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Special Focus on Control and Analysis for Stochastic Systems

Adaptive state-feedback stabilization of state-constrained stochastic high-order nonlinear systems

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Abstract This paper presents an adaptive state-feedback strategy for state-constrained stochastic high-order nonlinear systems. By adding a power integrator and adaptive backstepping techniques, a new adaptive controller is constructed without imposing feasibility conditions, which guarantees that all closed-loop signals are bounded almost surely, full-state constraints are not violated almost surely, and the trivial solution of the closed-loop system is stochastically asymptotically stable.

Keywords stochastic high-order nonlinear systems, full-state constraints, feasibility conditions, state-feedback stabilization, adaptive control

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1 Introduction

Because stochastic noise and nonlinearity widely exist in various systems, increasing attention has been paid to controller design and stability analysis of stochastic nonlinear systems (see [1–14] and other references).

Recently, stochastic high-order nonlinear systems have been studied extensively. Many actual systems can be modeled using such a system structure, such as benchmark mechanical systems [15] and single-link manipulators [16]. Notably, the Jacobian linearization of stochastic high-order nonlinear systems is neither controllable nor feedback linearized, which leads to more difficulties in the control design and analysis of such systems. Fortunately, by adding a power integrator and backstepping techniques, these inherent difficulties can be successfully handled. The adaptive control problem of stochastic high-order systems was first addressed in [17]. Subsequently, the conditions of [17] were relaxed in [18]. Recently, Ref. [19] studied finite-time stabilization of stochastic high-order nonlinear systems with stochastic inverse dynamics, and Ref. [20] considered the finite-time control of stochastic nonlinear systems with unknown time-varying powers. However, these studies did not consider state/output constraints.

Because of hardware limitations, performance requirements, or safety specifications, state/output constraints are always involved in many nonlinear systems. In actual operation, violation of state/output constraints may decrease system performance or even make the system unstable, which motivates this study. In the past decade, the barrier Lyapunov function (BLF) proposed by [21,22] has made a significant breakthrough in dealing with state/output constraints, and the BLF-based method has been gradually generalized to stochastic systems. Adaptive stabilization and tracking of full-state constrained stochastic nonlinear systems were studied in [23, 24]. Ref. [25] solved the adaptive event-triggered control problem of full-state constrained stochastic nonlinear systems with actuator faults. However, these results have two limitations: they are only suitable for feedback-linearizable nonlinear systems (i.e., the powers of the system are equal to one) and state-constrained controllers depend heavily on feasibility conditions.

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As implied in [21, 22], the offline verification of feasibility conditions may be extremely complicated or even unsolvable. Therefore, developing a non-BLF-based method to remove such conditions is necessary. Ref. [26] integrated nonlinear mappings into adaptive neural network control design to remove feasibility conditions for stochastic nonlinear systems with time-delays, and the finite-time control problem for stochastic high-order nonlinear systems with asymmetric output constraints was solved in [27]. However, they did not consider full-state constraints of stochastic high-order nonlinear systems.

Based on these discussion, a challenging question arises: Without imposing feasibility conditions, how to design a state feedback controller for state-constrained stochastic high-order nonlinear systems such that asymmetric full-state constraints are not violated almost surely and the trivial solution of the closed-loop system is asymptotically stable in probability? In this paper, we will solve this problem comprehensively. The main contributions and difficulties of this paper lie in three aspects.

(i) This paper is not a parallel generalization from output to full-state constraints, and the controllers in [27] cannot be applied to prevent the violation of asymmetric full-state constraints. Accordingly, a new method for controller design and analysis of state-constrained stochastic high-order nonlinear systems needs to be developed.

(ii) Although the main idea of removing feasibility conditions originates from [28, 29], extending the idea of removing feasibility conditions to the stochastic high-order nonlinear system is extremely difficult.

(iii) Multiple uncertainties, including a serious unknown parameter θ, unknown constant bounds $\gamma_1, \ldots, \gamma_n$ in Remark 4, diffusion, and Hessian terms, will produce many nonlinear terms. Skillfully dealing with these terms is another piece of nontrivial work.

Notations. $\mathbb{R}_{\rm odd} = \{\frac{p}{q} | p$ and q are positive odd integers$\}$. $\mathbb{R}_{\rm even} = \{\frac{p}{q} | p$ is a positive even integer and q is a positive odd integer$\}$. $\mathcal{C}^i$ denotes the family of all nonnegative functions with continuous i-th partial derivatives. For any vector x, $|x|$ denotes its norm. For simplicity, let $f(x)$ or f denote a function $f(x(t))$.

2 Problem statement

In this paper, we study a stochastic nonlinear system:

$$\begin{aligned} {\rm d}x_i &= \big(x_{i+1}^{p_i} + f_i(\theta, \bar{x}_i)\big){\rm d}t + g_i^{\rm T}(\theta, \bar{x}_i){\rm d}\omega,\ i = 1, \ldots, n-1, \\ {\rm d}x_n &= \big(u^{p_n} + f_n(\theta, x)\big){\rm d}t + g_n^{\rm T}(\theta, x){\rm d}\omega, \end{aligned} \tag{1}$$

with asymmetric full-state constraints:

$$x(t) \in \Omega_x = \{x(t) \in \mathbb{R}^n : -\underline{h}_i < x_i(t) < \bar{h}_i,\ i = 1, \ldots, n\},\ \forall t \geqslant 0, \tag{2}$$

where $x = (x_1, \ldots, x_n)^{\rm T} \in \mathbb{R}^n$ is system state; for $i = 1, \ldots, n$, $\bar{x}_i = (x_1, \ldots, x_i)^{\rm T}$, $x = \bar{x}_n$; $p_i \geqslant 1 \in \mathbb{R}_{\rm odd}$ is called the power of system; $\theta \in \mathbb{R}^l$ is an unknown parameter vector; $u \in \mathbb{R}$ is the control input; ω is a q-dimensional standard Wiener process. $f_i : \mathbb{R}^l \times \mathbb{R}^i \to \mathbb{R}$, $g_i : \mathbb{R}^l \times \mathbb{R}^i \to \mathbb{R}^q$ are local Lipschitz functions with $f_i(0, 0) = 0$, $g_i(0, 0) = 0$. $\underline{h}_i$ and $\bar{h}_i$ are pre-specified positive constants.

Control objective of this paper is to design a state feedback controller for system (1) such that all the closed-loop signals are bounded almost surely, asymmetric full-state constraints are not violated almost surely, and the trivial solution is asymptotically stable in probability.

For system (1), we need the following assumption.

Assumption 1. There exist nonnegative smooth functions $f_{i1}(\bar{x}_i)$ and $g_{i1}(\bar{x}_i)$, $i = 1, \ldots, n$, a constant $w \in \mathbb{R}_{\rm even} \geqslant d$ with $d = \max_{2 \leqslant i \leqslant n}\{1, \frac{\frac{1}{p_1 \cdots p_n}}{1 - \sum_{j=1}^{i} \frac{1}{p_j \cdots p_n}}\}$ and an unknown constant $\theta > 0$ such that

$$|f_i| \leqslant \theta f_{i1}(\bar{x}_i) \sum_{j=1}^{i} |x_j|^{\frac{r_i + w}{r_j}}, \quad |g_i| \leqslant \theta g_{i1}(\bar{x}_i) \sum_{j=1}^{i} |x_j|^{\frac{2r_i + w}{2r_j}}, \tag{3}$$

where $r_1 = 1$, $r_{i+1} = \frac{r_i + w}{p_i}$. Let $r = \max_{1 \leqslant i \leqslant n}\{r_i\}$, and then one of the conditions should be satisfied:

(i) if $\frac{r}{r_i} = 1$ or $\frac{r}{r_i} \geqslant 2$, then $r_n + w \geqslant r_i$, $i = 1, \ldots, n$;

(ii) otherwise, $r_n + w \geqslant 2r_i$.

Remark 1. Conditions (i) or (ii) of Assumption 1 will be used to guarantee the local Lipschitz condition of the closed-loop stochastic system.

