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Special Focus on Control and Analysis for Stochastic Systems

Robust SOF Stackelberg game for stochastic LPV systems

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Abstract A robust static output feedback (SOF) Stackelberg game with multiple followers in stochastic linear parameter varying (LPV) systems is investigated. The conditions for the existence of a robust SOF Stackelberg strategy set under H_∞ constraints are established by the cross-coupled matrix inequality (CCMI). To determine this strategy set, the optimization problems corresponding to the relevant cost bounds are defined, and their solution sets are derived using the Karush-Kuhn-Tucker conditions. The results show that the robust SOF Stackelberg strategy set can be obtained by solving higher-order cross-coupled matrix equations (CCMEs). Because CCMEs are complex and difficult to solve numerically, a heuristic algorithm is developed by combining the CCMEs with the CCMIs. The convergence property is proven using the Krasnoselskii-Mann (KM) iteration algorithm. Finally, two numerical examples are solved to demonstrate the reliability and usefulness of the proposed heuristic algorithm.

Keywords Stackelberg games, linear parameter varying (LPV) stochastic systems, cross-coupled matrix equations (CCMEs)

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1 Introduction

The leader-follower hierarchical control problems or Stackelberg games have found many practical applications in systems science and control engineering fields. In [1], the leader-follower stochastic differential game based on the open-loop strategy has been investigated. In [2], the robust semi-global leader-following practical consensus problem of a group of general linear systems has been investigated.

There have been many studies on Stackelberg games (e.g., [3] and reference therein). The basic concept of Stackelberg games is that players are assumed to be divided into two types: leaders and followers. The leaders announce their strategies ahead of the followers, and the followers determine their strategies later. To define a Stackelberg strategy for a game, a leader (or leaders as a group) is assumed to know the rationale in which the followers are playing. Additionally, the followers are assumed to be rational in the sense that they will play according to this rationale. Stackelberg games are usually used in a hierarchical decision situation where players have different objectives. In [4], an asymmetric information linear-quadratic stochastic Stackelberg differential game for a class of stochastic systems governed by mean-field type stochastic differential equation has been investigated.

Recently, Stackelberg games have been extensively investigated from practical perspectives. For example, a communication network with a single service provider and several users can be formulated as a Stackelberg game. The service provider sets the usage price as the leader to the users or followers. In a smart grid, optimal demand response management has been investigated within the framework of Stackelberg games. The utility company is assumed to be the leader, and the users are assumed to be the followers [5]. A scheduling problem in a packet switch operation in a ring architecture has been investigated where a central processor can decide the strategy such that the total throughput is maximized

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by inducing a large number of local controllers for each link [6]. From this background, systems theory based on the Stackelberg game theory has been developed to solve strategic decision problems in the past few decades. In [7], a linear-quadratic (LQ) stochastic Stackelberg differential game with asymmetric information has been investigated. In [8], the demand response management of multiple microgrids with different and overlapping sales areas has been investigated, where a Stackelberg game model of microgrids and users is designed. In particular, a noncooperative Stackelberg game between microgrids and users was discussed.

Uncertainties in systems make it difficult for people to design controllers. These uncertainties may be caused by modeling errors, system noise, and external disturbances. Researchers have developed various mathematical models to describe systems with uncertainties. Deterministic and stochastic linear parameter varying (LPV) systems are one such model that is useful for explicitly describing systems with uncertain parameters subject to arbitrary smooth or discontinuous variations explicitly (e.g., [9] and reference therein). It is well known that a gain-scheduled (GS) control is an effective and reliable control technique to deal with systems having uncertain parameters and to evaluate the performance of control efforts. In [10], a new robust stability controller based on the GS H_∞ control for an LPV model of the electric vehicle was investigated against the parameter uncertainties.

With advancements in the GS control technique in recent years, several dynamic games such as Nash games and Pareto optimal control problems for systems with uncertain parameters have been studied using GS strategies [11–14]. However, Stackelberg (or leader-follower) dynamic games, with a hierarchical structure for systems with uncertain parameters using GS strategies, have not been studied, except for [15]. Usually, studying a dynamic game with a hierarchical structure is more complex and difficult than studying a dynamic game with a parallel structure. Moreover, considering the application of dynamic game theory in the control field, for example, a robust control problem based on the dynamic game theory, it is impractical to consider a dynamic game under the assumption that players know all the state information in systems. In a dynamic game, it is more realistic to assume that players observe individual and/or local information. These are some of the challenges faced by researchers.

Motivated by the preceding challenges, a robust static output feedback (SOF) Stackelberg game with multiple followers is investigated for stochastic LPV systems in this study. The GS and mode-independent robust SOF Stackelberg strategy sets are studied. Compared with the GS (or mode-dependent) strategy, the mode-independent strategy has the advantage of easy design and implementation.

The contributions of this study are threefold. First, the existence conditions of the robust SOF Stackelberg strategy set with an H_∞ constraint are obtained in terms of the cross-coupled matrix inequalities (CCMIs). It is worth noting that the basic idea of this study derives from the H_2/H_∞ controller design technique [16, 17]. Second, an optimization problem is defined to determine the solvability conditions of the CCMIs, which are established using the Karush-Kuhn-Tucker (KKT) conditions. Third, a heuristic algorithm based on the linear matrix inequality (LMI)-based optimization problem is introduced to avoid the computational difficulty of finding the robust SOF Stackelberg strategy set.

Usually NP-hard (non-deterministic polynomial-time hard)[1)] bilinear matrix inequality (BMI) optimization problems must be solved if SOF strategies are used in a robust control problem. Thus, a computational algorithm for validating the heuristic algorithm is proposed using the Krasnoselskii-Mann (KM) iterative algorithm [18]. As a result, the robust SOF Stackelberg strategy set can be obtained by solving the optimization problems of cross-coupled matrix equations (CCMEs) and CCMIs recursively. Furthermore, it is proved that the convergence of the heuristic algorithm is guaranteed. Finally, two numerical examples are presented to demonstrate the reliability and usefulness of the proposed heuristic algorithm.

Notation. The following symbols are used in standard mathematical notations within the formulas. $\mathrm{E}[\cdot]$ denotes the mathematical expectation operator with respect to the stochastic measure. A^{T} denotes the transpose of the matrix A. **block diag** denotes the block diagonal matrix. $\|\cdot\|$ denotes the Euclidean vector norm or the Euclidean norm of a matrix. $\mathrm{N}(\mu, \sigma^2)$ denotes a normal distribution with mean μ and variance σ^2. $L^2_F([0,\ \infty),\ \mathbb{R}^k)$ denotes the space of nonanticipative stochastic processes with respect to the filter satisfying $\mathrm{E}[\int_0^\infty \|\phi(t)\|^2 \mathrm{d}t] < \infty$.

1) A problem is called NP-hard if an algorithm for solving it can be transformed into one for solving any non-deterministic polynomial-time problem.

2 Preliminary results

Let us consider a stochastic LPV system with disturbances:

$$\mathrm{d}x(t) = \big[A(\eta(t))x(t) + Bu(t) + B_v v(t)\big]\mathrm{d}t + A_p(\eta(t))x(t)\mathrm{d}w(t),\ x(0) = x^0, \tag{1a}$$

$$z(t) = E(\eta(t))x(t), \tag{1b}$$

$$y(t) = Cx(t), \tag{1c}$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector, $u(t) \in \mathbb{R}^m$ denotes the control input, and $v(t) \in \mathbb{R}^{n_v}$ denotes the external disturbance. $z(t) \in \mathbb{R}^{n_z}$ is the controlled output. $y(t) \in \mathbb{R}^p$ is the output vector. $w(t) \in \mathbb{R}$ denotes a one-dimensional standard Wiener process defined in the filtered probability space [16,17,19]. In this paper, it is assumed that $\mathrm{d}w(t)$ satisfies the independent increment property, such as $\mathrm{E}[x(t)\mathrm{d}w(t)] = 0$ and $\mathrm{E}[\mathrm{d}w(t)^2] = \sigma^2\mathrm{d}t$. $\eta(t) \in \mathbb{R}^r$ denotes the time-varying parameters. r is the number of time-varying parameters. Without loss of generality, it is supposed that the stochastic LPV system (1) admits a unique solution $x(t) = x(t, u_0, u_1, \ldots, u_N, v, x(0))$. The parameter dependent coefficient matrices $A(\eta(t))$, $A_p(\eta(t))$ and $E(\eta(t))$ can be expressed as

$$\begin{bmatrix} A(\eta(t)) & A_p(\eta(t)) \end{bmatrix} = \sum_{k=1}^{M} \delta_k(t) \begin{bmatrix} A_k & A_{pk} \end{bmatrix}, \tag{2a}$$

$$E(\eta(t)) = \sum_{k=1}^{M} \delta_k(t) E_k, \tag{2b}$$

where $\delta_k(t) \geqslant 0$, $\sum_{k=1}^{M} \delta_k(t) = 1$, $M = 2^r$.

The H_∞ norm is defined as follows [9].

Definition 1. The H_∞ norm of the mean-square stable stochastic LPV system (1) when $u(t) \equiv 0$ is defined as

$$\|L\|_\infty = \sup_{\substack{v \in L^2_F([0,\ \infty),\ \mathbb{R}^{n_v}),\\ v \neq 0,\ x^0 = 0}} \frac{\|z\|_2}{\|v\|_2}, \tag{3}$$

where

$$\|z\|_2^2 := \mathrm{E}\left[\int_0^\infty \|z(t)\|^2 \mathrm{d}t\right], \quad \|v\|_2^2 := \mathrm{E}\left[\int_0^\infty \|v(t)\|^2 \mathrm{d}t\right].$$

The following result has been proved [13].

