SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

October 2021, Vol. 64 200201:1–200201:16
https://doi.org/10.1007/s11432-021-3280-4

Special Focus on Control and Analysis for Stochastic Systems

Fault estimation and fault-tolerant control for linear discrete time-varying stochastic systems

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Received 15 February 2021/Revised 22 April 2021/Accepted 8 May 2021/Published online 14 September 2021

Abstract This paper presents a scheme for simultaneous fault estimation and fault-tolerant control of linear discrete time-varying stochastic systems. An observer is proposed to estimate the system state and the fault simultaneously. The estimation errors of both the system state and fault can achieve exponential stability in mean square sense even if the fault arbitrarily changes or is unbounded. The controllers of the drift term and diffusion term are designed separately, and then based on the estimated fault, the fault compensation is performed to realize fault tolerance. For the parameter design in the estimator and controllers, we provide two different algorithms via the cone complementarity linearization and the state transition matrix methods, respectively. As an extension, a class of quasi-linear systems is also discussed. A simulation example with two different fault types and an application in electromechanical servo systems are provided to illustrate the usefulness of the proposed scheme.

Keywords linear discrete stochastic systems, fault estimation, fault-tolerant control, state transition matrix, exponential stability in mean square

Citation Zhang T L, Deng F Q, Sun Y, et al. Fault estimation and fault-tolerant control for linear discrete time-varying stochastic systems. Sci China Inf Sci, 2021, 64(10): 200201, https://doi.org/10.1007/s11432-021-3280-4

1 Introduction

It is well-known that practical engineering systems are very common to suffer from unexpected changes due to the external environment influence, component damage, or signal mutation. For these reasons, a system that originally operates normally may exhibit various failures or imperfect behaviors. Various failures can seriously damage and reduce the reliability of the system to cause catastrophic accidents. Therefore, it is very important to find and handle the fault in time to improve the safety of control systems. It can be found that considerable efforts have been made in this direction and a great number of impressive results have been obtained [1, 2]. Parallel to fault detection and isolation, fault estimation is also a subject widely concerned by scholars. Many different observers have also been explored such as fuzzy observers [3], Luenberger-like interval observers [4], and sliding mode observers [5, 6]. Recently, an optimal state and fault estimation scheme was proposed by [7] for two-dimensional discrete systems.

In the process of actual industrial production, random interference is often everywhere. Therefore, stochastic system control has become a very attractive research field in the last decade; see [8] on stochastic synthetic gene networks, [9–13] on stochastic stability, and [14] on stochastic multi-objective optimization. On the other hand, fault diagnosis of stochastic systems has attracted lots of researchers' attention in recent years. For example, H_- sensitivity index for discrete time stochastic systems based on Riccati equations can be reviewed in [15]. Ref. [16] was devoted to achieving simultaneous fault estimations and non-fragile output feedback fault-tolerant control for continuous time Markovian jump systems in the presence of faults and disturbances. As for nonlinear stochastic systems, Refs. [17, 18] used the fuzzy method and Hamilton-Jacobi inequality to study the existence conditions of the fault detection filter of nonlinear switched/Itô stochastic systems, respectively. As said in [19–22], due to wide applications

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and the development of computer techniques, it is more valuable to carry out the study of discrete time stochastic systems. For linear discrete stochastic time-varying systems, a new method called "state transition matrix method" was introduced in [23], where two kinds of state transition matrices were presented. The approach in [23] has been successfully applied to the study of finite-time stability of linear discrete time-varying stochastic systems in [21]. Compared with the widely used Lyapunov function method, the state transition matrix method is more direct in analysis and the obtained results are less conservative in general. However, to the best of our knowledge, there has no research so far on the application of the state transition matrix method to stochastic fault diagnosis. In [24], a sensor fault estimation filter design for discrete-time linear time-varying systems was investigated, while in [25], a unified filter for simultaneous input and state estimation of linear discrete time systems was designed. Recently, an optimal state and fault estimation scheme was proposed by [7] for two-dimensional discrete systems. None of the above studies [7, 24, 25] on fault/input/state estimation considered the stochastic system with state, control input, and fault-dependent noise together. The fault diagnosis of discrete stochastic systems with multiplicative noise merits needs further study.

The above mentioned motivates us to carry out this research. This paper is mainly concerned with fault-tolerant control for linear discrete time-varying stochastic systems based on a simultaneous state and fault estimation. The main contributions of this article are as follows.

- Fault estimation and fault-tolerant control for linear discrete time-varying stochastic systems with state, control input, and fault signal dependent noise are considered based on a novel observer design scheme for both the state and fault. Through such an observer design scheme, we can take the fault estimation error and the state estimation error into account simultaneously. So, from the state estimation error to be stable or not, we are in a position to judge whether the fault estimation is accurate. There is no more strict restriction on the fault signal. In spite of an infinite energy signal or a random signal, it can be accurately estimated with the estimation error to achieve mean square exponential convergence.

- A stochastic fault-tolerant control strategy designed separately for drift and diffusion terms ensures that the system achieves mean square exponential stability. Compared with previous studies, the advantage of our designed control strategy lies in that it can accurately realize the fault compensation in the diffusion term.

- For the design of parameters in the estimator and controllers, we provide two different algorithms for a discrete periodic system based on the cone complementarity linearization method and the state transition matrix method, respectively. The cone complementarity linearization algorithm can be easily operated using the CVX toolbox in Matlab. The state transition matrix method transforms the parameter design into a convex optimization problem, which can be solved easily.

- For a class of quasi-linear discrete stochastic systems, we prove that similar results to linear discrete time-varying systems can be obtained.

The rest of this paper is organized as follows. In Section 2, we make some preliminaries by introducing some research backgrounds and basic tools. The convergence analysis of estimation errors and the structure of the controlled system are presented in Section 3. Section 4 develops two computational approaches to the design of parameters in the estimator and controllers. In particular, two theorems of quasi-linear systems that are locally and globally uniformly exponentially stable in mean square (UESMS) are presented in Subsection 4.3. Section 5 presents two simulation examples to verify the effectiveness of our main results. Section 6 concludes this paper with some remarks.