3 Main results

3.1 System transformation

Construct the nonlinear transformation:

$$\xi_i = H_i(x_i) = \frac{x_i}{h_i(x_i)}, \quad h_i(x_i) = (\underline{h}_i + x_i)(\bar{h}_i - x_i), \quad i = 1, \ldots, n. \tag{4}$$

For $i = 1, \ldots, n$, $\frac{\mathrm{d}\xi_i}{\mathrm{d}x_i} = \frac{\underline{h}_i \bar{h}_i + x_i^2}{h_i^2} > 0$ means that $\xi_i(t)$ is strictly increasing with respect to the constrained state $x_i(t)$. By the continuity of $x_i(t)$, it is easy to know that $\xi_i(t)$ is smooth with respect to the constrained state $x_i(t)$. Besides, $\xi_i \to \infty$ when $x_i \to \underline{h}_i$ or $x_i \to \bar{h}_i$. Hence, for any initial state $x(0) \in \Omega_x$, as long as ξ_i, $i = 1, \ldots, n$, are bounded almost surely, full-state constraints are not violated almost surely.

By (1) and (4), we obtain the unconstrained system:

$$\begin{aligned} \mathrm{d}\xi_i &= K_{\xi_i}\big(x_{i+1}^{p_i}\mathrm{d}t + f_i(\theta, \bar{x}_i)\mathrm{d}t + g_i^{\mathrm{T}}(\theta, \bar{x}_i)\mathrm{d}\omega\big), \quad i = 1, \ldots, n-1, \\ \mathrm{d}\xi_n &= K_{\xi_n}\big(u^{p_n}\mathrm{d}t + f_n(\theta, x)\mathrm{d}t + g_n^{\mathrm{T}}(\theta, x)\mathrm{d}\omega\big), \end{aligned} \tag{5}$$

where $K_{\xi_i} = \frac{\underline{h}_i \bar{h}_i + x_i^2}{h_i^2}$.

Remark 2. As mentioned in [30], the validness of (4) can be guaranteed if the zero divisions must be events of probability zero. This fact can be established by proving almost sure boundedness of ξ_i, $i = 1, \ldots, n$, i.e., $\underline{h}_i < x_i < \bar{h}_i$, $i = 1, \ldots, n$ almost sure in this paper.

Remark 3. In what follows, we first assume $x(t) \in \overline{\Omega}_{x_0} \in \Omega_x$, $\forall t \geqslant 0$, where $\overline{\Omega}_{x_0} \in \Omega_x$ is a closed set which will be given later. With this priori knowledge, the following control design and stability analysis are true and any given smooth function with respect to $x_1, \ldots, x_n$ has known lower and upper constant bounds in $\overline{\Omega}_{x_0}$. However, $x(t) \in \overline{\Omega}_{x_0}$ needs to be verified for a closed-loop system. By choosing $x(0) \in \Omega_x$ and a Lyapunov function, $x(t) \in \overline{\Omega}_{x_0}$ for any $t \geqslant 0$ can be proved in Theorem 1, and then the circular argument does not happen, where $\overline{\Omega}_{x_0}$ only depends on the initial values and the Lyapunov function.

Remark 4. In view of $x(t) \in \overline{\Omega}_{x_0} \in \Omega_x$, $\forall t \geqslant 0$, we derive $0 < b_i \leqslant h_i(x_i(t)) \leqslant a_i < \infty$, $\forall t \geqslant 0$, where a_i and b_i, $i = 1, \ldots, n$, are constants, and then $K_{\xi_i}(t) \geqslant \frac{\underline{h}_i \bar{h}_i}{a_i^2}$, $i = 1, \ldots, n$. Define $K_i(x_i, x_{i+1}) = K_{\xi_i} h_{i+1}^{p_i}$, $i = 1, \ldots, n-1$, and $K_n(x_n) = K_{\xi_n}$. From the continuity of K_i, it follows that $0 < \gamma_i \leqslant K_i(t)$, $\forall t \geqslant 0$, $i = 1, \ldots, n$, where γ_i, $i = 1, \ldots, n$, are positive constants. The introduction of γ_i is to compensate the influence of $p_i \geqslant 1$.

3.2 Controller design

To begin with, we specify a constant λ by the following way:

(I) If condition (i) of Assumption 1 holds, then $\max_{1\leqslant i\leqslant n}\{r_i\} = \lambda$;

(II) If condition (ii) of Assumption 1 holds, we can select $\varsigma \geqslant 1$ and $\lambda \in \mathbb{R}_{\mathrm{odd}}$ satisfying $\max_{1\leqslant i\leqslant n}\{\frac{r_i + w}{\varsigma}, 2r_i\} \leqslant \lambda \leqslant r_n + w$.

Step 1. The first Lyapunov function is chosen as $V_1 = \frac{r_1}{4\varsigma\lambda - w} z_1^{\frac{4\varsigma\lambda - w}{\lambda}} + \frac{1}{2}\widetilde{\Theta}^2(t)$, where $\xi_1^{\frac{\lambda}{r_1}} = z_1$, $\widetilde{\Theta} = \Theta - \widehat{\Theta}(t)$ is the estimation error and $\Theta = \max_{1\leqslant i\leqslant n}\{1, \theta, \theta^2, \theta^{\frac{4\varsigma\lambda}{w + r_i}}, \theta^{\frac{8\varsigma\lambda}{w + 2r_i}}\}$. In view of (5) and Itô's differentiation rule, one has

$$\begin{aligned} \mathcal{L}V_1 \leqslant{}& z_1^{\frac{4\varsigma\lambda - w - r_1}{\lambda}} K_1\big(\xi_2^{p_1} - \alpha_2^{p_1}\big) + z_1^{\frac{4\varsigma\lambda - w - r_1}{\lambda}} K_1 \alpha_2^{p_1} + z_1^{\frac{4\varsigma\lambda - w - r_1}{\lambda}} K_{\xi_1} f_1 - \widetilde{\Theta}\dot{\widehat{\Theta}} \\ &+ \frac{K_{\xi_1}^2 (4\varsigma\lambda - w - r_1)}{2r_1} z_1^{\frac{4\varsigma\lambda - w - 2r_1}{\lambda}} g_1^{\mathrm{T}} g_1. \end{aligned} \tag{6}$$

From Assumption 1, it follows that

$$z_1^{\frac{4\varsigma\lambda - w - r_1}{\lambda}} K_{\xi_1} f_1 \leqslant \theta K_{\xi_1} |z_1|^{\frac{4\varsigma\lambda - w - r_1}{\lambda}} f_{11} h_1^{\frac{w + r_1}{r_1}} |z_1|^{\frac{w + r_1}{\lambda}} \leqslant \Theta \beta_{1,1}(\xi_1) |z_1|^{4\varsigma}, \tag{7}$$

where $\beta_{1,1}(\xi_1)$ is a nonnegative smooth function. Similarly,

$$\frac{K_{\xi_1}^2 (4\varsigma\lambda - w - r_1)}{2r_1} z_1^{\frac{4\varsigma\lambda - w - 2r_1}{\lambda}} g_1^{\mathrm{T}} g_1 \leqslant \theta^2 \overline{g}_{11}(\xi_1) |z_1|^{\frac{4\varsigma\lambda - w - 2r_1}{\lambda}} |z_1|^{\frac{w + 2r_1}{\lambda}} \leqslant \Theta \beta_{1,2}(\xi_1) |z_1|^{4\varsigma}, \tag{8}$$

where $\overline{g}_{11}$ and $\beta_{1,2}(\xi_1)$ are nonnegative smooth functions. From (7) and (8), Eq. (6) is rewritten as

$$\mathcal{L}V_1 \leqslant z_1^{\frac{4\varsigma\lambda-w-r_1}{\lambda}} K_1\big(\xi_2^{p_1}-\alpha_2^{p_1}\big) + z_1^{\frac{4\varsigma\lambda-w-r_1}{\lambda}} K_1\alpha_2^{p_1} + \Theta\beta_1(\xi_1)|z_1|^{4\varsigma} - \widetilde{\Theta}\dot{\widehat{\Theta}}, \tag{9}$$

where $\beta_1(\xi_1)=\beta_{1,1}(\xi_1)+\beta_{1,2}(\xi_1)$. Note that $0<\gamma_1\leqslant K_1$ in Remark 4, the virtual controller is chosen as

$$\alpha_2^{p_1} = -z_1^{\frac{r_2p_1}{\lambda}}\left(\frac{1}{\gamma_1}\big(n+\widehat{\Theta}\beta_1(\xi_1)\big)\right) \triangleq -z_1^{\frac{r_2p_1}{\lambda}}\varphi_1(\widehat{\Theta},\xi_1)^{\frac{r_2p_1}{\lambda}}. \tag{10}$$

Substituting (10) into (9) yields

$$\mathcal{L}V_1 \leqslant z_1^{\frac{4\varsigma\lambda-w-r_1}{\lambda}} K_1\big(\xi_2^{p_1}-\alpha_2^{p_1}\big) - nz_1^{4\varsigma} + \left(\widetilde{\Theta}+\nu_1\right)\left(\beta_1(\xi_1)|z_1|^{4\varsigma} - \dot{\widehat{\Theta}}\right), \tag{11}$$

where $\nu_1=0$.