Lemma 1. Let us consider the stochastic LPV system (1) when $u(t) \equiv 0$. For a given positive parameter γ, if there exists a matrix $Y > 0$ satisfying the following LMIs (4), then the stochastic LPV system (1) is mean-square stable with $\|L\|_\infty < \gamma$ under $x^0 = 0$.

$$\begin{bmatrix} YA_k + A_k^{\mathrm{T}}Y & YB_v & A_{pk}^{\mathrm{T}}Y & E_k^{\mathrm{T}} \\ B_v^{\mathrm{T}}Y & -\gamma^2 I_{n_v} & 0 & 0 \\ YA_{pk} & 0 & -Y & 0 \\ E_k & 0 & 0 & -I_{n_z} \end{bmatrix} < 0,\ k = 1, \ldots, M. \tag{4}$$

Moreover, the worst-case disturbance is given by

$$v^*(t) = F_\gamma^* x(t) = \gamma^{-2} B_v^{\mathrm{T}} Y x(t). \tag{5}$$

In addition, an optimization problem of the upper bound of the cost functional when $v(t) \equiv 0$ or $B_v \equiv 0$ in (1a) is defined [11–13].

Definition 2. Let us consider the cost functional:

$$J(u, x^0) = \mathrm{E}\left[\int_0^\infty \big[x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)\big]\mathrm{d}t\right], \tag{6}$$

where $Q = Q^{\mathrm{T}} \geqslant 0$, $R = R^{\mathrm{T}} > 0$.

Then, the LQ control problem concerning the GS SOF strategy is to find a control gain $K(\eta(t))$:

$$u(t) = K(\eta(t))y(t) = K(\eta(t))Cx(t) = \sum_{k=1}^{M} \delta_k(t)K_kCx(t), \tag{7}$$

such that the bound of the quadratic cost functional (6) is minimized.

The following result is an extension of the existing results in [11–13].

Lemma 2. If there exists a matrix $P > 0$ satisfying the LMIs (8):

$$\begin{aligned}\Lambda_k(P, K_k) := P(A_k + BK_kC) + (A_k + BK_kC)^{\mathrm{T}}P + A_{pk}^{\mathrm{T}}PA_{pk} \\ + Q + C^{\mathrm{T}}K_k^{\mathrm{T}}RK_kC \leqslant 0, \quad k = 1, \ldots, M,\end{aligned} \tag{8}$$

then the stochastic LPV system is mean-square stable and the gain-scheduled SOF control (7) satisfies the following inequality:

$$J(u, x^0) \leqslant \mathrm{E}\big[x^{\mathrm{T}}(0)Px(0)\big]. \tag{9}$$

Proof. Let us define a candidate Lyapunov function:

$$V(x(t)) = x^{\mathrm{T}}(t)Px(t), \tag{10}$$

where $P = P^{\mathrm{T}} > 0$.

The closed-loop stochastic LPV system using the control (7) is given by

$$\mathrm{d}x(t) = \big[A(\eta(t)) + BK(\eta(t))C\big]x(t)\mathrm{d}t + A_p(\eta(t))x(t)\mathrm{d}w(t). \tag{11}$$

The result can be derived along the trajectories of the closed-loop stochastic LPV system (11) by applying Itô's formula.

$$\mathrm{d}V(x(t)) = \boldsymbol{L}V(x(t))\mathrm{d}t + 2x^{\mathrm{T}}A_p^{\mathrm{T}}(\eta(t))Px(t)\mathrm{d}w(t), \tag{12}$$

where

$$\begin{aligned}\boldsymbol{L}V(x(t)) := x^{\mathrm{T}}(t)\Big[P\big(A(\eta(t)) + BK(\eta(t))C\big) + \big(A(\eta(t)) + BK(\eta(t))C\big)^{\mathrm{T}}P \\ + A_p^{\mathrm{T}}(\eta(t))PA_p(\eta(t))\Big]x(t).\end{aligned}$$

If there exists a $P > 0$ such that

$$\begin{aligned}P\big(A(\eta(t)) + BK(\eta(t))\big) + \big(A(\eta(t)) + BK(\eta(t))\big)^{\mathrm{T}}P \\ + A_p^{\mathrm{T}}(\eta(t))PA_p(\eta(t)) + Q + C^{\mathrm{T}}K^{\mathrm{T}}(\eta(t))RK(\eta(t))C \leqslant 0,\end{aligned} \tag{13}$$

then, the following equation holds:

$$\mathrm{E}\big[\mathrm{d}V(x(t))\big] = \mathrm{E}\big[\boldsymbol{L}V(x(t))\mathrm{d}t\big] \leqslant -\mathrm{E}\Big[x^{\mathrm{T}}(t)\big(Q + C^{\mathrm{T}}K^{\mathrm{T}}(\eta(t))RK(\eta(t))C\big)x(t)\Big] < 0. \tag{14}$$

Therefore, the closed-loop stochastic LPV system is mean-square stable. Moreover, by integrating both sides of (14) from 0 to t_f, we obtain

$$V(x(t_f)) - V(x(0)) \leqslant -\mathrm{E}\left[\int_0^{t_f} \big[x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)\big]\mathrm{d}t\right]. \tag{15}$$

Because the closed-loop stochastic LPV system is mean-square stable, $\mathrm{E}[x(t_f)] \to 0$ as $t_f \to +\infty$ holds and the inequality on the cost bound (9) is obtained.

Conversely, applying the Schur complement to inequality (13) leads to the following matrix inequality:

$$\begin{bmatrix} P\big(A(\eta(t)) + BK(\eta(t))C\big) + \big(A(\eta(t)) + BK(\eta(t))C\big)^{\mathrm{T}}P + Q & A_p^{\mathrm{T}}(\eta(t))P & C^{\mathrm{T}}K^{\mathrm{T}}(\eta(t)) \\ PA_p(\eta(t)) & -P & 0 \\ K(\eta(t))C & 0 & -R^{-1} \end{bmatrix}$$

$$= \sum_{k=1}^{M} \delta_k(t)\Gamma_k(P, K_k) \leqslant 0, \tag{16}$$

where

$$\Gamma_k(P, K_k) := \begin{bmatrix} P\big(A_k + BK_kC\big) + \big(A_k + BK_kC\big)^{\mathrm{T}}P + Q & A_{pk}^{\mathrm{T}}P & C^{\mathrm{T}}K_k^{\mathrm{T}} \\ PA_{pk} & -P & 0 \\ K_kC & 0 & -R^{-1} \end{bmatrix}.$$

If inequalities $\Gamma_k(P, K_k) \leqslant 0$, $k = 1, \ldots, M$ are satisfied, then inequality (16) holds. Moreover, the Schur complement means that $\Gamma_k(P, K_k) \leqslant 0$, $k = 1, \ldots, M$ are equivalent to inequalities (8). Therefore, if inequalities (8) are satisfied respectively, the closed-loop stochastic LPV system is mean-square stable.

It should be noted that the feedback gain can be computed by applying the KKT conditions. The details will be discussed later.

3 Problem formulation

Consider the following stochastic LPV system with one leader $u_0(t)$ and N followers $u_i(t)$, $i = 1, \ldots, N$:

$$\mathrm{d}x(t) = \left[A(\eta(t))x(t) + B_0u_0(t) + \sum_{i=1}^{N} B_iu_i(t) + B_vv(t)\right]\mathrm{d}t + A_p(\eta(t))x(t)\mathrm{d}w(t), \tag{17a}$$

$$z(t) = \begin{bmatrix} E(\eta(t))x(t) \\ D_0u_0(t) \\ D_1u_1(t) \\ \vdots \\ D_Nu_N(t) \end{bmatrix}, \tag{17b}$$

$$y_i(t) = C_ix(t), \tag{17c}$$

where $y_i(t) \in \mathbb{R}^{r_i}$ denotes the i-th output vector. $u_0(t) \in \mathbb{R}^{m_0}$ denotes the leader's control input. $u_i(t) \in \mathbb{R}^{m_i}$, $i = 1, \ldots, N$ denotes the i-th follower's control input. In the following, P_0 is defined to represent the leader and P_i, $i = 1, \ldots, N$ to represent the i-th follower. Other variables are defined by stochastic equation (1). It should be noted that D_i is not dependent on time-varying parameters since a controller designer can choose a controlled output. Hence, without loss of generality, the following assumptions are made.

Assumption 1. $D_i^{\mathrm{T}}D_i = I_{m_i}$, $i = 0, 1, \ldots N$, $D_i \in \mathbb{R}^{g_i \times m_i}$.

The cost functionals are defined by

$$J_v(u_0, u_1, \ldots, u_N, v, x^0) = \mathrm{E}\left[\int_0^{\infty} \left[\gamma^2\|v(t)\|^2 - \|z(t)\|^2\right]\mathrm{d}t\right], \tag{18a}$$

$$J_i(u_0, u_1, \ldots, u_N, v, x^0) = \mathrm{E}\left[\int_0^{\infty} \left\{x^{\mathrm{T}}(t)Q_ix(t) + u_i^{\mathrm{T}}(t)R_iu_i(t)\right\}\mathrm{d}t\right], \tag{18b}$$

where $i = 0, 1, \ldots, N$, $Q_i = Q_i^{\mathrm{T}} \geqslant 0$, $R_i = R_i^{\mathrm{T}} > 0$.