For convenience, the notations adopted in this paper are as follows. S': the transpose of a vector or matrix S; $S > 0$ ($S < 0$): S is a positive definite (negative definite) symmetric matrix; I_m: $m \times m$ identity matrix; $\mathbb{R}^n$: the n-dimensional real Euclidean vector space with the norm $\|x\| = \sqrt{\sum_{k=0}^{n} x_k^2}$ for $x = [x_1, x_2, \ldots, x_n]'$; $\mathbb{R}^{n\times m}$: the $n \times m$ real matrix space; $\mathcal{N} := \{0, 1, 2, \ldots, \}$; $\bar{\mathcal{N}}_T := \{1, 2, \ldots, T\}$; $\mathcal{N}_T := \{0, 1, 2, \ldots, T\}$; $S^{\dagger}$: the Moore-Penrose pseudo inverse of $S \in \mathbb{R}^{n\times m}$; $\|S\|_2$: the spectral norm of the matrix S; $\lambda_{\max}(A)$: the maximum eigenvalue of real symmetric matrix A; E: the mathematical expectation operator.

2 Preliminaries

Consider the following linear discrete time-varying stochastic system:

$$\begin{cases} x_{k+1} = A_k x_k + B_{k,1} u_{k,1} + B_{k,2} f_k + (C_k x_k + D_{k,1} u_{k,2} + D_{k,2} f_k) w_k, \\ y_k = H_k x_k + M_k f_k, \ k \in \mathcal{N}, \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the system state vector, $y_k \in \mathbb{R}^{n_y}$ is the measurement output, $f_k \in \mathbb{R}^{n_f}$ stands for the fault signal, and $\{w_k\}_{k\in\mathcal{N}}$ represents the system internal noise driven by one-dimensional independent white noise processes defined on the complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_k\}_{k\in\mathcal{N}}, \mathcal{P})$ with $\mathcal{F}_k = \sigma(w_0, w_1, \ldots, w_{k-1})$, $\mathcal{F}_0 = \{\phi, \Omega\}$. Assume $E[w_k] = 0$, $E[w_k w_j] = \delta_{kj}$, where δ_{kj} is a Kronecker function defined by $\delta_{kj} = 0$ for $k \neq j$ while $\delta_{kj} = 1$ for $k = j$. $u_{k,1} \in \mathbb{R}^{n_{u,1}}$ and $u_{k,2} \in \mathbb{R}^{n_{u,2}}$ are the controllers appearing in the drift term and the diffusion term, respectively.

In this paper, we want to design $u_{k,1}$ and $u_{k,2}$ to stabilize system (1). However, the major obstacle to achieving our goal is that there exist unknown faults in the system (1) and the state variables are not available. Therefore, a practicable scheme is proposed to construct a simultaneous fault and state estimator, and then to design controllers based on these estimated values. When considering completely unknown fault signals, most of the existing studies deal with H_∞ fault estimators, which usually regard $\Delta f_k = f_{k+1} - f_k$ as a new disturbance signal and restrain the influence from Δf_k to the estimated values below a given disturbance attenuation level. Therefore, H_∞ fault estimators cannot reconstruct the system state and fault perfectly. In this paper, we design a novel state-fault estimator as follows:

$$\begin{cases} \hat{x}_{k+1} = A_k \hat{x}_k + B_{k,1} u_{k,1} + B_{k,2} \hat{f}_k + \hat{L}_k (y_k - \hat{y}_k), \\ \hat{y}_k = H_k \hat{x}_k + M_k \hat{f}_k, \\ \hat{f}_k = (M_k' M_k)^{-1} M_k' (y_k - H_k \hat{x}_k), \end{cases} \tag{2}$$

where $\hat{x}_k$ and $\hat{f}_k$ stand for the estimated values of x_k and f_k, respectively, and $\hat{L}_k$ is the observer parameter to be designed.

Remark 1. Observing the state-fault estimator (2), compared with some traditional estimators, the most significant difference is the construction of $\hat{f}_k$. Such a design offers undreamed of levels of advantage in analyzing the estimation error of f_k; see Lemma 2 of this paper for more details.

Remark 2. In this paper, for the sake of the existence of $(M_k' M_k)^{-1}$, a basic assumption is $\mathrm{rank}(M_k) = n_f$. It should be pointed out that such an assumption is common and reasonable in the fault diagnosis research. $\mathrm{rank}(M_k) = n_f$ means that system (1) is a non-wide system. When system (1) is wide, $\|\mathcal{L}_{y,f}\|_- = 0$, where $\mathcal{L}_{y,f}$ is defined as a perturbed operator from f to y and $\|\mathcal{L}_{y,f}\|_-$ is the corresponding H_- index with

$$\|\mathcal{L}_{y,f}\|_- = \inf_{x_0=0, f_k\in l_2, f_k\neq 0} \frac{\{\sum_{i=0}^{\infty} E\|y_k\|^2\}^{1/2}}{\{\sum_{i=0}^{\infty} E\|f_k\|^2\}^{1/2}}.$$

It is obvious that this case is not readily applicable for fault detection; see [15, 26] for its rigorous proof and detailed explanations.

Remark 3. It is a very common way to design different controllers for the drift term and diffusion term in stochastic control systems to attenuate the noise; see [27, 28].

Remark 4. In [23], the authors have studied uniform detectability, exact detectability, exact observability, and finite-time stability of the discrete time-varying stochastic system:

$$\begin{cases} x_{k+1} = A_k x_k + C_k x_k w_k, \\ y_k = H_k x_k \end{cases} \tag{3}$$

based on the state transition matrix approach. In this work, we also apply the state transition matrix method to parameter design for the first time.