Step i ($i=2,\ldots,n$). Suppose that Steps $2,\ldots,i-1$ have been completed and there is a set of virtual controllers:

$$\begin{aligned} &\alpha_1 = 0,\\ &\alpha_2 = -z_1^{\frac{r_2}{\lambda}}\varphi_1(\widehat{\Theta},\xi_1)^{\frac{r_2}{\lambda}},\\ &\ \vdots\\ &\alpha_i = -z_{i-1}^{\frac{r_i}{\lambda}}\varphi_{i-1}(\widehat{\Theta},\bar{\xi}_{i-1})^{\frac{r_i}{\lambda}}, \end{aligned} \tag{12}$$

such that

$$\begin{aligned} \mathcal{L}V_{i-1} \leqslant{}& z_{i-1}^{\frac{4\varsigma\lambda-w-r_{i-1}}{\lambda}} K_{i-1}\big(\xi_i^{p_{i-1}}-\alpha_i^{p_{i-1}}\big) + \left(\widetilde{\Theta}+\nu_{i-1}\right)\left(\sum_{j=1}^{i-1}\beta_j(\bar{\xi}_{j-1})|z_j|^{4\varsigma} - \dot{\widehat{\Theta}}\right)\\ &-(n+2-i)\sum_{j=1}^{i-1} z_j^{4\varsigma}, \end{aligned} \tag{13}$$

where $\nu_{i-1}=-\sum_{j=2}^{i-1}\frac{\partial W_j}{\partial\widehat{\Theta}}$. In what follows, we prove that Eq. (13) still holds at Step i.

The ith Lyapunov function is chosen as $V_i(\bar{\xi}_i)=V_{i-1}+W_i(\bar{\xi}_i)$, where

$$z_i = \xi_i^{\frac{\lambda}{r_i}} - \alpha_i^{\frac{\lambda}{r_i}}, \tag{14}$$

$$W_i = \int_{\alpha_i}^{\xi_i}\left(s^{\frac{\lambda}{r_i}} - \alpha_i^{\frac{\lambda}{r_i}}\right)^{\frac{4\varsigma\lambda-w-r_i}{\lambda}} \mathrm{d}s. \tag{15}$$

By (5), (13), (15) and Itô's formula, one has

$$\begin{aligned} \mathcal{L}V_i \leqslant{}& z_{i-1}^{\frac{4\varsigma\lambda-w-r_{i-1}}{\lambda}} K_{i-1}\big(\xi_i^{p_{i-1}}-\alpha_i^{p_{i-1}}\big) + \left(\widetilde{\Theta}+\nu_{i-1}\right)\left(\sum_{j=1}^{i-1}\beta_j(\bar{\xi}_{j-1})|z_j|^{4\varsigma} - \dot{\widehat{\Theta}}\right)\\ &+z_i^{\frac{4\varsigma\lambda-w-r_i}{\lambda}} K_i\big(\xi_{i+1}^{p_i}-\alpha_{i+1}^{p_i}\big) + z_i^{\frac{4\varsigma\lambda-w-r_i}{\lambda}} K_i\alpha_{i+1}^{p_i} + \sum_{j=1}^{i-1}\frac{\partial W_i}{\partial\xi_j}K_{\xi_j}\left(x_{j+1}^{p_j}+f_j\right)\\ &+\frac{\partial W_i}{\partial\widehat{\Theta}}\dot{\widehat{\Theta}} - (n+2-i)\sum_{j=1}^{i-1}z_j^{4\varsigma} + \frac{K_{\xi_{j_1}}K_{\xi_{j_2}}}{2}\sum_{j_1,j_2=1}^{i}\frac{\partial^2 W_i}{\partial\xi_{j_1}\partial\xi_{j_2}}g_{j_1}^{\mathrm{T}}g_{j_2}. \end{aligned} \tag{16}$$

For $i=2,\ldots,n$, a direct computation leads to

$$\frac{\partial W_i}{\partial\xi_j} = -\frac{4\varsigma\lambda-w-r_i}{\lambda}\frac{\partial\alpha_i^{\frac{\lambda}{r_i}}}{\partial\xi_j}\int_{\alpha_i}^{\xi_i}\left(s^{\frac{\lambda}{r_i}}-\alpha_i^{\frac{\lambda}{r_i}}\right)^{\frac{4\varsigma\lambda-w-r_i-\lambda}{\lambda}}\mathrm{d}s, \tag{17}$$

$$\frac{\partial W_i}{\partial\widehat{\Theta}} = -\frac{4\varsigma\lambda-w-r_i}{\lambda}\frac{\partial\alpha_i^{\frac{\lambda}{r_i}}}{\partial\widehat{\Theta}}\int_{\alpha_i}^{\xi_i}\left(s^{\frac{\lambda}{r_i}}-\alpha_i^{\frac{\lambda}{r_i}}\right)^{\frac{4\varsigma\lambda-w-r_i-\lambda}{\lambda}}\mathrm{d}s, \tag{18}$$

$$\frac{\partial W_i}{\partial \xi_i} = z_i^{\frac{4\varsigma\lambda - w - r_i}{\lambda}}, \tag{19}$$

$$\begin{aligned}\frac{\partial^2 W_i}{\partial \xi_{j_1}\partial \xi_{j_2}} = & -\Big(\frac{4\varsigma\lambda - w - r_i}{\lambda}\Big)\Big(\frac{\partial^2 \alpha_i^{\frac{\lambda}{r_i}}}{\partial \xi_{j_1}\partial \xi_{j_2}}\Big)\int_{\alpha_i}^{\xi_i}\Big(s^{\frac{\lambda}{r_i}} - \alpha_i^{\frac{\lambda}{r_i}}\Big)^{\frac{4\varsigma\lambda - w - r_i - \lambda}{\lambda}}\mathrm{d}s \\ & +\Big(\frac{4\varsigma\lambda - w - r_i}{\lambda}\Big)\cdot\Big(\frac{4\varsigma\lambda - w - r_i - \lambda}{\lambda}\Big)\cdot\frac{\partial \alpha_i^{\frac{\lambda}{r_i}}}{\partial \xi_{j_1}}\cdot\frac{\partial \alpha_i^{\frac{\lambda}{r_i}}}{\partial \xi_{j_2}} \\ & \cdot\int_{\alpha_i}^{\xi_i}\Big(s^{\frac{\lambda}{r_i}} - \alpha_i^{\frac{\lambda}{r_i}}\Big)^{\frac{4\varsigma\lambda - w - r_i - 2\lambda}{\lambda}}\mathrm{d}s, \end{aligned} \tag{20}$$

$$\frac{\partial^2 W_i}{\partial \xi_i \partial \xi_j} = \left(\frac{4\varsigma\lambda - w - r_i}{\lambda}\right) z_i^{\frac{4\varsigma\lambda - w - r_i - \lambda}{\lambda}}\left(\frac{\partial z_i}{\partial \xi_j}\right), \tag{21}$$