The problem of finding the robust SOF Stackelberg strategy with the H_∞ constraint for the stochastic LPV system (17) is formulated as follows.

Problem 1. For any given $\gamma > 0$, find the GS SOF strategies $u_i(t) = u_i^*(t) \in L_F^2([0, \ \infty), \mathbb{R}^{m_i})$, $i = 0, 1, \ldots, N$ such that

(i) When the leader's strategy $u_0(t) = K_0(\eta(t))C_0x(t)$, follower's strategy $u_i(t) = K_i(\eta(t), K_0)C_ix(t)$, $i = 1, \ldots, N$, and a disturbance $v(t) = v^*(t) = F_\gamma^*x(t)$ are applied, a follower's strategy set $(u_1(u_0), \ldots, u_N(u_0))$ satisfies the following Nash equilibrium condition:

$$\bar{J}_i(u_1^\dagger(u_0), \ldots, u_i^\dagger(u_0), \ldots, u_N^\dagger(u_0), v^*, x^0) \leqslant \bar{J}_i(u_1^\dagger(u_0), \ldots, u_i(u_0), \ldots, u_N^\dagger(u_0), v^*, x^0), \tag{19}$$

where $u_i^{\dagger}(u_0)$, $i=1,\ldots,N$ denotes the Nash equilibrium strategy, which depends on the leader strategy $u_0(t)=K_0(\eta(t))x(t)$ and

$$\begin{aligned}
&J_i(u_1(u_0),\ldots,u_N(u_0),v^*,x^0)=\mathrm{E}\left[\int_0^{\infty}\big[x^{\mathrm{T}}(t)Q_ix(t)+u_i^{\mathrm{T}}(u_0,t)R_iu_i(u_0,t)\big]\mathrm{d}t\right]\\
&\quad\leqslant \bar{J}_i(u_1(u_0),\ldots,u_N(u_0),v^*,x^0)=\mathrm{E}\big[x^{\mathrm{T}}(0)P_ix(0)\big],\\
&P_i\left(A_k+B_0K_{0k}C_0+\sum_{j=1}^{N}B_jK_{jk}(K_{0k})C_j+B_vF_{\gamma}^*\right)\\
&\quad+\left(A_k+B_0K_{0k}C_0+\sum_{j=1}^{N}B_jK_{jk}(K_{0k})C_j+B_vF_{\gamma}^*\right)^{\mathrm{T}}P_i\\
&\quad+A_{pk}^{\mathrm{T}}P_iA_{pk}+Q_i+C_i^{\mathrm{T}}K_{ik}^{\mathrm{T}}R_iK_{ik}C_i\leqslant 0;
\end{aligned}$$

(ii) When the worst-case disturbance $v(t)=v^*(t)$ is implemented in a stochastic LPV system (17), $u_i(t)=u_i^*(t)$, $i=1,\ldots,N$ satisfies the following inequality (20):

$$\bar{J}_0(u_0^*,u_1^*,\ldots,u_N^*,v^*,x^0)=\min_{u_0}\bar{J}_0(u_0,u_1^{\dagger}(u_0),\ldots,u_N^{\dagger}(u_0),v^*,x^0), \tag{20}$$

where

$$\begin{aligned}
&u_i^*(t)=u_i^{\dagger}(u_0^*)=K_i^*(\eta(t),K_0^*)C_ix(t),\\
&J_0(u_0,u_1(u_0),\ldots,u_N(u_0),v,x^0)=\mathrm{E}\left[\int_0^{\infty}\big[x^{\mathrm{T}}(t)Q_0x(t)+u_0^{\mathrm{T}}(t)R_0u_0(t)\big]\mathrm{d}t\right]\\
&\quad\leqslant \bar{J}_0(u_1(u_0),\ldots,u_N(u_0),v,x^0)=\mathrm{E}\big[x^{\mathrm{T}}(0)P_0x(0)\big],\\
&P_0\left(A_k+B_0K_{0k}C_0+\sum_{j=1}^{N}B_jK_{jk}(K_{0k})C_j+B_vF_{\gamma}^*\right)\\
&\quad+\left(A_k+B_0K_{0k}C_0+\sum_{j=1}^{N}B_jK_{jk}(K_{0k})C_j+B_vF_{\gamma}^*\right)^{\mathrm{T}}P_0\\
&\quad+A_{pk}^{\mathrm{T}}P_0A_{pk}+Q_0+C_0^{\mathrm{T}}K_{0k}^{\mathrm{T}}R_0K_{0k}C_0\leqslant 0;
\end{aligned}$$

(iii) For the closed-loop stochastic LPV system, the H_{∞} norm conditions are satisfied such that

$$J_v(u_1^*,\ldots,u_N^*,v,x^0)\geqslant 0 \tag{21}$$

and $u_i(t)=u_i^*(t)$, $i=0,1,\ldots,N$.

A critical issue for performing the feedback strategy in practical dynamic games is that players can only access their individual and/or local state information. Although observer-based state estimation techniques are often used when perfect state information cannot be obtained, a state estimator design is required. This makes the controller structure more complicated and costly. However, different players may access different state information. It is desirable to design a controller based on individual and local state information of each player. An SOF strategy with a simple structure is one of the most important solutions in control theory for solving this problem. However, designing an SOF strategy faces the challenge of solving non-convex optimization problems, such as BMI optimization problems. The related BMI optimization problem is a well-known NP-hard problem for its numerical difficulty. To avoid the computational difficulty of finding the robust SOF Stackelberg strategy set, a heuristic algorithm based on the LMI-based optimization problem is newly developed.

A new framework, called "Chemical Game Theory (CGT)", uses well-known, rigorous principles from chemical engineering to solve strategic decision problems [20]. In strategic decisions, players each can choose from among two or more alternative possibilities, and the outcome depends upon the collective choices from all players. Specifically, a game of N players is represented by N reactors in parallel. The concentrations that enter a reactor correspond to the bias that a player enters the game with. The reactions occurring in the reactors are comparable to the decision making process of each player. The concentrations of the final products represent the likelihood of each outcome given the preexisting biases for the situation. In this way, CGT mathematically solves problems that are difficult for humans to adjust.

In the following, we first consider the robust SOF Stackelberg strategy set by using GS control and then the mode-independent control.

4 Gain-scheduled Stackelberg strategy set

In this section, the existence conditions of the GS robust SOF Stackelberg strategy set are derived using the preliminary results.

4.1 Follower's Nash strategy set

First, we consider the follower's Nash strategy set. For a fixed strategy set

$$u_i(t) = K_i(\eta(t), K_0)C_i x(t) = \sum_{k=1}^{M} \delta_k(t) K_{ik} C_i x(t),\ i = 0, 1, \ldots, j-1, j+1, \ldots, N, \tag{22a}$$

$$v(t) = F_\gamma x(t), \tag{22b}$$

consider the following closed-loop stochastic LPV systems with the i-th cost functional:

$$\mathrm{d}x(t) = \left(\left[A(\eta(t)) + \sum_{j=0, j\neq i}^{N} B_j K_j(\eta(t)) C_j + B_v F_\gamma\right] x(t) + B_i u_i(t)\right)\mathrm{d}t + A_p(\eta(t)) x(t)\mathrm{d}w(t), \tag{23a}$$

$$J_i(u_i(K_{0k}), x^0) = \mathrm{E}\left[\int_0^\infty \left[x^{\mathrm{T}}(t) Q_i x(t) + u_i^{\mathrm{T}}(K_{0k}) R_i u_i(K_{0k})\right]\mathrm{d}t\right]. \tag{23b}$$

Using the preliminary Lemma 2 and making a term-wise comparison between (23a), (23b) and (1), (6) with

$$A(\eta(t)) \leftarrow A(\eta(t)) + \sum_{j=0, j\neq i}^{N} B_j K_j(\eta(t)) C_j + B_v F_\gamma,\ B \leftarrow B_i,\ u(t) \leftarrow u_i(t),\ Q \leftarrow Q_i,\ R \leftarrow R_i,$$

we arrive at the following CCMIs:

$$\Phi_{ik}(P_i, K_{0k}, K_{1k}, \ldots, K_{Nk}) \leqslant 0, \quad k = 1, \ldots, M, \tag{24}$$

where

$$\Phi_{ik}(P_i, K_{0k}, K_{1k}, \ldots, K_{Nk}) := P_i \Xi_{\gamma k} + \Xi_{\gamma k}^{\mathrm{T}} P_i + A_{pk}^{\mathrm{T}} P_i A_{pk} + Q_i + C_i^{\mathrm{T}} K_{ik}^{\mathrm{T}} R_i K_{ik} C_i,$$

$$\Xi_{\gamma k} := A_k + \sum_{j=0}^{N} B_j K_{jk} C_j + B_v F_\gamma.$$

Second, the following optimization problem is defined:

$$\begin{aligned} &\min_{\Sigma_i} \mathrm{Tr}\ \left[M_0 P_i\right],\ \Sigma_i := (P_i, K_{i1}, \ldots, K_{iM}) \\ &\text{s.t. } \Phi_{ik}(P_i, K_{0k}, K_{1k}, \ldots, K_{Nk}) \leqslant 0,\ k = 1, \ldots, M, \end{aligned} \tag{25}$$

where, the matrices inside the set of Σ_i mean the optimization variables in this problem. Furthermore, K_{0k}, K_{1k}, $\ldots$, $K_{(i-1)k}$, $K_{(i+1)k}$, $\ldots$, K_{Nk}, $k = 1, \ldots, M$ are fixed.