This paper aims to study the fault-tolerant control of (1) based on the state-fault estimator (2). The diagram of the proposed fault-tolerant scheme is shown in Figure 1. Because $\hat{x}_k$ and $\hat{f}_k$ are component

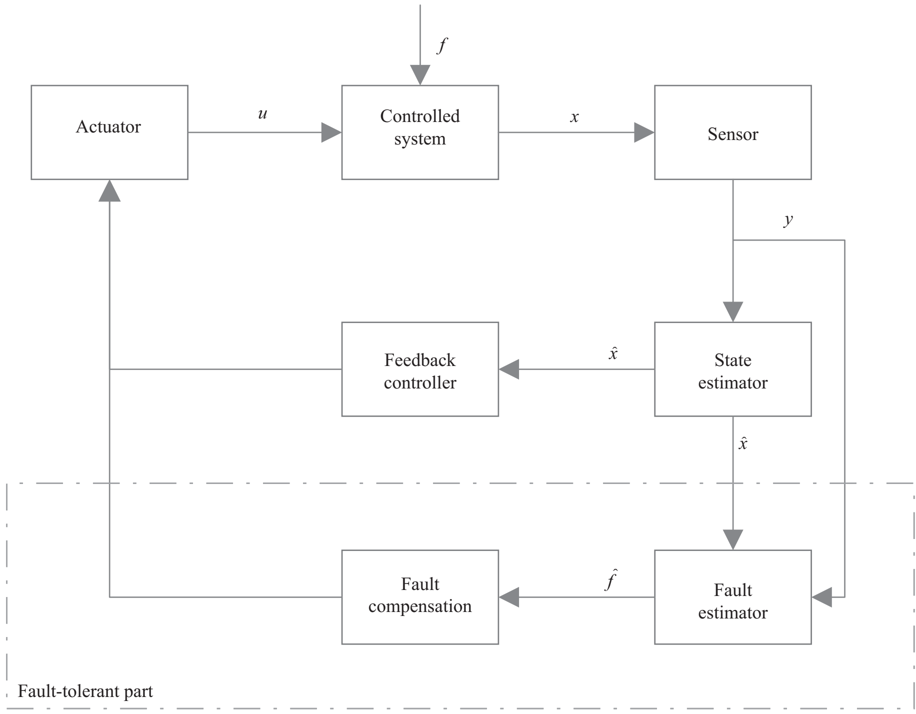

Figure 1 The diagram of the proposed fault-tolerant scheme.

parts of fault-tolerant controllers $u_{k,1}$ and $u_{k,2}$, when estimator (2) has satisfactory estimation functions, it can stabilize system (1). So, the controller $u_{k,i}$ is designed as

$$u_{k,i} = K_{k,i,1}\hat{x}_k + K_{k,i,2}\hat{f}_k, \quad i = 1, 2. \tag{4}$$

Let $e_{x_k} = x_k - \hat{x}_k$ stand for the estimation error of x_k. By defining $\tilde{x}_k = \left[\begin{smallmatrix} x_k \\ e_{x_k} \end{smallmatrix}\right]$ and $\tilde{f}_k = \left[\begin{smallmatrix} f_k \\ \hat{f}_k \end{smallmatrix}\right]$, the closed-loop augmented system is obtained as

$$\tilde{x}_{k+1} = \tilde{A}_k\tilde{x}_k + \tilde{B}_{k,1}u_{k,1} + \tilde{B}_{k,2}\tilde{f}_k + (\tilde{C}_k\tilde{x}_k + \tilde{D}_{k,1}u_{k,2} + \tilde{D}_{k,2}\tilde{f}_k)w_k, \tag{5}$$

where

$$\tilde{A}_k = \begin{bmatrix} A_k & 0 \\ 0 & A_k - \hat{L}_k H_k \end{bmatrix}, \; \tilde{B}_{k,1} = \begin{bmatrix} B_{k,1} \\ 0 \end{bmatrix}, \; \tilde{B}_{k,2} = \begin{bmatrix} B_{k,2} & 0 \\ B_{ik}^2 - \hat{L}_k M_k & \hat{L}_k M_k - B_{k,2} \end{bmatrix},$$
$$\tilde{C}_k = \begin{bmatrix} C_k & 0 \\ C_k & 0 \end{bmatrix}, \; \tilde{D}_{k,1} = \begin{bmatrix} D_{k,1} \\ D_{k,1} \end{bmatrix}, \; \tilde{D}_{k,2} = \begin{bmatrix} D_{k,2} & 0 \\ D_{k,2} & 0 \end{bmatrix}.$$

In the following, we give an essential definition and a useful lemma.

Definition 1 ([29]). System (3) is said to be UESMS if there exist $\beta \geqslant 1$ and $\lambda \in (0, 1)$ such that for any $0 \leqslant k_0 \leqslant k < +\infty$, the following inequality holds:

$$E\|x_k\|^2 \leqslant \beta\lambda^{(k-k_0)}\|x_{k_0}\|^2.$$

Lemma 1 ([30]). Let matrices L, M and N be given with appropriate sizes. Then the following matrix equation

$$LXM = N \tag{6}$$

has a solution X if and only if

$$LL^\dagger NMM^\dagger = N.$$

Moreover, in this case, any solution to (6) can be represented by

$$X = L^\dagger NM^\dagger + Y - L^\dagger LYMM^\dagger,$$

where Y is a matrix with an appropriate size.

3 The structure of the closed-loop system

This section will discuss the structure of the closed-loop system (5).

Lemma 2. For estimator (2) and system (1), let $e_{f_k} = f_k - \hat{f}_k$ stand for the estimation error of f_k. If M_K, H_k, and $(M_k'M_k)^{-1}$ are uniformly bounded for all $k \in \mathcal{N}$, then $\{e_{f_k}\}_k$ is UESMS if $\{e_{x_k}\}_k$ is UESMS.

Proof. According to (2) and (1), it is easy to see that

$$\begin{aligned}\hat{f}_k &= (M_k'M_k)^{-1}M_k'(y_k - H_k\hat{x}_k) = (M_k'M_k)^{-1}M_k'(H_kx_k + M_kf_k - H_k\hat{x}_k)\\ &= (M_k'M_k)^{-1}M_k'H_ke_{x_k} + f_k.\end{aligned}$$

Thus,

$$-e_{f_k} = (M_k'M_k)^{-1}M_k'H_ke_{x_k}, \tag{7}$$

from which the conclusion is obvious.