$$\frac{\partial^2 W_i}{\partial \xi_i^2} = \left(\frac{4\varsigma\lambda - w - r_i}{\lambda}\right) z_i^{\frac{4\varsigma\lambda - w - r_i - \lambda}{\lambda}}\left(\frac{\partial z_i}{\partial \xi_i}\right), \tag{22}$$

where $j,\, j_1,\, j_2 = 1,\ldots,i-1$. Observing $\alpha_i = -z_{i-1}^{\frac{r_i}{\lambda}}\varphi_{i-1}^{\frac{r_i}{\lambda}}$ and $z_{i-1} = \xi_{i-1}^{\frac{\lambda}{r_{i-1}}} - \alpha_{i-1}^{\frac{\lambda}{r_{i-1}}}$, $i = 2,\ldots,n$, we get $\alpha_i^{\frac{\lambda}{r_i}} = -\varphi_{i-1}\xi_{i-1}^{\frac{\lambda}{r_{i-1}}} - \varphi_{i-1}\varphi_{i-2}z_{i-2} = -\varphi_{i-1}\xi_{i-1}^{\frac{\lambda}{r_{i-1}}} - \varphi_{i-1}\varphi_{i-2}\xi_{i-2}^{\frac{\lambda}{r_{i-2}}} - \cdots - \varphi_{i-1}\varphi_{i-2}\cdots\varphi_1\xi_1^{\frac{\lambda}{r_1}}$. From (14) and Lemma 2 in [18], it follows that

$$\left|\frac{\partial \alpha_i^{\frac{\lambda}{r_i}}}{\partial \xi_j}\right| \leqslant \psi_{i,1}(\bar{\xi}_{i-1}, \widehat{\Theta})\sum_{k=1}^{i-1}|z_k|^{\frac{\lambda - r_j}{\lambda}}, \tag{23}$$

$$\left|\frac{\partial^2 \alpha_i^{\frac{\lambda}{r_i}}}{\partial \xi_{j_1}\partial \xi_{j_2}}\right| \leqslant \psi_{i,2}(\bar{\xi}_{i-1}, \widehat{\Theta})\sum_{k=1}^{i-1}|z_k|^{\frac{\lambda - r_{j_1} - r_{j_2}}{\lambda}}, \tag{24}$$

where $j, j_1, j_2 = 1,\ldots,i-1$, $\psi_{i,k}(\bar{\xi}_{i-1}, \widehat{\Theta})$, $k = 1, 2$, are some nonnegative smooth functions. By using (14) and Lemmas 2–4 in [18],

$$\begin{aligned} z_{i-1}^{\frac{4\varsigma\lambda - w - r_{i-1}}{\lambda}} K_{i-1}\big(\xi_i^{p_{i-1}} - \alpha_i^{p_{i-1}}\big) & \leqslant 2^{1 - \frac{p_{i-1}r_i}{\lambda}} K_{i-1} z_{i-1}^{\frac{4\varsigma\lambda - w - r_{i-1}}{\lambda}} z_i^{\frac{w + r_{i-1}}{\lambda}} \\ & \leqslant \frac{1}{8}\sum_{j=1}^{i-1}|z_j|^{4\varsigma} + \Theta\beta_{i,1}(\bar{\xi}_i, \widehat{\Theta})|z_i|^{4\varsigma}, \end{aligned} \tag{25}$$

where $\beta_{i,1}(\bar{\xi}_i, \widehat{\Theta})$ is a smooth nonnegative function. From (14), (17), (23), Assumption 1 and Lemmas 2–4 in [18], it follows that

$$\begin{aligned} \sum_{j=1}^{i-1}\frac{\partial W_i}{\partial \xi_j}K_{\xi_j}\left(x_{j+1}^{p_j} + f_j\right) \leqslant & \sum_{j=1}^{i-1}\left|\frac{4\varsigma\lambda - w - r_i}{\lambda}\cdot\frac{\partial \alpha_i^{\frac{\lambda}{r_i}}}{\partial \xi_j}\cdot\int_{\alpha_i}^{\xi_i}\left(s^{\frac{\lambda}{r_i}} - \alpha_i^{\frac{\lambda}{r_i}}\right)^{\frac{4\varsigma\lambda - w - r_i - \lambda}{\lambda}}\mathrm{d}s\right| \\ & \cdot\bar{f}_{j1}\left(|z_j\varphi_j + z_{j+1}|^{\frac{w + r_j}{\lambda}} + \theta\sum_{k=1}^{j}|z_k|^{\frac{w + r_j}{\lambda}}\right) \\ \leqslant & \sum_{j=1}^{i-1}\left|\frac{4\varsigma\lambda - w - r_i}{\lambda}\cdot\frac{\partial \alpha_i^{\frac{\lambda}{r_i}}}{\partial \xi_j}\cdot|z_i|^{\frac{4\varsigma\lambda - w - \lambda}{\lambda}}\right| \\ & \cdot\bar{f}_{j1}\left(|z_j\varphi_j|^{\frac{w + r_j}{\lambda}} + |z_{j+1}|^{\frac{w + r_j}{\lambda}} + \theta\sum_{k=1}^{j}|z_k|^{\frac{w + r_j}{\lambda}}\right) \\ \leqslant & \frac{1}{8}\sum_{j=1}^{i-1}|z_j|^{4\varsigma} + \Theta\beta_{i,2}(\bar{\xi}_i, \widehat{\Theta})|z_i|^{4\varsigma}, \end{aligned} \tag{26}$$

where $\bar{f}_{j1}$ and $\beta_{i,2}$ are some nonnegative smooth functions. Similar to (26),

$$\frac{\partial W_i}{\partial \xi_i}K_{\xi_i}f_i \leqslant \theta\bar{f}_{i1}|z_i|^{\frac{4\varsigma\lambda - w - r_i}{\lambda}}\sum_{k=1}^{i}|z_k|^{\frac{w + r_i}{\lambda}} \leqslant \frac{1}{8}\sum_{j=1}^{i-1}|z_j|^{4\varsigma} + \Theta\beta_{i,3}(\bar{\xi}_i, \widehat{\Theta})|z_i|^{4\varsigma}, \tag{27}$$

where $\bar{f}_{i1}$ and $\beta_{i,3}$ are some nonnegative smooth functions. By (14), (21), Assumption 1 and Lemmas 2–4 in [18],

$$\begin{aligned}\sum_{j=1}^{i-1} K_{\xi_i}K_{\xi_j}\frac{\partial^2 W_i}{\partial\xi_i\partial\xi_j}g_i^{\mathrm{T}}g_j \leqslant{}& \sum_{j=1}^{i-1}K_{\xi_i}K_{\xi_j}\left|\Big(\frac{4\varsigma\lambda-w-r_i}{\lambda}\Big)|z_i|^{\frac{4\varsigma\lambda-w-r_i-\lambda}{\lambda}}\frac{\partial z_i}{\partial\xi_j}\right| \\ &\cdot\left(\theta g_{i1}\sum_{k=1}^{i}|x_k|^{\frac{w+2r_i}{2r_k}}\right)^{\mathrm{T}}\cdot\left(\theta g_{j1}\sum_{k=1}^{j}|x_k|^{\frac{w+2r_j}{2r_k}}\right) \\ \leqslant{}& \sum_{j=1}^{i-1}\theta^2\tilde{g}_{ji}\left|\left(\frac{4\varsigma\lambda-w-r_i}{\lambda}\right)|z_i|^{\frac{4\varsigma\lambda-w-r_i-\lambda}{\lambda}}\frac{\partial\alpha_i^{\frac{\lambda}{r_i}}}{\partial\xi_j}\right|\sum_{k=1}^{i}|z_k|^{\frac{w+r_j+r_i}{\lambda}} \\ \leqslant{}& \frac{1}{8}\sum_{j=1}^{i-1}|z_j|^{4\varsigma}+\Theta\beta_{i,4}(\bar{\xi}_i,\widehat{\Theta})|z_i|^{4\varsigma},\end{aligned}\tag{28}$$

where $\tilde{g}_{ji}$ and $\beta_{i,4}$ are some nonnegative smooth functions. By (14), (22), Assumption 1 and Lemmas 2–4 in [18], one gets

