0), v_0(0)]^{\mathrm{T}} = [2, 1]^{\mathrm{T}}$; the initial states and velocities of two followers are chosen as $[x_1(0), x_2(0), v_1(0), v_2(0)]^{\mathrm{T}} = [3.1, 6.1, 1, 2.7]^{\mathrm{T}}$.

According to the design procedure, we first obtain the explicit formula of the cost function when $N = 2$. The next step is to use Algorithm B1 to find optimal gain parameters. During this process, the number of particles are $M = 25$, the initial position K_m and velocity ν_m are randomly set to different values, and ϵ is set to 0.0001. Under these conditions, the obtained $J_{min} = 50.3593$ and corresponding optimal gain parameters are $[k_1, l_1, k_2, l_2] = [0.25, 0.65, 0.71, 1.72]$. By MATLAB, Fig. 1 is presented to exhibit the changing of the cost functional J with respect to any one of gain parameters when the other gain parameters take the optimal values, by which the effectiveness of the theoretical results is further illustrated. Moreover, Fig. 2-3 are obtained to demonstrate the trajectories of all signals of the closed-loop systems.

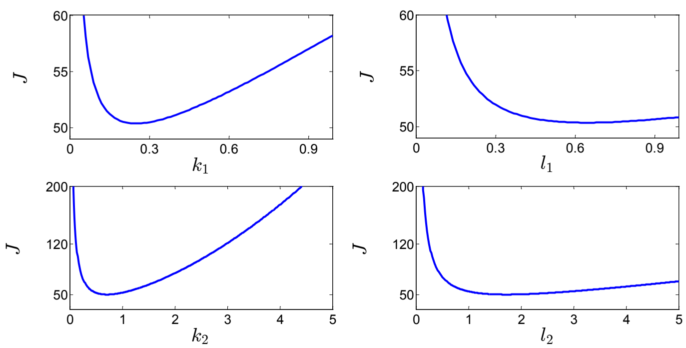

Fig. 1. Trajectory of cost functional J.

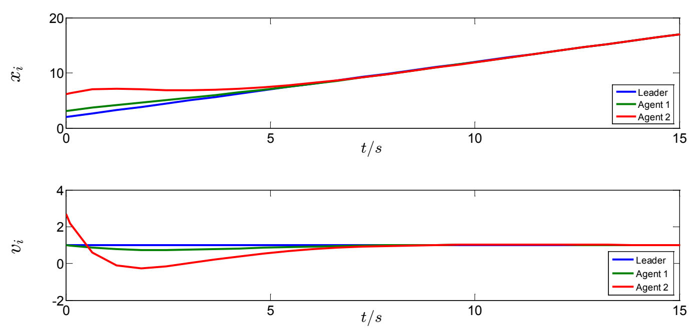

Fig. 2. Trajectories of x_0, x_i and v_i, $i = 1, 2$.

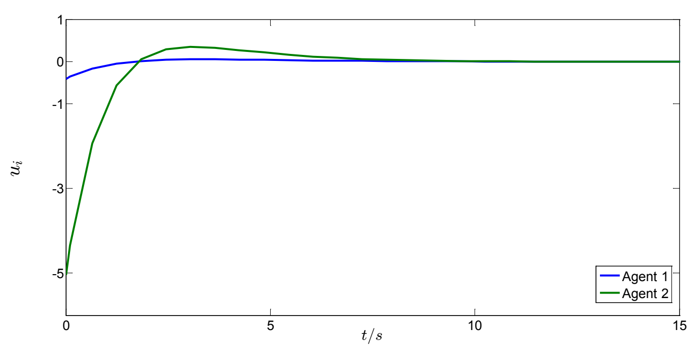

Fig. 3. Trajectories of u_1 and u_2.