Remark 5. From Lemma 2, the relationship between e_{f_k} and e_{x_k} is considered. On this basis, we just have to concentrate on the convergence of e_{x_k} below, which directly ensures the convergence of e_{f_k}. In addition, when analyzing the stability of the controlled system, we can also transform the convergence study of e_{f_k} into that of e_{x_k} to avoid the existence of unknown input items. Note that, only if $K_{k,i,2}\hat{f}_k$ perfectly offsets fault f_k, $e_{x_{k+1}}$ and x_{k+1} will not be affected by any unknown information. This situation makes it possible for us to obtain the stabilization of the closed-loop system (5). So, it is very crucial to design appropriate control gain matrices $K_{k,i,2}$, $k \in \mathcal{N}$, $i = 1, 2$. Motivated by the aforementioned reason, the following theorem is an answer to how to design $K_{k,i,2}$ based on Lemma 1.

The following is an assumption about system (1) throughout the paper.

Assumption 1. Suppose that $D_{k,1}D_{k,1}^{\dagger}D_{k,2} = D_{k,2}$, $B_{k,1}B_{k,1}^{\dagger}B_{k,2} = B_{k,2}$.

Theorem 1. Under Assumption 1, $K_{k,1,2}$ and $K_{k,2,2}$ can be designed as

$$\begin{aligned}K_{k,1,2} &= -B_{k,1}^{\dagger}B_{k,2} + Y_k - B_{k,1}^{\dagger}B_{k,1}Y_k,\\ K_{k,2,2} &= -D_{k,1}^{\dagger}D_{k,2} + Y_k - D_{k,1}^{\dagger}D_{k,1}Y_k,\end{aligned} \tag{8}$$

where Y_k is any matrix with an appropriate size. Moreover, the augmented system $\tilde{x}_k = \left[\begin{smallmatrix} x_k \\ e_{x_k}\end{smallmatrix}\right]$ can be rewritten as

$$\tilde{x}_{k+1} = \bar{A}_k\tilde{x}_k + \bar{C}_k\tilde{x}_kw_k, \tag{9}$$

where

$$\begin{aligned}\bar{A}_k &= \begin{bmatrix} A_k + B_{k,1}K_{k,1,1} & -B_{k,1}K_{k,1,1} - B_{k,2}(M_k'M_k)^{-1}M_k'H_k \\ 0 & A_k - \hat{L}_kH_k - (B_{k,2} - \hat{L}_kM_k)(M_k'M_k)^{-1}M_k'H_k \end{bmatrix},\\ \bar{C}_k &= \begin{bmatrix} C_k + D_{k,1}K_{k,2,1} & -D_{k,1}K_{k,2,1} - D_{k,2}(M_k'M_k)^{-1}M_k'H_k \\ C_k + D_{k,1}K_{k,2,1} & -D_{k,1}K_{k,2,1} - D_{k,2}(M_k'M_k)^{-1}M_k'H_k \end{bmatrix}.\end{aligned}$$

Proof. Considering system (1) with estimator (2), we have

$$\begin{aligned}e_{x_{k+1}} &= x_{k+1} - \hat{x}_{k+1}\\ &= A_kx_k + B_{k,1}u_{k,1} + B_{k,2}f_k + (C_kx_k + D_{k,1}u_{k,2} + D_{k,2}f_k)w_k - A_k\hat{x}_k - B_{k,1}u_{k,1}\\ &\quad - B_{k,2}\hat{f}_k - \hat{L}_k(H_kx_k + M_kf_k - H_k\hat{x}_k - M_k\hat{f}_k)\\ &= A_ke_{x_k} + B_{k,2}e_{f_k} - \hat{L}_kH_ke_{x_k} - \hat{L}_kM_ke_{f_k} + (C_kx_k + D_{k,1}K_{k,2,1}\hat{x}_k + D_{k,1}K_{k,2,2}\hat{f}_k + D_{k,2}f_k)w_k.\end{aligned} \tag{10}$$

In (10),

$$\begin{aligned}C_kx_k + D_{k,1}K_{k,2,1}\hat{x}_k &= -D_{k,1}K_{k,2,1}x_k + D_{k,1}K_{k,2,1}\hat{x}_k + (D_{k,1}K_{k,2,1} + C_k)x_k\\ &= -D_{k,1}K_{k,2,1}e_{x_k} + (D_{k,1}K_{k,2,1} + C_k)x_k.\end{aligned} \tag{11}$$

Then, one can get

$$\begin{aligned}e_{x_{k+1}} =& A_k e_{x_k} + B_{k,2} e_{f_k} - \hat{L}_k H_k e_{x_k} - \hat{L}_k M_k e_{f_k}\\ &+ [(D_{k,1}K_{k,2,1} + C_k)x_k - D_{k,1}K_{k,2,1}e_{x_k} + D_{k,1}K_{k,2,2}\hat{f}_k + D_{k,2}f_k]w_k.\end{aligned}$$

The key to merging f_k and $\hat{f}_k$ is to choose an appropriate control gain matrix $K_{k,2,2}$, $k \in \mathcal{N}$. Due to $D_{k,1}D^{\dagger}_{k,1}D_{k,2} = (-D_{k,1})(-D^{\dagger}_{k,1})D_{k,2} = D_{k,2}$, by Lemma 1, there must exist $K_{k,2,2}$ such that $-D_{k,1}K_{k,2,2} = D_{k,2}$. Moreover, $K_{k,2,2}$ can be expressed as in (8). Therefore, we have

$$D_{k,1}K_{k,2,2}\hat{f}_k + D_{k,2}f_k = D_{k,2}e_{f_k}$$

and

$$\begin{aligned}&e_{x_{k+1}}\\ &= A_k e_{x_k} + B_{k,2}e_{f_k} - \hat{L}_k H_k e_{x_k} - \hat{L}_k M_k e_{f_k} + [(D_{k,1}K_{k,2,1} + C_k)x_k - D_{k,1}K_{k,2,1}e_{x_k} + D_{k,2}e_{f_k}]w_k\\ &= (A_k - \hat{L}_k H_k)e_{x_k} + (B_{k,2} - \hat{L}_k M_k)e_{f_k} + [-D_{k,1}K_{k,2,1}e_{x_k} + (D_{k,1}K_{k,2,1} + C_k)x_k + D_{k,2}e_{f_k}]w_k.\end{aligned}$$