$$\begin{aligned}\frac{K_{\xi_i}^2}{2}\frac{\partial^2 W_i}{\partial\xi_i^2}g_i^{\mathrm{T}}g_i \leqslant{}& \frac{K_{\xi_i}^2}{2}\left|\Big(\frac{4\varsigma\lambda-w-r_i}{\lambda}\Big)|z_i|^{\frac{4\varsigma\lambda-w-r_i-\lambda}{\lambda}}\frac{\partial z_i}{\partial\xi_i}\right| \\ &\cdot\left(\theta g_{i1}\sum_{k=1}^{i}|x_k|^{\frac{w+2r_i}{2r_k}}\right)^{\mathrm{T}}\cdot\left(\theta g_{i1}\sum_{k=1}^{i}|x_k|^{\frac{w+2r_i}{2r_k}}\right) \\ \leqslant{}& \theta^2\tilde{g}_{ii}\left|\left(\frac{4\varsigma\lambda-w-r_i}{\lambda}\right)|z_i|^{\frac{4\varsigma\lambda-w-r_i-\lambda}{\lambda}}\frac{\lambda}{r_i}\xi_i^{\frac{\lambda}{r_i}-1}\right|\sum_{k=1}^{i}|z_k|^{\frac{w+2r_i}{\lambda}} \\ \leqslant{}& \frac{1}{8}\sum_{j=1}^{i-1}|z_j|^{4\varsigma}+\Theta\beta_{i,5}(\bar{\xi}_i,\widehat{\Theta})|z_i|^{4\varsigma},\end{aligned}\tag{29}$$

where $\tilde{g}_{ii}$ and $\beta_{i,5}$ are some nonnegative smooth functions. In view of (14), (20), (23), (24), Assumption 1 and Lemmas 2–4 in [18], we have

$$\begin{aligned}&\sum_{j=1}^{i-1}\frac{K_{\xi_j}^2}{2}\frac{\partial^2 W_i}{\partial\xi_j^2}g_j^{\mathrm{T}}g_j \\ &\leqslant\sum_{j=1}^{i-1}\frac{K_{\xi_j}^2}{2}\cdot\left|\int_{\alpha_i}^{\xi_i}\left(s^{\frac{\lambda}{r_i}}-\alpha_i^{\frac{\lambda}{r_i}}\right)^{\frac{4\varsigma\lambda-w-r_i-\lambda}{\lambda}}\mathrm{d}s\cdot\left(\frac{\partial^2\alpha_i^{\frac{\lambda}{r_i}}}{\partial\xi_j^2}\right)\left(\frac{4\varsigma\lambda-w-r_i}{\lambda}\right)\right. \\ &\quad+\left(\frac{4\varsigma\lambda-w-r_i}{\lambda}\right)\left(\frac{4\varsigma\lambda-w-r_i-\lambda}{\lambda}\right)\left(\frac{\partial\alpha_i^{\frac{\lambda}{r_i}}}{\partial\xi_j}\right)^2 \\ &\quad\left.\cdot\int_{\alpha_i}^{\xi_i}\left(s^{\frac{\lambda}{r_i}}-\alpha_i^{\frac{\lambda}{r_i}}\right)^{\frac{4\varsigma\lambda-w-r_i-2\lambda}{\lambda}}\mathrm{d}s\right|\cdot\left(\theta g_{j1}\sum_{k=1}^{j}|x_k|^{\frac{w+2r_j}{2r_k}}\right)^{\mathrm{T}}\cdot\left(\theta g_{j1}\sum_{k=1}^{j}|x_k|^{\frac{w+2r_j}{2r_k}}\right) \\ &\leqslant\sum_{j=1}^{i-1}\theta^2\tilde{g}_{jj}\cdot\left|\left(\frac{4\varsigma\lambda-w-r_i}{\lambda}\right)\cdot|z_i|^{\frac{4\varsigma\lambda-w-\lambda}{\lambda}}\cdot\left(\psi_{i,j}\sum_{k=1}^{i-1}|z_k|^{\frac{\lambda-2r_j}{\lambda}}\right)\right. \\ &\quad\left.+\left(\frac{4\varsigma\lambda-w-r_i}{\lambda}\right)\left(\frac{4\varsigma\lambda-w-r_i-\lambda}{\lambda}\right)\cdot\left(\bar{\psi}_{i,j}\sum_{k=1}^{i-1}|z_k|^{\frac{\lambda-r_j}{\lambda}}\right)^2\cdot|z_i|^{\frac{4\varsigma\lambda-w-2\lambda}{\lambda}}\right|\cdot\sum_{k=1}^{j}|z_k|^{\frac{w+2r_j}{\lambda}} \\ &\leqslant\frac{1}{8}\sum_{j=1}^{i-1}|z_j|^{4\varsigma}+\Theta\beta_{i,6}(\bar{\xi}_i,\widehat{\Theta})|z_i|^{4\varsigma},\end{aligned}\tag{30}$$

where $\tilde{g}_{jj}$, $\psi_{i,j}$, $\bar{\psi}_{i,j}$ and $\beta_{i,6}$ are some nonnegative smooth functions. Without loss of generality, we assume $i_1>i_2$. Similar to (30), we have

$$\sum_{i_1,i_2=1,i_1\neq i_2}^{i-1}K_{\xi_{i_1}}K_{\xi_{i_2}}\frac{\partial^2 W_i}{\partial\xi_{i_1}\partial\xi_{i_2}}g_{i_1}^{\mathrm{T}}g_{i_2}$$

$$
\begin{aligned}
&\leqslant \sum_{i_1,i_2=1,i_1\neq i_2}^{i-1} \theta^2\cdot\tilde{g}_{i_1i_2}\left|\int_{\alpha_i}^{\xi_i}\left(s^{\frac{\lambda}{r_i}}-\alpha_i^{\frac{\lambda}{r_i}}\right)^{\frac{4\varsigma\lambda-w-r_i-\lambda}{\lambda}}\mathrm{d}s\cdot\left(\psi_{i,i1}(\bar{\xi}_{i-1},\widehat{\Theta})\sum_{k=1}^{i-1}|z_k|^{\frac{\lambda-r_{i_1}-r_{i_2}}{\lambda}}\right)\right.\\
&\quad\cdot\left(\frac{4\varsigma\lambda-w-r_i}{\lambda}\right)+\left(\frac{4\varsigma\lambda-w-r_i}{\lambda}\right)\left(\frac{4\varsigma\lambda-w-r_i-\lambda}{\lambda}\right)\\
&\quad\cdot\left(\psi_{i,i2}(\bar{\xi}_{i-1},\widehat{\Theta})\sum_{k=1}^{i-1}|z_k|^{\frac{\lambda-r_{i_1}}{\lambda}}\right)\cdot\left(\psi_{i,i3}(\bar{\xi}_{i-1},\Theta)\sum_{k=1}^{i-1}|z_k|^{\frac{\lambda-r_{i_2}}{\lambda}}\right)\\
&\quad\left.\cdot\int_{\alpha_i}^{\xi_i}\left(s^{\frac{\lambda}{r_i}}-\alpha_i^{\frac{\lambda}{r_i}}\right)^{\frac{4\varsigma\lambda-w-r_i-2\lambda}{\lambda}}\mathrm{d}s\right|\cdot\sum_{k=1}^{i_1}|z_k|^{\frac{w+r_{i_1}+r_{i_2}}{\lambda}}\\
&\leqslant\frac{1}{8}\sum_{j=1}^{i-1}|z_j|^{4\varsigma}+\Theta\beta_{i,7}(\bar{\xi}_i,\widehat{\Theta})|z_i|^{4\varsigma},
\end{aligned}
\tag{31}
$$

where $\tilde{g}_{i_1i_2}$, $\psi_{i,i1}$, $\psi_{i,i2}$, $\psi_{i,i3}$ and $\beta_{i,7}$ are some nonnegative smooth functions. By (25)–(31), Eq. (16) can be rewritten as