In this paper, we investigate Stackelberg differential games with a single leader and several followers. The leader globally dominates over the followers for the entire duration in the sense that before the start of the game, he chooses and then announces his strategy to the followers who play a Nash game. The multiple Nash followers choose their optimal strategies noncooperatively and simultaneously based on the leader's observed strategy [5]. Therefore, when considering the follower's optimization problem, fixing the strategy sets of other players and formulating the follower's Nash problem as an individual optimization problem, the optimization problem (25) can be defined. Furthermore, we will show that the strategy set can be calculated independently using an iterative method in the layer section.

In order to avoid the dependence of the initial value, it is assumed that

$$\mathrm{E}[x(0)x^{\mathrm{T}}(0)] = M_0. \tag{26}$$

The optimization problem (25) can be solved using the KKT conditions. The following Lagrangian ℓ_i is defined to establish the necessary conditions:

$$\begin{aligned} \ell_i &:= \ell_i(P_i, G_{i1}, \ldots, G_{iM}, K_{0k}, K_{1k}, \ldots, K_{Nk}) \\ &= \mathrm{Tr}\ \big[M_0 P_i\big] + \sum_{k=1}^{M} \mathrm{Tr}\big[G_{ik}\Phi_{ik}(P_i, K_{0k}, K_{1k}, \ldots, K_{Nk})\big] \\ &= \mathrm{Tr}\ \big[M_0 P_i\big] + \sum_{k=1}^{M} \mathrm{Tr}\big[G_{ik}(P_i\Xi_{\gamma k} + \Xi_{\gamma k}^{\mathrm{T}} P_i + A_{pk}^{\mathrm{T}} P_i A_{pk} + Q_i + C_i^{\mathrm{T}} K_{ik}^{\mathrm{T}} R_i K_{ik} C_i)\big], \end{aligned} \tag{27}$$

where G_{ik}, $k = 1, \ldots, M$ are the symmetric matrix of the Lagrange multipliers.

By using the KKT conditions [21] and the formula $\frac{\partial}{\partial X}\mathrm{Tr}\ \big[AXB\big] = A^{\mathrm{T}}B^{\mathrm{T}}$ where $X = X^{\mathrm{T}}$, the following coupled-matrix equations are established:

$$G_{ik}\Phi_{ik}(P_i, K_{0k}, K_{1k}, \ldots, K_{Nk}) = 0,$$
$$G_{ik} \geqslant 0, \quad \Phi_{ik}(P_i, K_{0k}, K_{1k}, \ldots, K_{Nk}) \leqslant 0, \quad i = 1, \ldots, N, \quad k = 1, \ldots, M, \tag{28a}$$
$$\frac{\partial \ell_i}{\partial P_i} = \Delta_i(K_{01}, \ldots, K_{NM}, G_{i1}, \ldots, G_{iM}) = 0, \quad i = 1, \ldots, N, \tag{28b}$$
$$\frac{1}{2}\cdot\frac{\partial \ell_i}{\partial K_{ik}} = (B_i^{\mathrm{T}} P_i + R_i K_{ik} C_i) G_{ik} C_i^{\mathrm{T}} = 0, \quad i = 1, \ldots, N, \tag{28c}$$

where

$$\Delta_i(K_{01}, \ldots, K_{NM}, G_{i1}, \ldots, G_{iM}) := M_0 + \sum_{k=1}^{M}\big[\Xi_{\gamma k} G_{ik} + G_{ik}\Xi_{\gamma k}^{\mathrm{T}} + A_{pk} G_{ik} A_{pk}^{\mathrm{T}}\big].$$

It is worth noting that the CCMEs (28b) are the generalized stochastic Sylvester equation with many unknown variables. The discussion on the solvability of the generalized stochastic Sylvester equation (28b) is not easy. It is beyond the subject of this paper. Therefore, we assume the solvability of this equation.

If the generalized stochastic Sylvester equation (28b) has a positive definite solution set such that $\sum_{k=1}^{M} C_i G_{ik} C_i^{\mathrm{T}} > 0$, the fixed follower's strategy set can be obtained as follows:

$$u_i(t) = u_i^*(t) = K_i^*(\eta(t), K_0) C_i x(t) = \sum_{k=1}^{M} \delta_k(t) K_{ik}^* C_i x(t), \quad i = 1, \ldots, N, \tag{29}$$

where

$$K_{ik}^* = -R_i^{-1} B_i^{\mathrm{T}} P_i G_{ik} C_i^{\mathrm{T}} \big(C_i G_{ik} C_i^{\mathrm{T}}\big)^{-1}.$$

4.2 Leader's strategy set

Second, consider the leader's strategy set. Assume that the leader uses the strategy with the following form:

$$u_0(t) = K_{0k}(\eta(t)) C_0 x(t) = \sum_{k=1}^{M} \delta_k(t) K_{0k} C_0 x(t). \tag{30}$$

The closed-loop stochastic LPV systems become

$$\mathrm{d}x(t) = \left(\left[A(\eta(t)) + \sum_{j=1}^{N} B_j K_{jk}(\eta(t)) C_j + B_v F_\gamma\right] x(t) + B_0 u_0(t)\right)\mathrm{d}t + A_p(\eta(t)) x(t)\mathrm{d}w(t), \tag{31a}$$

$$J_0(K_{0k}x, K_{1k}(K_{0k})x, \ldots, K_{Nk}(K_{0k})x, v, x^0) = \mathrm{E}\bigg[\int_0^{\infty}\big[x^{\mathrm{T}}(t)Q_0x(t) + u_0^{\mathrm{T}}R_0u_0\big]\mathrm{d}t\bigg]. \tag{31b}$$

By using a technique similar to that for deriving the follower's strategy set, the following CCMIs are derived:

$$\Phi_{0k}(P_0, K_{0k}, K_{1k}, \ldots, K_{Nk}) \leqslant 0, \quad k = 1, \ldots, M, \tag{32}$$

where

$$\Phi_{0k}(P_0, K_{0k}, K_{1k}, \ldots, K_{Nk}) := P_0\Xi_{\gamma k} + \Xi_{\gamma k}^{\mathrm{T}}P_0 + A_{pk}^{\mathrm{T}}P_0A_{pk} + Q_0 + C_0^{\mathrm{T}}K_{0k}^{\mathrm{T}}R_0K_{0k}C_0.$$

Therefore, the optimization problem for the leader's cost bound can be defined:

$$\begin{aligned}&\min_{\Sigma_0}\mathrm{Tr}\ \big[M_0P_0\big],\ \Sigma_0 := (P_0, P_1, \ldots, P_N, K_{01}, \ldots, K_{NM}, G_{11}, \ldots, G_{NM})\\ &\text{s.t. } \Phi_{0k}(P_0, K_{0k}, K_{1k}, \ldots, K_{Nk}) \leqslant 0,\ k = 1, \ldots, M,\ \text{and (28)},\end{aligned} \tag{33}$$

where the matrices inside the set of Σ_0 mean the optimization variables in this problem.

It should be noted that for each fixed predetermined follower's optimal strategy set satisfying the Nash equilibrium condition, the leader's optimization problem can be defined.

In order to derive the necessary conditions, the following Lagrangian ℓ_0 is defined:

$$\begin{aligned}\ell_0 &:= \ell_0(P_0, P_1, \ldots, P_N, G_{01}, \ldots, G_{NM}, K_{01}, \ldots, K_{NM}, U_{11}, \ldots, U_{NM})\\ &= \mathrm{Tr}\ \big[M_0P_0\big] + \sum_{k=1}^{M}\mathrm{Tr}\big[G_{0k}\Phi_{0k}(P_0, K_{0k}, K_{1k}, \ldots, K_{Nk})\big]\\ &\quad + \sum_{j=1}^{N}\sum_{k=1}^{M}\mathrm{Tr}\big[U_{jk}G_{jk}\Phi_{jk}(P_j, K_{0k}, K_{1k}, \ldots, K_{Nk})\big]\\ &\quad + \sum_{j=1}^{N}\mathrm{Tr}\big[V_j\Delta_j(K_{01}, \ldots, K_{NM}, G_{j1}, \ldots, G_{jM})\big]\\ &\quad + \sum_{j=1}^{N}\sum_{k=1}^{M}\mathrm{Tr}\Big[\big(B_j^{\mathrm{T}}P_j + R_jK_{jk}C_j\big)G_{jk}C_j^{\mathrm{T}}W_{jk}\Big],\end{aligned} \tag{34}$$

where U_{ik}, V_i, W_{ik}, G_{0k}, $k = 1, \ldots, M$ are the symmetric matrices of the Lagrange multipliers.