According to the relationship between e_{f_k} and e_{x_k} in (7), the dynamic equation of $e_{x_{k+1}}$ can be rewritten as

$$\begin{aligned}e_{x_{k+1}} =& [A_k - \hat{L}_k H_k - (B_{k,2} - \hat{L}_k M_k)(M_k' M_k)^{-1}M_k' H_k]e_{x_k}\\ &+ [(-D_{k,1}K_{k,2,1} - D_{k,2}(M_k' M_k)^{-1}M_k' H_k)e_{x_k} + (D_{k,1}K_{k,2,1} + C_k)x_k]w_k.\end{aligned} \tag{12}$$

Subsequently, we consider the reconstruction of x_{k+1}. Note that

$$\begin{aligned}x_{k+1} = & A_k x_k + B_{k,1}K_{k,1,1}\hat{x}_k + B_{k,1}K_{k,1,2}\hat{f}_k + B_{k,2}f_k\\ &+ (C_k x_k + D_{k,1}K_{k,2,1}\hat{x}_k + D_{k,1}K_{k,2,2}\hat{f}_k + D_{k,2}f_k)w_k.\end{aligned}$$

Analogously, by Lemma 1 as well as $B_{k,1}B^{\dagger}_{k,1}B_{k,2} = B_{k,2}$, $-B_{k,1}K_{k,1,2} = B_{k,2}$ is solvable with $K_{k,1,2}$ given in (8). Substituting $-B_{k,1}K_{k,1,2} = B_{k,2}$ and $-D_{k,1}K_{k,2,2} = D_{k,2}$ into the expression of x_{k+1}, it yields that

$$\begin{aligned}x_{k+1} =& A_k x_k + B_{k,1}K_{k,1,1}\hat{x}_k + B_{k,1}K_{k,1,2}\hat{f}_k + B_{k,2}f_k\\ &+ (C_k x_k + D_{k,1}K_{k,2,1}\hat{x}_k + D_{k,1}K_{k,2,2}\hat{f}_k + D_{k,2}f_k)w_k\\ =& - B_{k,1}K_{k,1,1}e_{x_k} + (A_k + B_{k,1}K_{k,1,1})x_k + B_{k,2}e_{f_k}\\ &+ \big[- D_{k,1}K_{k,2,1}e_{x_k} + (C_k + D_{k,1}K_{k,2,1})x_k + D_{k,2}e_{f_k}\big]w_k\\ =& [-B_{k,1}K_{k,1,1} - B_{k,2}(M_k' M_k)^{-1}M_k' H_k]e_{x_k} + (A_k + B_{k,1}K_{k,1,1})x_k\\ &+ \big[(-D_{k,1}K_{k,2,1} - D_{k,2}(M_k' M_k)^{-1}M_k' H_k)e_{x_k} + (C_k + D_{k,1}K_{k,2,1})x_k\big]\, w_k.\end{aligned} \tag{13}$$

Therefore, combining (12) and (13), we can get (9).

Remark 6. Assumption 1 means that $D_{k,2}$ and $B_{k,2}$ are the linear combination of column vectors in $D_{k,1}$ and $B_{k,1}$, respectively. So $\mathrm{rank}([D_{k,1}, D_{k,2}]) = \mathrm{rank}(D_{k,1})$ and $\mathrm{rank}([B_{k,1}, B_{k,2}]) = \mathrm{rank}(B_{k,1})$. Meanwhile, by Lemma 1, $\mathrm{rank}([D_{k,1}, D_{k,2}]) = \mathrm{rank}(D_{k,1})$ and $\mathrm{rank}([B_{k,1}, B_{k,2}]) = \mathrm{rank}(B_{k,1})$ can guarantee Assumption 1. Therefore, Assumption 1 is equivalent to $\mathrm{rank}([D_{k,1}, D_{k,2}]) = \mathrm{rank}(D_{k,1})$ and $\mathrm{rank}([B_{k,1}, B_{k,2}]) = \mathrm{rank}(B_{k,1})$. This is a common hypothesis in existing references on fault-tolerant control [2, 31], which implies that the fault f_k simultaneously exists in the input channels $u_{k,1}$ and $u_{k,2}$. In this case, applying Lemma 1, it is not difficult to prove that any control gain matrices $K_{k,1,2}$ and $K_{k,2,2}$ matched with the fault can be represented by (8).

4 Parameter design

In this section, we consider the parameter design for the fault estimator and controllers.

4.1 A cone complementarity linearization algorithm

In this subsection, we will utilize the Lyapunov theorem about discrete time-varying stochastic systems to obtain the stabilization of system (5). The following Lyapunov-type theorem can be found in [23, 29].

Theorem 2. Suppose $\{\bar{A}_k\}_{k\in\mathcal{N}}$ and $\{\bar{C}_k\}_{k\in\mathcal{N}}$ in (9) are uniformly bounded. If there exist a uniformly bounded positive definite matrix sequence $\{P_k\}_{k\in\mathcal{N}}$ and an $\varepsilon > 0$, such that

$$\bar{A}_k' P_{k+1}\bar{A}_k + \bar{C}_k' P_{k+1}\bar{C}_k - P_k < -\varepsilon I, \quad k \in \mathcal{N}, \tag{14}$$

then the linear discrete time-varying stochastic system (9) is UESMS.

For the periodic system (9), we have the following corollary of Theorem 2.