$$
\begin{aligned}
\mathcal{L}V_i\leqslant&\left(\widetilde{\Theta}+\nu_{i-1}\right)\left(\sum_{j=1}^{i-1}\beta_j(\bar{\xi}_{j-1})|z_j|^{4\varsigma}-\dot{\widehat{\Theta}}\right)-\left(n+\frac{9}{8}-i\right)\sum_{j=1}^{i-1}z_j^{4\varsigma}+\Theta\beta_i(\bar{\xi}_i,\widehat{\Theta})|z_i|^{4\varsigma}\\
&+z_i^{\frac{4\varsigma\lambda-w-r_i}{\lambda}}K_i(\xi_{i+1}^{p_i}-\alpha_{i+1}^{p_i})+z_i^{\frac{4\varsigma\lambda-w-r_i}{\lambda}}K_i\alpha_{i+1}^{p_i}+\frac{\partial W_i}{\partial\widehat{\Theta}}\dot{\widehat{\Theta}},
\end{aligned}
\tag{32}
$$

where $\beta_i=\sum_{j=1}^{7}\beta_{ij}$. From (18) and Lemma 3 in [18], there is a nonnegative function κ_i such that

$$
\left|-\nu_{i-1}\beta_i(\bar{\xi}_i,\widehat{\Theta})|z_i|^{4\varsigma}+\frac{\partial W_i}{\partial\widehat{\Theta}}\sum_{j=1}^{i}\beta_jz_j^{4\varsigma}\right|\leqslant\sum_{j=2}^{i-1}\left|\frac{\partial W_j}{\partial\widehat{\Theta}}\right|\beta_iz_i^{4\varsigma}+\left|\frac{\partial W_i}{\partial\widehat{\Theta}}\right|\sum_{j=1}^{i}\beta_jz_j^{4\varsigma}\leqslant\frac{1}{8}\sum_{j=1}^{i-1}|z_j|^{4\varsigma}+\kappa_iz_i^{4\varsigma}. \tag{33}
$$

Hence,

$$
\begin{aligned}
&\left(\widetilde{\Theta}+\nu_{i-1}\right)\left(\sum_{j=1}^{i-1}\beta_j(\bar{\xi}_{j-1})|z_j|^{4\varsigma}-\dot{\widehat{\Theta}}\right)+\frac{\partial W_i}{\partial\widehat{\Theta}}\dot{\widehat{\Theta}}\\
&\quad\leqslant\left(\widetilde{\Theta}+\nu_i\right)\left(\sum_{j=1}^{i}\beta_j(\bar{\xi}_{j-1})|z_j|^{4\varsigma}-\dot{\widehat{\Theta}}\right)+\frac{1}{8}\sum_{j=1}^{i-1}|z_j|^{4\varsigma}+\kappa_iz_i^{4\varsigma}-\widetilde{\Theta}\beta_i|z_i|^{4\varsigma},
\end{aligned}
\tag{34}
$$

where $\nu_i=-\sum_{j=2}^{i}\frac{\partial W_i}{\partial\widehat{\Theta}}$. The virtual controller is chosen as

$$
\alpha_{i+1}^{p_i}=-z_i^{\frac{r_{i+1}p_i}{\lambda}}\left(\frac{1}{\gamma_i}\left(n+1-i+\kappa_i+\widehat{\Theta}\beta_i\right)\right). \tag{35}
$$

Substituting (34) and (35) into (32) yields

$$
\begin{aligned}
\mathcal{L}V_i\leqslant&\ z_i^{\frac{4\varsigma\lambda-w-r_{i+1}}{\lambda}}K_i(\xi_{i+1}^{p_i}-\alpha_{i+1}^{p_i})+\left(\widetilde{\Theta}+\nu_i\right)\left(\sum_{j=1}^{i}\beta_j(\bar{\xi}_{j-1})|z_j|^{4\varsigma}-\dot{\widehat{\Theta}}\right)\\
&-(n+1-i)\sum_{j=1}^{i}z_j^{4\varsigma}.
\end{aligned}
\tag{36}
$$

At Step n, we choose $V_n(\xi)=V_{n-1}(\bar{\xi}_{n-1})+W_n(\xi)$ and design the controller and the adaptive law:

$$
u^{p_n}=-z_n^{\frac{r_{n+1}p_n}{\lambda}}\left(\frac{1}{\gamma_n}\left(1+\vartheta_n+\widehat{\Theta}\beta_n\right)\right)\triangleq-z_n^{\frac{r_{n+1}p_n}{\lambda}}\varphi_n^{\frac{r_{n+1}p_n}{\lambda}}, \tag{37}
$$

$$
\dot{\widehat{\Theta}}=\sum_{j=1}^{n}\beta_jz_j^{4\varsigma}, \tag{38}
$$

such that

$$\mathcal{L}V_n \leqslant -\sum_{j=1}^{n} z_j^{4\varsigma}. \tag{39}$$

3.3 Stability analysis

Theorem 1. If Assumption 1 holds for the system (1), then for all initial values $x(0) \in \Omega_x$, there exists a state-feedback controller such that

(1) The closed-loop system has a unique strong solution on $[0, \infty)$;

(2) All the closed-loop signals are bounded almost surely and asymmetric full-state constraints are not violated almost surely;

(3) The trivial solution of the closed-loop system (1), (4), (37) and (38) is asymptotically stable in probability and $P\{\lim_{t\to\infty} \widehat{\Theta}(t)$ exists and is finite$\} = 1$.

Proof. (1) Substituting (37) and (38) into (5) yields

$$\mathrm{d}\xi = h(\theta, \xi, u)\mathrm{d}t + d(\theta, \xi)\mathrm{d}w, \tag{40}$$

where $h(\theta, \xi, u)$ and $d(\theta, \xi)$ are continuous functions with $h(0, 0, 0) = 0$ and $d(0, 0) = 0$. By (37),

$$\begin{aligned}
\frac{\partial u^{p_n}}{\partial \widehat{\Theta}} &= -\frac{r_{n+1}p_n}{\lambda}\left(\frac{\partial \varphi_n}{\partial \widehat{\Theta}}\varphi_n^{\frac{r_{n+1}p_n}{\lambda}-1} z_n^{\frac{r_{n+1}p_n}{\lambda}} + \frac{\partial z_n}{\partial \widehat{\Theta}} z_n^{\frac{r_{n+1}p_n}{\lambda}-1}\varphi_n^{\frac{r_{n+1}p_n}{\lambda}}\right),\\
\frac{\partial u^{p_n}}{\partial \xi_j} &= -\frac{r_{n+1}p_n}{\lambda}\left(\frac{\partial \varphi_n}{\partial \xi_j}\varphi_n^{\frac{r_{n+1}p_n}{\lambda}-1} z_n^{\frac{r_{n+1}p_n}{\lambda}} + \frac{\partial z_n}{\partial \xi_j} z_n^{\frac{r_{n+1}p_n}{\lambda}-1}\varphi_n^{\frac{r_{n+1}p_n}{\lambda}}\right).
\end{aligned} \tag{41}$$

Since φ_j is a smooth function with respect to $\widehat{\Theta}, \bar{\xi}_j$, $j = 1, \ldots, n$, and

$$\begin{aligned}
\frac{\partial z_n}{\partial \widehat{\Theta}} &= \frac{\partial \varphi_{n-1}}{\partial \widehat{\Theta}} z_{n-1} + \varphi_{n-1}\frac{\partial \varphi_{n-2}}{\partial \widehat{\Theta}} z_{n-2} + \cdots + \varphi_{n-1}\cdots\varphi_2\frac{\partial \varphi_1}{\partial \widehat{\Theta}} z_1,\\
\frac{\partial z_n}{\partial \xi_j} &= \frac{\partial \varphi_{n-1}}{\partial \xi_j} z_{n-1} + \varphi_{n-1}\frac{\partial \varphi_{n-2}}{\partial \xi_j} z_{n-2} + \cdots + \varphi_{n-1}\cdots\varphi_j\frac{\lambda}{r_j}\xi_j^{\frac{\lambda-r_j}{r_j}},\\
\frac{\partial z_n}{\partial \xi_n} &= \frac{\lambda}{r_n}\xi_n^{\frac{\lambda-r_n}{r_n}},
\end{aligned} \tag{42}$$

by the continuity of $\frac{\partial \varphi_i}{\partial \widehat{\Theta}}$, $\frac{\partial \varphi_i}{\partial \xi_j}$ and $\frac{\partial z_j}{\partial \xi_j}$, $i = 1, \ldots, n-1$, $j = 1, \ldots, n$, we know that $u^{p_n} \in \mathcal{C}^1$. Since f_k and g_k, $k = 1, \ldots, n$, satisfy local Lipschitz conditions and Eq. (4) is an equivalent transformation, by Lemma 1 in [17], the closed-loop system (1), (4), (37) and (38) has a unique strong solution on $[0, \infty)$.