Using the KKT conditions [21], the following coupled-matrix equations can be established.

$$\begin{aligned}&G_{0k}\Phi_{0k}(P_0, K_{0k}, K_{1k}, \ldots, K_{Nk}) = 0,\\ &G_{0k} \geqslant 0,\ \Phi_{0k}(P_0, K_{0k}, K_{1k}, \ldots, K_{Nk}) \leqslant 0, \quad k = 1, \ldots, M,\end{aligned} \tag{35a}$$

$$\frac{\partial \ell_0}{\partial P_i} = \sum_{k=1}^{M}\big[\Xi_{\gamma k}G_{ik}U_{ik} + G_{ik}U_{ik}\Xi_{\gamma k}^{\mathrm{T}} + A_{pk}G_{ik}U_{ik}A_{pk}^{\mathrm{T}} + B_iW_{ik}^{\mathrm{T}}C_iG_{ik}\big] = 0,\ i = 1, \ldots, N, \tag{35b}$$

$$\begin{aligned}\frac{\partial \ell_0}{\partial G_{ik}} &= \Phi_{ik}(P_i, K_{0k}, K_{1k}, \ldots, K_{Nk})U_{ik} + V_i\Xi_{\gamma k} + \Xi_{\gamma k}^{\mathrm{T}}V_i + A_{pk}^{\mathrm{T}}V_iA_{pk}\\ &\quad + \big(B_i^{\mathrm{T}}P_i + R_iK_{ik}C_i\big)^{\mathrm{T}}W_{ik}^{\mathrm{T}}C_i = 0, \quad i = 1, \ldots, N, \quad k = 1, \ldots, M,\end{aligned} \tag{35c}$$

$$\frac{\partial \ell_0}{\partial P_0} = M_0 + \sum_{k=1}^{M}\big[\Xi_{\gamma k}G_{0k} + G_{0k}\Xi_{\gamma k}^{\mathrm{T}} + A_{pk}G_{0k}A_{pk}^{\mathrm{T}}\big] = 0, \tag{35d}$$

$$\begin{aligned}\frac{1}{2}\cdot\frac{\partial \ell_0}{\partial K_{ik}} &= \big(B_i^{\mathrm{T}}P_0G_{0k} + R_iK_{ik}C_iG_{ik}U_{ik}\big)C_i^{\mathrm{T}} + \sum_{j=1}^{N}B_i^{\mathrm{T}}\big(P_jG_{jk}U_{jk} + V_jG_{jk}\big)C_i^{\mathrm{T}}\\ &\quad + R_iW_{ik}^{\mathrm{T}}C_iG_{ik}C_i^{\mathrm{T}} = 0, \quad i = 1, \ldots, N, \quad k = 1, \ldots, M,\end{aligned} \tag{35e}$$

$$\frac{1}{2}\cdot\frac{\partial \ell_0}{\partial K_{0k}} = \big(B_0^{\mathrm{T}}P_0 + R_0K_{0k}C_0\big)G_{0k}C_0^{\mathrm{T}} + \sum_{j=1}^{N}B_0^{\mathrm{T}}\big(P_jG_{jk}U_{jk} + V_jG_{jk}\big)C_0^{\mathrm{T}} = 0, \quad k = 1, \ldots, M. \tag{35f}$$

If the generalized stochastic Sylvester equation (35d) have a positive definite solution set such that $\sum_{k=1}^{M} C_0 G_{0k} C_0^{\mathrm{T}} > 0$, the leader's strategy set can be obtained as follows.

$$u_0(t) = K_0^*(\eta(t))C_i x(t) = \sum_{k=1}^{M} \delta_k(t) K_{0k}^* C_i x(t), \tag{36}$$

where

$$K_{0k}^* = -R_0^{-1} B_0^{\mathrm{T}} \left[P_0 G_{0k} + \sum_{j=1}^{N} \left(P_j G_{jk} U_{jk} + V_j G_{jk} \right) \right] C_0^{\mathrm{T}} \left(C_0 G_{0k} C_0^{\mathrm{T}} \right)^{-1}.$$

It should be noted that the CCMEs (35d) are the same generalized stochastic Sylvester equations as (28). Additionally, we assume the solvability of this equation.

4.3 Disturbance attenuation condition

Finally, the disturbance attenuation condition is investigated using Lemma 2. If the strategy sets $u_0(t) = u_0^*(t) = K_0^*(\eta(t))C_0 x(t)$, $u_i(t) = u_i^*(t) = K_i^*(\eta(t))C_i x(t)$, $i = 1, \ldots, N$ exist, the following stochastic LPV system can be considered:

$$\mathrm{d}x(t) = \left[\left(A(\eta(t)) + \sum_{j=0}^{N} B_j K_j^*(\eta(t)) C_j \right) x(t) + B_v v(t) \right] \mathrm{d}t + A_p(\eta(t)) x(t) \mathrm{d}w(t), \tag{37a}$$

$$z(t) = \begin{bmatrix} E(\eta(t)) \\ D_0 K_0(\eta(t)) \\ D_1 K_1(\eta(t)) \\ \vdots \\ D_N K_N(\eta(t)) \end{bmatrix} x(t). \tag{37b}$$

By comparing (1) with (37), the following conditions can be easily established.

For a given disturbance attenuation level $\gamma > 0$, assume that there exists a real symmetric matrix $Y = Y^{\mathrm{T}} > 0$ such that the following LMIs are satisfied:

$$\begin{bmatrix} Y\Xi_{Kk} + \Xi_{Kk}^{\mathrm{T}} Y & Y B_v & A_{pk}^{\mathrm{T}} Y & E_{Kk}^{\mathrm{T}} \\ B_v^{\mathrm{T}} Y & -\gamma^2 I_{n_v} & 0 & 0 \\ Y A_{pk} & 0 & -Y & 0 \\ E_{Kk} & 0 & 0 & -I_{n_z} \end{bmatrix} < 0, \quad k = 1, \ldots, M, \tag{38}$$

where

$$\Xi_{Kk} := A_k + \sum_{j=0}^{N} B_j K_{jk}^* C_j, \quad E_{Kk} := \begin{bmatrix} E_k \\ D_0 K_{0k}^* \\ D_1 K_{1k}^* \\ \vdots \\ D_N K_{Nk}^* \end{bmatrix}.$$

Then, the worst-case disturbance can be calculated as follows:

$$v(t) = F_\gamma^* x(t) = \gamma^{-2} B_v^{\mathrm{T}} Y x(t). \tag{39}$$

4.4 Existence condition

We are now in a position to state the main result of this paper.

Theorem 1. Suppose that the CCMEs (29) with $C_iG_{ik}C_i^{\mathrm{T}} > 0$ and CCMEs (35) with $C_0G_{0k}C_0^{\mathrm{T}} > 0$ admit a solution set, respectively. Then, the strategy sets (29), (36) and the worst-case disturbance (39) are the solutions to Problem 1. Furthermore, these strategy sets satisfy inequalities (19) and (21).

Proof. The detailed proof of this theorem is omitted as the proof can be completed by tracing the procedure before the theorem.

5 Mode-independent Stackelberg strategy set

In Section 4, mode-dependent (or GS) strategy sets were used. However, because it is complicated and costly to implement the mode-dependent strategies, mode-independent strategy sets have become more attractive and effective. In this section, we derive mode-independent strategy sets.

Assume the players adopt the following mode-independent strategy sets:

$$u_i(t) = K_iC_ix(t), \quad i = 0, 1, \ldots, N, \tag{40a}$$

$$v(t) = F_\gamma x(t). \tag{40b}$$

By using the same techniques as those in Section 4, the following CCMEs can be derived:

$$H_{ik}\Psi_{ik}(X_i, K_0, K_1, \ldots, K_N) = 0, \quad H_{ik} \geqslant 0, \quad \Psi_{ik}(X_i, K_0, K_1, \ldots, K_N) \leqslant 0, \quad i = 1, \ldots, N, \quad k = 1, \ldots, M, \tag{41a}$$

$$M_0 + \sum_{k=1}^{M}\big[\Theta_{\gamma k}H_{ik} + H_{ik}\Theta_{\gamma k}^{\mathrm{T}} + A_{pk}H_{ik}A_{pk}^{\mathrm{T}}\big] = 0, \quad i = 1, \ldots, N, \quad k = 1, \ldots, M, \tag{41b}$$

$$H_{0k}\Psi_{0k}(X_0, K_0, K_1, \ldots, K_N) = 0, \quad H_{0k} \geqslant 0, \quad \Psi_{0k}(X_i, K_0, K_1, \ldots, K_N) \leqslant 0, \quad i = 1, \ldots, N, \tag{41c}$$

$$M_0 + \sum_{k=1}^{M}\big[\Theta_{\gamma k}H_{0k} + H_{0k}\Theta_{\gamma k} + A_{pk}H_{0k}A_{pk}^{\mathrm{T}}\big] = 0, \tag{41d}$$

$$\sum_{k=1}^{M}\big[\Theta_{\gamma k}H_{ik}T_{ik} + H_{ik}T_{ik}\Theta_{\gamma k}^{\mathrm{T}} + A_{pk}H_{ik}T_{ik}A_{pk}^{\mathrm{T}} + B_iW_i^{\mathrm{T}}C_iH_{ik}\big] = 0, \quad i = 1, \ldots, N, \tag{41e}$$

$$\Psi_{ik}(X_i, K_0, K_1, \ldots, K_N)T_{ik} + Z_i\Theta_{\gamma k} + \Theta_{\gamma k}^{\mathrm{T}}Z_i + A_{pk}^{\mathrm{T}}Z_iA_{pk} + (B_i^{\mathrm{T}}X_i + R_iK_iC_i)^{\mathrm{T}}W_i^{\mathrm{T}}C_i = 0, \quad i = 1, \ldots, N, \quad k = 1, \ldots, M, \tag{41f}$$

$$\sum_{k=1}^{M}\big(B_i^{\mathrm{T}}X_0H_{0k} + R_iK_iC_iH_{ik}T_{ik}\big)C_i^{\mathrm{T}} + \sum_{j=1}^{N}\sum_{k=1}^{M}B_i^{\mathrm{T}}\big(X_jH_{jk}T_{jk} + Z_jH_{jk}\big)C_i^{\mathrm{T}} + R_iW_i^{\mathrm{T}}C_i\left(\sum_{k=1}^{M}H_{ik}\right)C_i^{\mathrm{T}} = 0, \quad i = 1, \ldots, N, \tag{41g}$$