Corollary 1. Assume system (9) is a periodic system with the period T, $0 < T \in \mathcal{N}$. Under Assumption 1, if there exists a positive definite matrix sequence $\{P_k\}_{k\in\mathcal{N}_{T-1}}$, such that

$$\bar{A}_k' P_{k+1}\bar{A}_k + \bar{C}_k' P_{k+1}\bar{C}_k - P_k < 0, \quad k \in \mathcal{N}_{T-1}, \tag{15}$$

then system (9) is UESMS.

By Lemma 2 and the expression of $\tilde{x}_k$, for the fault estimator (2), if system (9) is UESMS, then so do e_{f_k}, x_k and e_{x_k}. Using Schur's complement, Eq. (15) is equivalent to

$$\Phi_k = \begin{bmatrix} -P_k & \bar{A}_k' & \bar{C}_k' \\ \bar{A}_k & -X_{k+1} & 0 \\ \bar{C}_k & 0 & -X_{k+1} \end{bmatrix} < 0, \quad k \in \mathcal{N}_{T-1}, \tag{16}$$

where $X_k = P_k^{-1}$. Therefore, we can get the following theorem.

Theorem 3. Considering Assumption 1 and the periodic system (9) with a period $T > 0$, if there exist some matrix sequences $\{P_k\}_{k\in\mathcal{N}_{T-1}}$, $\{X_k\}_{k\in\mathcal{N}_{T-1}}$ with $X_k P_k = I$, $\{K_{k,1,1}\}_{k\in\mathcal{N}_{T-1}}$, $\{K_{k,2,1}\}_{k\in\mathcal{N}_{T-1}}$ and $\{\hat{L}_k\}_{k\in\mathcal{N}_{T-1}}$, such that the matrix inequality (16) holds, then systems (9) and (1) are UESMS, where the control gain matrices $\{K_{k,1,2}\}_{k\in\mathcal{N}_{T-1}}$ and $\{K_{k,2,2}\}_{k\in\mathcal{N}_{T-1}}$ are given in (8).

Now we use the cone complementarity linearization algorithm to solve (16). Based on the cone complementarity linearization algorithm proposed in [32], the non-convex problem formulated by (16) can be converted into the following nonlinear minimization problem:

$$\min_{P_k, X_k, k\in\mathcal{N}_{T-1}} \sum_{k=0}^{T-1} \operatorname{trace}(P_k X_k)$$

subject to linear matrix inequality (LMI) (16) and

$$\begin{bmatrix} P_k & I \\ I & X_k \end{bmatrix} \geqslant 0, \ k \in \mathcal{N}_{T-1}.$$

We summarize our linearization algorithm as Algorithm 1, which transforms the design problems of fault-tolerant controllers and fault estimator of system (1) into solving a convex optimization problem.

4.2 State transition matrix method

For linear discrete systems, except the well-known Lyapunov function method, another important method called the state transition matrix method can also be used to investigate in-depth various stabilities. Below we explore the parameter design of the controllers (4) and estimator (2) based on the state transition matrix method. In the following lemma, we give a new state transition matrix for system (9).

Lemma 3. For system (9), $\forall j \geqslant i$, we have $E\|\tilde{x}_j\|^2 = E\|\varphi_{j,i}\tilde{x}_i\|^2$, where $\varphi_{j,i} = I_{2n}$ when $j = i$, and for $j \geqslant i$,

$$\varphi_{j,i} = \left(I_{2^{j-i-1}} \otimes \begin{bmatrix} \bar{A}_{j-1} \\ \bar{C}_{j-1} \end{bmatrix} \right) \varphi_{j-1,i}.$$

Algorithm 1 A cone complementarity linearization algorithm

1: Find a set of feasible solutions of LMI (16) denoted by $\{P_{k,0}\}_{k\in\mathcal{N}_{T-1}}$, $\{X_{k,0}\}_{k\in\mathcal{N}_{T-1}}$; if there is none, the procedure exits. Set $i=0$.

2: Solve the convex optimization problem to obtain $P_{k,i+1}$ and $X_{k,i+1}$:

$$\min_{X_{k,i+1},P_{k,i+1}} \sum_{k=0}^{T-1}\{\text{trace}(P_{k,i}X_{k,i+1})+\text{trace}(X_{k,i}P_{k,i+1})\}$$

subject to LMIs

$$\begin{bmatrix} -P_{k,i+1} & \bar{A}_k' & \bar{C}_k' \\ \bar{A}_k & -X_{k+1,i+1} & 0 \\ \bar{C}_k & 0 & -X_{k+1,i+1} \end{bmatrix} < 0,\ k\in\mathcal{N}_{T-1},$$

and

$$\begin{bmatrix} P_{k,i+1} & I \\ I & X_{k,i+1} \end{bmatrix} \geqslant 0,\ k\in\mathcal{N}_{T-1}.$$

3: If a stopping criterion $|\sum_{k=0}^{T-1}\{\text{trace}(P_{k,i}X_{k,i+1})+\text{trace}(X_{k,i}P_{k,i+1})\}-4nT|<\epsilon$ for a prescribed $\epsilon>0$ is satisfied, the procedure stops. Otherwise, set $i=i+1$ and go back to step 2.