(2) It is divided into two parts.

Part I. Because $V_n(\xi)$ is positive definite and radially unbounded, we have $\alpha_3(|\xi|) \leqslant V_n(\xi) \leqslant \alpha_4(|\xi|)$, where α_3 and α_4 are $\mathcal{K}_\infty$ functions. Define the closed set $C = \{\xi(t) \in \mathbb{R}^n : |\xi(t)| \leqslant \alpha_3^{-1}\alpha_4(|\xi(0)|)\}$, the stopping times $T = \inf\{t \geqslant 0; |\xi(t)| > \alpha_3^{-1}\alpha_4(|\xi(0)|)\}$ and $\tau_n = \inf\{t \geqslant 0; |\xi(t)| \geqslant n\}$. The sample space Ω can be decomposed into the following two mutually exclusive events:

$$\Omega_1 = \{\xi(t) \in C, \forall t \geqslant 0\}, \quad \Omega_2 = \{\xi(t) \notin C, \forall t \geqslant 0\}. \tag{43}$$

Next, we prove $P\{\Omega_2\} = 0$. Suppose that $P\{\Omega_2\} > 0$. For almost all $\omega \in \Omega_2$, by (14) and Dynkin formula, one has

$$\mathrm{E}V_n(\xi(\tau_n \wedge T \wedge t)) = \mathrm{E}V_n\big(\xi(0)\big) + \mathrm{E}\int_0^{\tau_n \wedge T \wedge t} \mathcal{L}V_n(\xi(s))\mathrm{d}s \leqslant V_n\big(\xi(0)\big), \quad \forall t \in [0, T]. \tag{44}$$

Setting $n \to \infty$, we have $P(T < t)V_n(\xi(T)) \leqslant \mathrm{E}V_n(\xi(T \wedge t)) \leqslant V_n(\xi(0)) \leqslant \alpha_4(|\xi(0)|)$. However, by the definition of the stopping time, $P(T < t)V_n(\xi(T)) > P(T < t)\alpha_4(|\xi(0)|)$, we have $P(T < t)\alpha_4(|\xi(0)|) < P(T < t)V_n(\xi(T)) \leqslant \mathrm{E}V_n(\xi(T \wedge t)) \leqslant V_n(\xi(0)) \leqslant \alpha_4(|\xi(0)|)$, which means that $P(T < t) = 0$. Let $t \to \infty$, and then $P(T < \infty) = 0$, i.e., $P\{\Omega_2\} = 0$.

For any given initial value $x(0) \in \Omega_x$ and the corresponding Lyapunov function, $x(t) \in \overline{\Omega}_{x_0} \in \Omega_x$ almost surely is proved for any $t \geqslant 0$, where $\overline{\Omega}_{x_0} = \{x(t) \in \mathbb{R}^n : |x(t)| \leqslant |H^{-1}\alpha_1^{-1}\alpha_2(|\xi(0)|)|\}$ and $H^{-1}\alpha_1^{-1}\alpha_2(|\xi(0)|) = (H_1^{-1}\alpha_1^{-1}\alpha_2(|\xi(0)|), \ldots, H_n^{-1}\alpha_1^{-1}\alpha_2(|\xi(0)|))$. By choosing $x(0) \in \Omega_x$ and repeating the process from (6) to (37), $x(t) \in \overline{\Omega}_{x_0}$ almost surely for $\forall t \geqslant 0$ can be guaranteed, which prevents the circular argument. Hence, control design and analysis are valid. It is worth mentioning that $\bar{\Omega}_{x_0}$ can be chosen prior to control design when Lyapunov function and initial values are specified.

Part II. We first prove

$$c_{i,1}|\xi_i - \alpha_i|^{\frac{4\varsigma\lambda - w}{r_i}} \leqslant W_i \leqslant c_{i,2}|z_i|^{\frac{4\varsigma\lambda - w}{\lambda}}, \tag{45}$$

where $c_{i,1} = \frac{r_i}{4\varsigma\lambda - w} 2^{1-\frac{\lambda}{r_i}}$ and $c_{i,2} = 2^{1-\frac{r_i}{\lambda}}$ are positive constants. By Lemma 4 in [18], one gets

$$W_i \leqslant 2^{1-\frac{r_i}{\lambda}} |z_i|^{\frac{4\varsigma\lambda - w}{\varsigma\lambda}}. \tag{46}$$

We divide into two cases to prove the left-hand side of (45).

Case (i). If $\alpha_i \leqslant \xi_i$, by Lemma 4 in [18],

$$\begin{aligned} \int_{\alpha_i}^{\xi_i} \left(s^{\frac{\lambda}{r_i}} - \alpha_i^{\frac{\lambda}{r_i}}\right)^{\frac{4\varsigma\lambda - r_i - w}{\lambda}} \mathrm{d}s &\geqslant 2^{1-\frac{\lambda}{r_i}} \int_{\alpha_i}^{\xi_i} (s-\alpha_i)^{\frac{4\varsigma\lambda - r_i - w}{r_i}} \mathrm{d}(s-\alpha_i) \\ &= \frac{r_i}{4\varsigma\lambda - w} 2^{1-\frac{\lambda}{r_i}} (s-\alpha_i)^{\frac{4\varsigma\lambda - w}{r_i}} |_{\alpha_i}^{\xi_i} \\ &= \frac{r_i}{4\varsigma\lambda - w} 2^{1-\frac{\lambda}{r_i}} |\xi_i - \alpha_i|^{\frac{4\varsigma\lambda - w}{r_i}}. \end{aligned} \tag{47}$$

Case (ii). If $\alpha_i \geqslant \xi_i$. Similar to Case (i),

$$\begin{aligned} \int_{\alpha_i}^{\xi_i} \left(s^{\frac{\lambda}{r_i}} - \alpha_i^{\frac{\lambda}{r_i}}\right)^{\frac{4\varsigma\lambda - r_i - w}{\lambda}} \mathrm{d}s &\geqslant -2^{1-\frac{\lambda}{r_i}} \int_{\xi_i}^{\alpha_i} (\alpha_i - s)^{\frac{4\varsigma\lambda - r_i - w}{r_i}} \mathrm{d}(\alpha_i - s) \\ &= -\frac{r_i}{4\varsigma\lambda - w} 2^{1-\frac{\lambda}{r_i}} (\alpha_i - s)^{\frac{4\varsigma\lambda - w}{r_i}} |_{\xi_i}^{\alpha_i} \\ &= \frac{r_i}{4\varsigma\lambda - w} 2^{1-\frac{\lambda}{r_i}} |\xi_i - \alpha_i|^{\frac{4\varsigma\lambda - w}{r_i}}. \end{aligned} \tag{48}$$

Hence the inequality (45) holds. By Dynkin's formula, one gets

$$\begin{aligned} \mathrm{E}\{V_n(\xi(\tau_n \wedge t))\} &= V_n(\xi(0)) + \mathrm{E}\int_0^{\tau_n \wedge t} \mathcal{L}V_n(\xi(s))\mathrm{d}s \\ &\leqslant V_n\big(\xi(0)\big). \end{aligned} \tag{49}$$

From (45) and $V_n = \sum_{i=1}^n V_i$, it follows that $V_n(\xi) \geqslant \frac{1}{2}\widetilde{\Theta}^2(t) + c_{1,1}|\xi_1|^{\frac{4\varsigma\lambda - w}{r_1}} + c_{2,1}|\xi_2 - \alpha_2|^{\frac{4\varsigma\lambda - w}{r_2}} + c_{3,1}|\xi_3 - \alpha_3|^{\frac{4\varsigma\lambda - w}{r_3}} + \cdots + c_{n,1}|\xi_n - \alpha_n|^{\frac{4\varsigma\lambda - w}{r_n}}$. By setting $n \to \infty$ and (49), $\xi_1(t)$ and $\widetilde{\Theta}(t)$ are bounded almost surely. Hence, $\alpha_2(t)$ is bounded almost surely by its continuity. It is easy to deduce that $\xi_2(t)$ is bounded almost surely. Recursively, $\alpha_3(t), \xi_3(t), \ldots, u(t)$ are bounded almost surely. By the discussion in (4) below and one-to-one mapping between $\xi(t)$ and $x(t)$, asymmetric full-state constraints are not violated almost surely.