where

$$\Psi_{ik}(X_i, K_0, K_1, \ldots, K_N) := X_i\Theta_{\gamma k} + \Theta_{\gamma k}^{\mathrm{T}}X_i + A_{pk}^{\mathrm{T}}X_iA_{pk} + Q_i + C_i^{\mathrm{T}}K_i^{\mathrm{T}}R_iK_iC_i,$$

$$\Theta_{\gamma k} := A_k + \sum_{j=0}^{N}B_jK_jC_j + B_vF_\gamma.$$

If the CCMEs (41) have a positive definite solution set such that $\sum_{k=1}^{M}C_iH_{ik}C_i^{\mathrm{T}} > 0$, $i = 0, 1, \ldots, N$ and $\sum_{k=1}^{M}C_0H_{0k}C_0^{\mathrm{T}} > 0$, the strategy set for each player can be obtained as follows:

$$u_i(t) = K_i^*C_ix(t), \quad i = 0, 1, \ldots, N, \tag{42}$$

where

$$K_i^* = -R_i^{-1}B_i^{\mathrm{T}}X_i\left(\sum_{k=1}^{M}H_{ik}\right)C_i^{\mathrm{T}}\left(\sum_{k=1}^{M}C_iH_{ik}C_i^{\mathrm{T}}\right)^{-1}, \quad i=1,\dots,N,$$

$$K_0^* = -R_0^{-1}B_0^{\mathrm{T}}\left[X_0\sum_{k=1}^{M}H_{0k}+\sum_{j=1}^{N}\sum_{k=1}^{M}\big(X_jH_{jk}T_{jk}+Z_jH_{jk}\big)\right]C_0^{\mathrm{T}}\left(\sum_{k=1}^{M}C_0H_{0k}C_0^{\mathrm{T}}\right)^{-1}.$$

The next theorem guarantees the existence of the mode-independent robust SOF Stackelberg strategy set.

Theorem 2. Assume that the CCMEs (41) with $\sum_{k=1}^{M}C_iH_{ik}C_i^{\mathrm{T}}>0$ and $\sum_{k=1}^{M}C_0H_{0k}C_0^{\mathrm{T}}>0$ admit a solution set. Then, the strategy set (42) and the worst-case disturbance (39) are the solutions to Problem 1. Furthermore, these strategy sets satisfy inequalities (19) and (21).

Proof. The detailed proof is omitted because it can be proved by using a procedure similar to that in Section 4.

6 Numerical algorithm

In this section, a heuristic algorithm for solving CCMEs is presented. Because the CCMEs (35) in Section 4 are the same as the CCMEs (41), only the numerical algorithm for solving CCMEs (41) is considered.

Step 1. Set the initial conditions: choose $K_i^{(0)}$, $i=0,1,\dots,N$ and $F_\gamma^{(0)}$ such that the following closed-loop stochastic LPV system is mean-square stable:

$$\mathrm{d}x(t)=\left[A(\eta(t))+\sum_{i=0}^{N}B_iK_i^{(0)}+B_vF_\gamma^{(0)}\right]x(t)\mathrm{d}t+A_p(\eta(t))x(t)\mathrm{d}w(t). \tag{43}$$

Furthermore, $T_{ik}^{(0)}=I_n$, $Z_i^{(0)}=I_n$, $i=1,\dots,N$, $k=1,\dots,M$.

Step 2. For all i, $i=1,\dots,N$, carry out the following steps from Steps 2-1 to 2-3, where $\rho=0,1,2,\dots$.

Step 2-1. Solve the following optimization problem for $X_i^{(\rho+1)}$, subject to the LMI:

$$\min_{X_i^{(\rho+1)}}\ \mathrm{Tr}\left[M_0X_i^{(\rho+1)}\right]\quad \text{s.t. } \Psi_{ik}(X_i^{(\rho+1)},K_0^{(\rho)},K_1^{(\rho)},\dots,K_N^{(\rho)})\leqslant 0,\quad k=1,\dots,M. \tag{44}$$

Step 2-2. Solve the following CCMEs for $H_{ik}^{(\rho+1)}$, $k=1,\dots,M$:

$$H_{ik}^{(\rho+1)}\Psi_{ik}(X_i^{(\rho+1)},K_0^{(\rho)},K_1^{(\rho)},\dots,K_N^{(\rho)})=0, \tag{45a}$$

$$M_0+\sum_{k=1}^{M}\big[\Theta_{\gamma k}^{(\rho)}H_{ik}^{(\rho+1)}+H_{ik}^{(\rho+1)}\Theta_{\gamma k}^{(\rho)\mathrm{T}}+A_{pk}H_{ik}^{(\rho+1)}A_{pk}^{\mathrm{T}}\big]=0, \tag{45b}$$

where

$$\Theta_{\gamma k}^{(\rho)}:=A_k+\sum_{j=0}^{N}B_jK_j^{(\rho)}C_j+B_vF_\gamma^{(\rho)}.$$

Step 2-3. If $\sum_{k=1}^{M}C_iH_{ik}^{(\rho+1)}C_i^{\mathrm{T}}>0$, then compute $K_i^{(\rho+1)}$:

$$K_i^{(\rho+1)}=-R_i^{-1}B_i^{\mathrm{T}}X_i^{(\rho+1)}\left(\sum_{k=1}^{M}H_{ik}^{(\rho+1)}\right)C_i^{\mathrm{T}}\left(\sum_{k=1}^{M}C_iH_{ik}^{(\rho+1)}C_i^{\mathrm{T}}\right)^{-1}. \tag{46}$$

Step 3. Solve the following optimization problem for $X_0^{(\rho+1)}$ subject to the LMI:

$$\min_{X_0^{(\rho+1)}}\ \mathrm{Tr}\left[M_0X_0^{(\rho+1)}\right]\quad \text{s.t. } \Psi_{0k}(X_0^{(\rho+1)},K_0^{(\rho)},K_1^{(\rho)},\dots,K_N^{(\rho)})\leqslant 0,\quad k=1,\dots,M. \tag{47}$$

Step 4. Solve the following CCMEs for $H_{0k}^{(\rho+1)}$, $k=1,\ldots,M$:

$$H_{0k}^{(\rho+1)}\Psi_{0k}(X_0^{(\rho+1)},K_0^{(\rho)},K_1^{(\rho)},\ldots,K_N^{(\rho)})=0, \tag{48a}$$

$$M_0+\sum_{k=1}^{M}\big[\Theta_{\gamma k}^{(\rho)}H_{0k}^{(\rho+1)}+H_{0k}^{(\rho+1)}\Theta_{\gamma k}^{(\rho)\mathrm{T}}+A_{pk}H_{0k}^{(\rho+1)}A_{pk}^{\mathrm{T}}\big]=0. \tag{48b}$$

Step 5. Compute $W_i^{(\rho+1)}$, $i=1,\ldots,M$:

$$W_i^{(\rho+1)\mathrm{T}}=-R_i^{-1}\left(\sum_{k=1}^{M}(B_i^{\mathrm{T}}X_0^{(\rho+1)}H_{0k}^{(\rho+1)}+\sum_{j=1}^{N}\sum_{k=1}^{M}B_i^{\mathrm{T}}\big(X_j^{(\rho+1)}H_{jk}^{(\rho+1)}T_{jk}^{(\rho)}+Z_j^{(\rho)}H_{jk}^{(\rho+1)}\big)\right.$$
$$\left.+R_iK_i^{(\rho+1)}C_iH_{ik}^{(\rho+1)}T_{ik}^{(\rho)}\right)C_i^{\mathrm{T}}\left(\sum_{k=1}^{M}C_iH_{ik}^{(\rho+1)}C_i^{\mathrm{T}}\right)^{-1}. \tag{49}$$

Step 6. Solve the following CCMEs for $T_{ik}^{(\rho+1)}$, $Z_i^{(\rho+1)}$, $i=1,\ldots,N$, $k=1,\ldots,M$:

$$\sum_{k=1}^{M}\big[\Theta_{\gamma k}^{(\rho)}H_{ik}^{(\rho+1)}T_{ik}^{(\rho+1)}+H_{ik}^{(\rho+1)}T_{ik}^{(\rho+1)}\Theta_{\gamma k}^{(\rho)\mathrm{T}}+A_{pk}H_{ik}^{(\rho+1)}T_{ik}^{(\rho+1)}A_{pk}^{\mathrm{T}}$$
$$+B_iW_i^{(\rho+1)\mathrm{T}}C_iH_{ik}^{(\rho+1)}\big]=0, \tag{50a}$$

$$\Psi_{ik}(X_i^{(\rho+1)},K_0^{(\rho)},K_1^{(\rho)},\ldots,K_N^{(\rho)})T_{ik}^{(\rho+1)}+Z_i^{(\rho+1)}\Theta_{\gamma k}^{(\rho)}+\Theta_{\gamma k}^{(\rho)\mathrm{T}}Z_i^{(\rho+1)}+A_{pk}^{\mathrm{T}}Z_i^{(\rho+1)}A_{pk}$$
$$+(B_i^{\mathrm{T}}X_i^{(\rho+1)}+R_iK_i^{(\rho+1)}C_i)^{\mathrm{T}}W_i^{(\rho+1)\mathrm{T}}C_i=0. \tag{50b}$$