Proof. Based on the results in [21,23], for $j\geqslant i\geqslant 0$, we have $E\|x_j\|^2=E\|\psi_{j,i}\tilde{x}_i\|^2$ with $\psi_{j,i}=\left[\begin{smallmatrix}\psi_{j,i+1}\bar{A}_i\\ \psi_{j,i+1}\bar{C}_i\end{smallmatrix}\right]$. Therefore, it only needs to prove $\varphi_{j,i}=\psi_{j,i}$. This proof can be shown by induction. In the case of $j=i+1$, the equation $\varphi_{j,i}=\psi_{j,i}$ is obvious. Assume $\varphi_{j,i}=\psi_{j,i}$ holds for any $j=i+m$, i.e.,

$$\left(I_{2^{m-1}}\otimes\begin{bmatrix}\bar{A}_{i+m-1}\\ \bar{C}_{i+m-1}\end{bmatrix}\right)\varphi_{i+m-1,i}=\begin{bmatrix}\psi_{i+m,i+1}\bar{A}_i\\ \psi_{i+m,i+1}\bar{C}_i\end{bmatrix}. \tag{17}$$

Then we need to judge whether this relationship still holds when $j=i+m+1$. Note that the arbitrariness of i directly leads to $\varphi_{i+m+1,i+1}=\psi_{i+m+1,i+1}$. Therefore,

$$\begin{aligned}\psi_{i+m+1,i}=&(I_2\otimes\psi_{j+m+1,i+1})\begin{bmatrix}\bar{A}_i\\ \bar{C}_i\end{bmatrix}=\left(I_2\otimes\left[I_{2^{m-1}}\otimes\begin{bmatrix}\bar{A}_{i+m}\\ \bar{C}_{i+m}\end{bmatrix}\right]\right)\begin{bmatrix}\psi_{i+m,i+1}\bar{A}_i\\ \psi_{i+m,i+1}\bar{C}_i\end{bmatrix}\\ =&\left(I_{2^m}\otimes\begin{bmatrix}\bar{A}_{i+m}\\ \bar{C}_{i+m}\end{bmatrix}\right)\psi_{i+m,i}=\varphi_{i+m+1,i}.\end{aligned}$$

Theorem 4. System (9) is UESMS if there exist two positive numbers $\alpha\geqslant 1$, $0<\eta<1$ such that $\|\varphi_{k,k_0}\|_2^2\leqslant\alpha\eta^{k-k_0}$, $\forall k_0\in\mathcal{N}, k\geqslant k_0$.

Proof. From Lemma 3, the relation $E\|\tilde{x}_j\|^2=E\|\varphi_{j,i}\tilde{x}_i\|^2$ leads to $E\|\tilde{x}_k\|^2\leqslant\|\varphi_{k,k_0}\|_2^2E\|\tilde{x}_{k_0}\|^2$. Due to $\|\varphi_{k,k_0}\|_2^2\leqslant\alpha\eta^{k-k_0}$, the conclusion is obvious.

Remark 7. The state transition matrix-based exponential stability and uniformly exponential stability criteria were provided in [33] for discrete time linear time-varying systems.

Corollary 2. For system (9), if for some $0<\eta<1$,

$$\left\|\begin{bmatrix}\bar{A}_j\\ \bar{C}_j\end{bmatrix}\right\|_2^2\leqslant\eta,\quad j\geqslant k_0\geqslant 0, \tag{18}$$

then system (9) is UESMS.

Proof. By Lemma 3 and condition (18), we have

$$\|\varphi_{k_0+1,k_0}\|_2^2=\left\|\left(1\otimes\begin{bmatrix}\bar{A}_{k_0}\\ \bar{C}_{k_0}\end{bmatrix}\right)\varphi_{k_0,k_0}\right\|_2^2\leqslant\left\|\begin{bmatrix}\bar{A}_{k_0}\\ \bar{C}_{k_0}\end{bmatrix}\right\|_2^2\leqslant\eta,$$

$$\|\varphi_{k_0+2,k_0}\|_2^2=\left\|\left(1_2\otimes\begin{bmatrix}\bar{A}_{k_0+1}\\ \bar{C}_{k_0+1}\end{bmatrix}\right)\varphi_{k_{0+1},k_0}\right\|_2^2\leqslant\left\|I_2\otimes\begin{bmatrix}\bar{A}_{k_0+1}\\ \bar{C}_{k_0+1}\end{bmatrix}\right\|_2^2\cdot\|\varphi_{k_0+1,k_0}\|_2^2$$

$$= \lambda_{\max}(I_2 \otimes (A'A + C'C))\|\varphi_{k_0+1,k_0}\|_2^2 \leqslant \left\| \begin{bmatrix} \bar{A}_{k_0} \\ \bar{C}_{k_0} \end{bmatrix} \right\|_2^2 \eta \leqslant \eta^2.$$

By induction, under the condition of (18), if $\varphi_{j,k_0} \leqslant \eta^{(j-k_0)}$, then

$$\|\varphi_{j+1,k_0}\|_2^2 = \left\| \left(I_{2^{j-k_0}} \otimes \begin{bmatrix} \bar{A}_j \\ \bar{C}_j \end{bmatrix} \right) \varphi_{j,k_0} \right\|_2^2 \leqslant \left\| \left(I_{2^{j-k_0}} \otimes \begin{bmatrix} \bar{A}_j \\ \bar{C}_j \end{bmatrix} \right) \right\|_2^2 \eta^{j-k_0}$$
$$\leqslant \left\| \begin{bmatrix} \bar{A}_j \\ \bar{C}_j \end{bmatrix} \right\|_2^2 \eta^{j-k_0} \leqslant \eta^{(j+1-k_0)}.$$

Hence, by Theorem 4 with $\alpha = 1$, system (9) is UESMS.

By the definition of the matrix 2-norm, we know that

$$\left\| \begin{bmatrix} \bar{A}_j \\ \bar{C}_j \end{bmatrix} \right\|_2^2 = \lambda_{\max}(\bar{A}_j'\bar{A}_j + \bar{C}_j'\bar{C}_j),$$

where $\lambda_{\max}(M)$ represents the largest eigenvalue of a real symmetric matrix M. As $\lambda_{\max}(\bar{A}_j'\bar{A}_j+\bar{C}_j'\bar{C}_j)I \geqslant \bar{A}_j'\bar{A}_j+\bar{C}_j'\bar{C}_j$, the largest eigenvalue $\lambda_{\max}(\bar{A}_j'\bar{A}_j+\bar{C}_j'\bar{C}_j)$ can be computed by solving the following convex optimization problem:

$$\min \lambda$$

subject to

$$\begin{bmatrix} \lambda I & \bar{A}_j' & \bar{C}_j' \\ \bar{A}_j & I & 0 \\ \bar{C}_j & 0 & I \end{bmatrix} \geqslant 0.$$

Combining Corollary 2 and Theorem 4, we immediately obtain the following easily used theorem.