(3) By almost sure boundedness of $\xi_1(t), \ldots, \xi_n(t)$ and (38), $P\{\lim_{t\to\infty} \widehat{\Theta}(t)$ exists and is finite$\} = 1$. Since $V_n(\xi)$ is a $\mathcal{C}^2$, positive definite and radially unbounded function, it follows from (14) and Lemma 1 in [18] that

$$P\Big\{\lim_{t\to\infty} (|\xi_1(t)| + \cdots + |\xi_n(t)|) = 0\Big\} = 1. \tag{50}$$

Since Eq. (4) is an equivalent coordinate transformation, the equilibrium of the closed-loop system (1), (4), (37) and (38) is asymptotically stable in probability.

4 Simulation example

Consider a stochastic nonlinear system:

$$\begin{aligned} \mathrm{d}x_1 &= (x_2^{p_1} + f_1)\,\mathrm{d}t + g_1 \mathrm{d}\omega, \\ \mathrm{d}x_2 &= (u^{p_2} + f_2)\,\mathrm{d}t + g_2 \mathrm{d}\omega, \end{aligned} \tag{51}$$

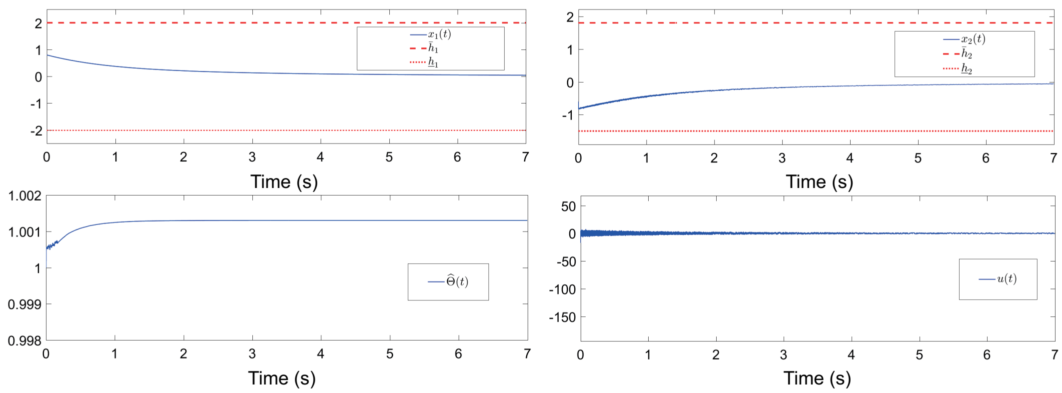

Figure 1 (Color online) Responses of the closed-loop system (51)–(53).

with asymmetric full-state constraints:

$$x(t) \in \Omega_x = \{x(t) \in \mathbb{R}^2 : -\underline{h}_i < x_i(t) < \bar{h}_i,\ i = 1, 2\},\ \forall t \geqslant 0, \tag{52}$$

where x_1 and x_2 are system states, u is control input, ω is a 1-dimensional independent standard Wiener process, $\underline{h}_1 = 2$, $\bar{h}_1 = 2$, $\underline{h}_2 = -1.5$, $\bar{h}_2 = 1.8$, $p_1 = \frac{5}{3}$, $p_2 = 3$, $f_1 = \frac{1}{8}\theta\sin(x_1)x_1^{\frac{5}{3}}$, $g_1 = 0$, $f_2 = \frac{1}{3}\theta\cos(x_1)x_2^{\frac{5}{3}}$ and $g_2 = \frac{1}{6}\theta x_2^{\frac{4}{3}}$. By choosing $r_1 = 1$, $w = \frac{2}{3}$, $r_2 = 1$ and $r_3 = \frac{5}{9}$, Assumption 1 holds.

We set $\lambda = 1$, $\tau = 1$ and $\xi_i = \frac{x_i}{h_i(x_i)}$, $i = 1, 2$, where $h_i(x_i) = (\underline{h}_i + x_i)(\bar{h}_i - x_i)$, and choose the Lyapunov function $V_1(z_1) = \frac{1}{4-w}z_1^{4-w} + \frac{1}{2}\tilde{\Theta}^2(t)$ with $\xi_1 = x_1$. The first virtual controller $\alpha_2^{\frac{5}{3}} = -\frac{1}{\gamma_1}z_1^{\frac{5}{3}}(2 + \hat{\Theta})$ leads to $\mathcal{L}V_1 \leqslant z_1^{\frac{7}{3}}K_1(\xi_2^{\frac{5}{3}} - \alpha_2^{\frac{5}{3}}) - 2z_1^4 + (\tilde{\Theta} + \nu_1)(\beta_1(\xi_1)|z_1|^4 - \dot{\hat{\Theta}})$. By choosing $V_2 = V_1 + \int_{\alpha_2}^{\xi_2}(s - \alpha_2)^{\frac{7}{3}}\mathrm{d}s$, the controller and the adaptive law:

$$u = -\frac{1}{\gamma_2}z_2^{\frac{5}{9}}\left(1 + \sum_{j=1}^{8}\beta_{2j}\hat{\Theta}\right)^{\frac{1}{3}},$$

$$\dot{\hat{\Theta}} = \sum_{j=1}^{9}\beta_{2j}z_2^4 + z_1^4, \quad \hat{\Theta}(0) = 1, \tag{53}$$

such that $\mathcal{L}V_1 \leqslant -z_1^4 - z_2^4$, where $K_{\xi_i} = \frac{\underline{h}_i\bar{h}_i + x_i^2}{h_i^2}$, $i = 1, 2$, $K_1 = K_{\xi_1}h_2^{p_1}$, $K_2 = K_{\xi_2}$, $\beta_{21} = K_1^{\frac{12}{5}}$, $\beta_{22} = 2K_{\xi_2}h_2^{\frac{5}{3}}$, $\beta_{23} = 20(K_{\xi_2}h_2^{\frac{5}{3}}\psi_1^{\frac{5}{3}})^{\frac{12}{5}}$, $\beta_{24} = 3(K_{\xi_1}h_2^{\frac{5}{3}}\psi_1^{\frac{5}{3}}\psi_2)^{\frac{12}{7}}$, $\beta_{25} = 3K_{\xi_1}h_2^{\frac{5}{3}}\psi_2$, $\beta_{26} = 3(K_{\xi_1}h_1^{\frac{5}{3}}\psi_2)^{\frac{12}{7}}$, $\beta_{27} = 2K_{\xi_2}^2h_2^{\frac{8}{3}}$, $\beta_{28} = 2(K_{\xi_2}^2h_2^{\frac{8}{3}})^3$ and $\psi_1 = \psi_2 = 2 + \hat{\Theta}$.

By choosing $x_1(0) = 0.8$, $x_2(0) = -0.6$, $\theta = 1$, $\gamma_1 = \frac{1}{2}$ and $\gamma_2 = \frac{1}{3}$, Figure 1 shows the effectiveness of this control scheme.

5 Conclusion

A new scheme for the adaptive state-feedback control of state-constrained stochastic high-order nonlinear systems without feasibility conditions is provided in this paper.

Two problems are considered: (1) How to design an output feedback controller for a state-constrained stochastic high-order nonlinear system (1) under more general assumptions compared with those in [18]? (2) For constrained multiple systems in [31], can we design an output feedback tracking controller?

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