Step 7. If $\sum_{k=1}^{M}C_0H_{0k}^{(\rho+1)}C_0^{\mathrm{T}}>0$, then compute $K_0^{(\rho+1)}$:

$$K_0^{(\rho+1)}=-R_0^{-1}B_0^{\mathrm{T}}\left[X_0^{(\rho+1)}\sum_{k=1}^{M}H_{0k}^{(\rho+1)}+\sum_{j=1}^{N}\sum_{k=1}^{M}\big(X_j^{(\rho+1)}H_{jk}^{(\rho+1)}T_{jk}^{(\rho+1)}+Z_j^{(\rho+1)}H_{jk}^{(\rho+1)}\big)\right]C_0^{\mathrm{T}}$$
$$\times\left(\sum_{k=1}^{M}C_0H_{0k}^{(\rho+1)}C_0^{\mathrm{T}}\right)^{-1}. \tag{51}$$

Step 8. Solve the following optimization problem for $Y^{(\rho+1)}$ subject to the LMI:

$$\min_{Y^{(\rho+1)}}\ \mathrm{Tr}\ \big[Y^{(\rho+1)}\big]$$
$$\text{s.t.}\ \begin{bmatrix}Y^{(\rho+1)}\Theta_{Kk}^{(\rho+1)}+\Theta_{Kk}^{(\rho+1)\mathrm{T}}Y^{(\rho+1)} & Y^{(\rho+1)}B_v & A_{pk}^{\mathrm{T}}Y^{(\rho+1)} & E_{Kk}^{(\rho+1)\mathrm{T}}\\ B_v^{\mathrm{T}}Y^{(\rho+1)} & -\gamma^2I_{n_v} & 0 & 0\\ Y^{(\rho+1)}A_{pk} & 0 & -Y^{(\rho+1)} & 0\\ E_{Kk}^{(\rho+1)\mathrm{T}} & 0 & 0 & -I_{n_z}\end{bmatrix}<0,\quad k=1,\ldots,M, \tag{52}$$

where

$$\Theta_{Kk}^{(\rho)}:=A_k+\sum_{j=0}^{N}B_jK_j^{(\rho)}C_j,\quad E_{Kk}^{(\rho)}=\begin{bmatrix}E_k\\ D_0K_0^{(\rho)}C_0\\ D_1K_1^{(\rho)}C_1\\ \vdots\\ D_NK_N^{(\rho)}C_N\end{bmatrix}.$$

Step 9. Set

$$\boldsymbol{z}^{(\rho+1)}=(1-c_\rho)\boldsymbol{z}^{(\rho)}+c_\rho\mathscr{T}(\boldsymbol{z}^{(\rho)}),\quad \rho=0,1,2,\ldots, \tag{53}$$

where $\boldsymbol{z}^{(\rho+1)} = \mathscr{T}(\boldsymbol{z}^{(\rho)})$ with

$$
\begin{aligned}
& 0 \leqslant c_\rho \leqslant 1, \quad \sum_{\rho=1}^{\infty} c_\rho(1-c_\rho) = \infty, \\
& \boldsymbol{z}^{(\rho)} = \left[\boldsymbol{X}^{(\rho)}\ \boldsymbol{K}^{(\rho)}\ \boldsymbol{H}^{(\rho)}\ \boldsymbol{T}^{(\rho)}\ \boldsymbol{Z}^{(\rho)}\ \boldsymbol{W}^{(\rho)}\ Y^{(\rho)}\right], \\
& \boldsymbol{X}^{(\rho)} = \left(X_0^{(\rho)}\ X_1^{(\rho)}\ \cdots\ X_N^{(\rho)}\right), \ \boldsymbol{K}^{(\rho)} = \left(K_0^{(\rho)}\ K_1^{(\rho)}\ \cdots\ K_N^{(\rho)}\ \right), \\
& \boldsymbol{H}^{(\rho)} = \left(H_{01}^{(\rho)}\ \cdots\ H_{0M}^{(\rho)}\ H_{11}^{(\rho)}\ \cdots\ H_{1M}^{(\rho)}\ \cdots\ H_{N1}^{(\rho)}\ \cdots\ H_{NM}^{(\rho)}\right), \\
& \boldsymbol{T}^{(\rho)} = \left(T_{01}^{(\rho)}\ \cdots\ T_{0M}^{(\rho)}\ T_{11}^{(\rho)}\ \cdots\ T_{1M}^{(\rho)}\ \cdots\ T_{N1}^{(\rho)}\ \cdots\ T_{NM}^{(\rho)}\right), \\
& \boldsymbol{Z}^{(\rho)} = \left(Z_1^{(\rho)}\ \cdots\ Z_N^{(\rho)}\right), \ \boldsymbol{W}^{(\rho)} = \left(W_1^{(\rho)}\ \cdots\ W_N^{(\rho)}\right).
\end{aligned}
$$

Step 10. If the recursive algorithm consisting of Steps 2 to 9 converges, the required strategy set is obtained; otherwise, if the number of iterations reaches a preset threshold, declare that there is no strategy set. Stop.

Finally, the following convergence characteristics can be concluded in terms of the well-known results of [18].

Theorem 3. Let C be a closed convex subset of a uniformly convex Banach space. If $\mathscr{T}: C \to C$ is a nonexpansive mapping with a fixed point, then $\{\boldsymbol{z}^{(\rho)}\}$ converges weakly to a fixed point.

Proof. The proof is omitted because it is derived directly from the result of Theorem 2 [18] under the assumption of the sequence $\{c_\rho\}_{\rho=1}^{\infty}$.

It should be noted that $c_\rho = 1/\rho$ can be chosen simply, and the numerical section of Section 7 is selected as it is.

7 Numerical examples

In order to demonstrate the reliability and usefulness of the proposed heuristic algorithm, two numerical examples are solved.

7.1 Academic example

Let us consider the following values for the coefficient and weighted matrices in (17) and (18), respectively:

$$
\begin{aligned}
& M = 2,\ N = 2,\ A_1 = \begin{bmatrix} -1 & 0 \\ -4 & -5 \end{bmatrix},\ A_2 = \begin{bmatrix} -1 & 0 \\ -1 & -5 \end{bmatrix},\ A_{pk} = 0.3A_k,\ k = 1, 2, \\
& B_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.1 \\ 0.8 \end{bmatrix},\ B_v = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},\ x(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix},\ C_0 = \begin{bmatrix} 1 & 0 \end{bmatrix},\ C_1 = C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \\
& E_1 = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix},\ E_2 = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix},\ G_0 = G_1 = G_2 = 1, \\
& Q_0 = \mathrm{diag}\left(4\ 2\right),\ Q_1 = \mathrm{diag}\left(2\ 1\right),\ Q_2 = \mathrm{diag}\left(1.5\ 0.5\right),\ R_0 = 3,\ R_1 = 2,\ R_2 = 1,\ \gamma = 4.
\end{aligned}
$$

Using the heuristic algorithm in Section 6, the mode-dependent SOF feedback gains K_{ik}^*, $i = 0, 1, 2$, $k = 1, 2$ in (29), (36) and F_γ^* in (5) are computed as follows:

$$
\begin{aligned}
& K_{01}^* = \left[-6.7177 \times 10^{-1}\right], \quad K_{02}^* = \left[-6.0638 \times 10^{-1}\right], \\
& K_{11}^* = \left[-9.3002 \times 10^{-2}\right], \quad K_{12}^* = \left[-5.8561 \times 10^{-2}\right], \\
& K_{21}^* = \left[-3.4615 \times 10^{-3}\right], \quad K_{22}^* = \left[-7.2467 \times 10^{-2}\right],
\end{aligned}
$$

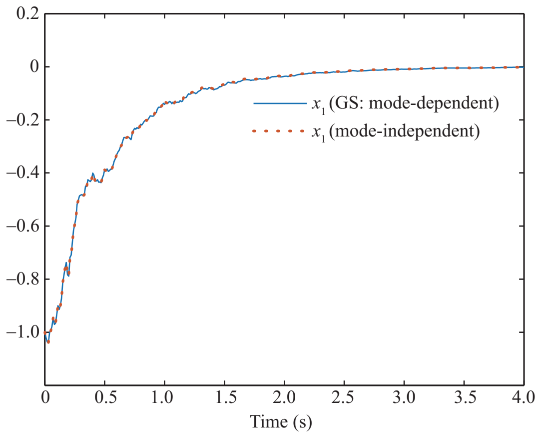

Figure 1 (Color online) State trajectories for $x_1(t)$.

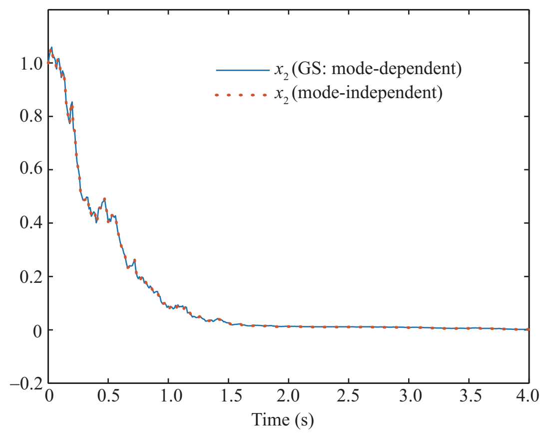

Figure 2 (Color online) State trajectories for $x_2(t)$.

$$F_{\gamma}^{*} = \begin{bmatrix} -1.0301 \times 10^{-4} & 9.1647 \times 10^{-4} \end{bmatrix}.$$

The mode-independent SOF feedback gains K_i^*, $i = 0, 1, 2$ in (42) and F_γ are also computed as follows:

$$K_0^* = \begin{bmatrix} -6.6865 \times 10^{-1} \end{bmatrix}, \quad K_1^* = \begin{bmatrix} -7.9350 \times 10^{-2} \end{bmatrix}, \quad K_2^* = \begin{bmatrix} -2.9529 \times 10^{-2} \end{bmatrix},$$
$$F_{\gamma}^{*} = \begin{bmatrix} -9.9391 \times 10^{-5} & 9.1909 \times 10^{-4} \end{bmatrix}.$$

Using the GS Stackelberg strategy set (mode-dependent Stackelberg strategy set) and mode-independent Stackelberg strategy set, the time histories under $\delta_1(t) = \cos^2 t$ and $\delta_2(t) = \sin^2 t$ for the state are depicted as Figures 1 and 2.

It should be noted that the increment $w(t+h) - w(t)$, $w(0) = 0$ is with the property $\mathrm{N}(0, h)$, $h = 0.01$. From these figures, it is observed that the difference between these strategy sets is very small, and the mode-independent strategy set is more suitable and realistic because the implementation is very easy.

Figure 3 shows the result of multiplying the variance h by 100 in order to display the comparison with the previous results better. It should be noted here that state $x_2(t)$ is oscillatory because the variance to 100 times in the original one-dimensional Wiener process is performed. However, it is confirmed that the mode-independent control strategy quickly compensates to the equilibrium point.

7.2 Williams-Otto process

In this subsection, a numerical example of the linearized hierarchical model of a chemical refining process [22] is presented as a practical case. Without loss of generality, $u_0(t)$ is assumed to be the leader of the control input. In contrast, $u_i(t)$, $i = 1, 2$ are assumed to be the followers' control input. A detailed description of the system and parameters is given in [23]. The system matrices are given below:

$$M = 2, \ N = 2,$$
$$A_1 = \begin{bmatrix} -4.93 & -1.01 & 0 & 0 \\ -3.20 & -5.30 & -12.8 & 0 \\ 6.40 & 0.347 & -32.5 & -1.04 \\ 0 & 0.833 & 11 & -3.96 \end{bmatrix}, \ A_2 = \begin{bmatrix} -4.93 & -1.01 & 0 & 0 \\ -3.20 & -5.30 & -12.8 & 0 \\ 6.40 & 0.347 & -27.5 & -1.04 \\ 0 & 0.833 & 5 & -3.96 \end{bmatrix},$$
$$A_{pk} = 0.15A_k, \ k = 1, 2, \ B_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ B_v = \begin{bmatrix} 0.1 \\ -0.2 \\ -1 \\ 0.5 \end{bmatrix}, \ x(0) = \begin{bmatrix} 0.4 \\ 1.2 \\ -0.5 \\ 0.5 \end{bmatrix},$$
$$C_0 = C_1 = C_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \ E_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}, \ E_2 = \begin{bmatrix} 0 & 2 & 0 & 2 \end{bmatrix}, \ G_0 = G_1 = G_2 = 1.$$

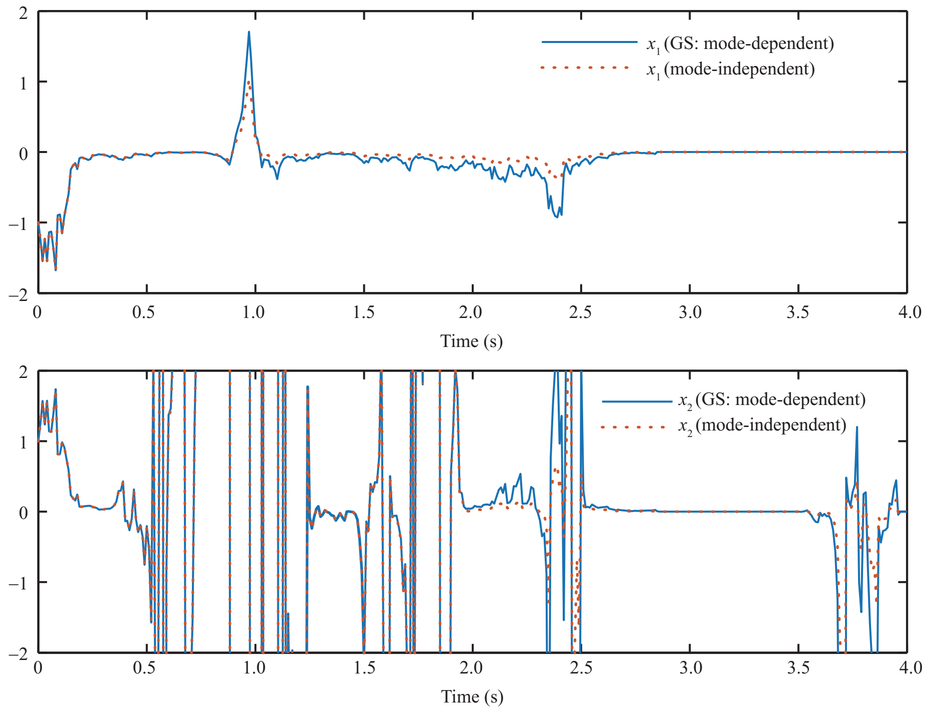

Figure 3 (Color online) State trajectories for a large noise.

The weight matrices are given by

$$Q_0 = \mathrm{diag}\left(4\ 2\ 2\ 5\right),\ Q_1 = \mathrm{diag}\left(2\ 1\ 1\ 3\right),\ Q_2 = \mathrm{diag}\left(1.5\ 0.5\ 1.5\ 1\right),\ R_0 = 3,\ R_1 = 2,\ R_2 = 4.$$

The disturbance attenuation level γ is chosen as four. The gains of the GS Stackelberg strategy set in (29), (36) are computed after 22 iterations. These values are given below.

$$\begin{aligned} K_{01}^* &= \left[-4.6159 \times 10^{-2}\right], \quad K_{02}^* = \left[5.1076 \times 10^{-3}\right], \\ K_{11}^* &= \left[-1.0771 \times 10^{-2}\right], \quad K_{12}^* = \left[-3.8137 \times 10^{-2}\right], \\ K_{21}^* &= \left[-5.4961 \times 10^{-3}\right], \quad K_{22}^* = \left[-5.8009 \times 10^{-3}\right], \\ F_\gamma^* &= \left[-6.4811 \times 10^{-3}\ 1.6257 \times 10^{-2}\ -9.9784 \times 10^{-3}\ 1.6676 \times 10^{-2}\right]. \end{aligned}$$

Conversely, the mode-independent SOF Stackelberg strategy set cannot be obtained because of the infeasibility of the optimization problem for the leader. Therefore, although the mode-independent SOF Stackelberg strategy has easy implementation as a feature, its solvability is conservative. The state trajectories are shown in Figure 4, where $\delta_1(t) = \cos^2 t$, $\delta_2(t) = \sin^2 t$. From this result, even if the SOF strategy is implemented and parameter variations exist, all the states are mean-square stable.

8 Conclusion

A robust SOF Stackelberg game with multiple followers for stochastic LPV systems was investigated. In view of practical implementation, it is assumed that players can access only their individual SOF information in the game. Two types of strategy sets, that is, the GS (or mode-dependent) strategy set and the mode-independent strategy set, were studied. After the existence conditions of the robust SOF Stackelberg strategy set were established, the optimization problems corresponding to the relevant cost bounds were defined to determine the robust SOF Stackelberg strategy set, which were solved using the KKT conditions. Consequently, it is shown that the robust SOF Stackelberg strategy set can be obtained by solving higher-order CCMEs. Because CCMEs are complex and difficult to solve numerically, a

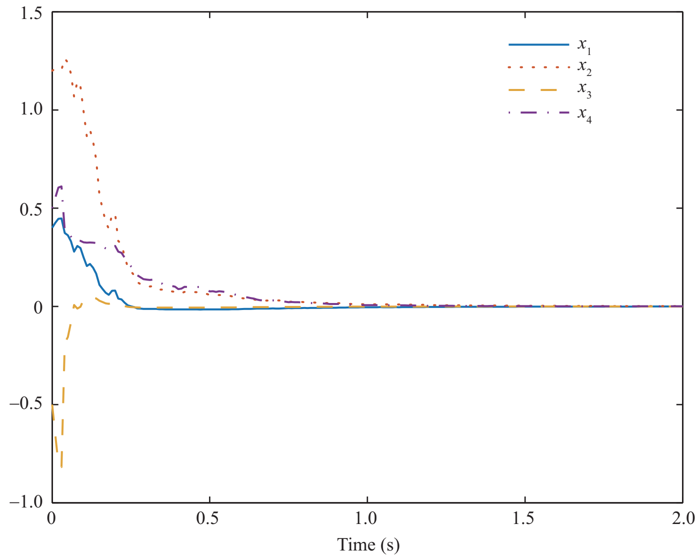

Figure 4 (Color online) State trajectories.

heuristic algorithm was developed by combining CCMEs with CCMIs. The convergence property was proved using the Krasnoselskii-Mann iterative algorithm. Finally, two numerical examples were solved to demonstrate the reliability and usefulness of the proposed heuristic algorithm. From these examples, it was also found that the difference between the state trajectory using the mode-dependent strategy set and the mode-independent strategy set is small and they are reliable.

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