Theorem 5. If there exist the control gain matrix sequence $\{K_{k,1,1}, K_{k,2,1}\}$ and observer parameter sequence $\{\hat{L}_k\}$, such that the following convex optimization problem:

$$\min_{K_{k,1,1},K_{k,2,1},\hat{L}_k} \lambda$$

subject to

$$\begin{bmatrix} \lambda I & \bar{A}_k' & \bar{C}_k' \\ \bar{A}_k & I & 0 \\ \bar{C}_k & 0 & I \end{bmatrix} \geqslant 0, \quad 0 < \lambda < 1, \quad k \in \mathcal{N}$$

is feasible, then system (9) is UESMS.

Corollary 3. Consider the periodic system (9) with a period $T > 0$. If there exist the control gain matrix sequence $\{K_{k,1,1}, K_{k,2,1}\}_{k\in\mathcal{N}_{T-1}}$ and observer parameter sequence $\{\hat{L}_k\}_{k\in\mathcal{N}_{T-1}}$, such that the following convex optimization problem:

$$\min_{K_{k,1,1},K_{k,2,1},\hat{L}_k} \lambda$$

subject to

$$\begin{bmatrix} \lambda I_{2n} & \bar{A}_k' & \bar{C}_k' \\ \bar{A}_k & I & 0 \\ \bar{C}_k & 0 & I \end{bmatrix} \geqslant 0, \quad 0 < \lambda < 1, \quad k = 0, 1, \ldots, T-1$$

is feasible, then system (9) is UESMS.

Remark 8. By constructing the fault estimator and using the fault compensation method, the original fault-tolerant control problem can be transformed into the stability problem of standard linear time-varying discrete-time stochastic systems. However, because the parameters of the closed-loop system (9) are relatively complex, it is not easy to directly use the Lyapunov function method to solve the control and observation matrices. Therefore, constructing a suitable Lyapunov function through the cone-complementary linearization algorithm is a relatively straightforward processing method, but such a method will increase the complexity in the process of constructing the Lyapunov function. The state transition matrix method proposed in this paper can obtain the control and observation matrices directly by calculating the system parameters, which is more efficient.

Remark 9. Note that some latest studies [34, 35] have tried to solve the fault diagnosis problem of discrete systems. However, comparatively speaking, the results of this paper have more technical advantages. On one hand, this paper considers the more complex time-varying discrete stochastic systems. On the other hand, the assumption of the fault signal does not include the limit of increment or limited energy.

4.3 Quasi-linear discrete stochastic systems

Since a quasi-linear discrete stochastic system can approximate a general nonlinear discrete stochastic system by Taylor's series expansion [36], it is valuable to consider quasi-linear discrete stochastic systems, and such results are important to fault-tolerant control of general nonlinear systems. Consider the following quasi-linear discrete stochastic system:

$$\begin{cases} x_{k+1} = A_k x_k + F_{k,1}(x_k) + B_{k,1}u_{k,1} + B_{k,2}f_k + (C_k x_k + F_{k,2}(x_k) + D_{k,1}u_{k,2} + D_{k,2}f_k)w_k, \\ y_k = H_k x_k + M_k f_k. \end{cases} \tag{19}$$

In this case, the state and fault estimator (2) is replaced accordingly by

$$\begin{cases} \hat{x}_{k+1} = A_k\hat{x}_k + F_{k,1}(\hat{x}_k) + B_{k,1}u_{k,1} + B_{k,2}\hat{f}_k + \hat{L}_k(y_k - \hat{y}_k), \\ \hat{y}_k = H_k\hat{x}_k + M_k\hat{f}_k, \\ \hat{f}_k = (M_k'M_k)^{-1}M_k'(y_k - H_k\hat{x}_k). \end{cases} \tag{20}$$

Now we generalize Theorem 2 to system (19).

Definition 2 ([36])**.** System (19) is said to be locally UESMS if there exist $\beta \geqslant 1$ and $\lambda \in (0,1)$, such that for any $0 \leqslant k_0 \leqslant k < +\infty$, $\|x_{k_0}\| \leqslant \varepsilon(\beta,\lambda)$, the following inequality holds:

$$E\|x_k\|^2 \leqslant \beta\lambda^{(k-k_0)}\|x_{k_0}\|^2. \tag{21}$$

If inequality (21) holds for any $x_{k_0} \in \mathcal{R}^n$, then system (19) is said to be globally UESMS.

Theorem 6. Suppose that the control gain matrix sequence $\{K_{k,1,1}, K_{k,1,2}, K_{k,2,1}, K_{k,2,2}\}$ and observer parameter sequence $\{\hat{L}_k\}$ designed by Theorem 2 ensure that the system (1) is UESMS. If $\lim_{x\to 0}\frac{\|F_{k,1}(x)\|+\|F_{k,2}(x)\|}{\|x\|} = 0$, then the state and fault estimator (20) and controllers (4) can guarantee that Eq. (19) is locally UESMS.

Proof. This theorem can be obtained based on the similar analysis as in Theorem 4 of [36].

Theorem 7. Under Assumption 1 and

$$\|F_{k,1}(x) - F_{k,1}(y)\| \vee \|F_{k,2}(x) - F_{k,2}(y)\| \leqslant \lambda\|x-y\|,\ \forall x,y \in \mathbb{R}^n,$$

if there exist positive definite matrix sequence $\{P_k\}$ with $P_k \leqslant \beta I$, control gain matrix sequence $\{K_{k,1,1}, K_{k,1,2}, K_{k,2,1}, K_{k,2,2}\}$, observer parameter sequence $\{\hat{L}_k\}$, and three positive constants $0 < \alpha_1$, α_2, $\alpha_3 < 1$ satisfying

$$(1+\alpha_1^{-1})\bar{A}_k'P_{k+1}\bar{A}_k + (1+\alpha_2^{-1})\bar{C}_k'P_{k+1}\bar{C}_k + (1+\alpha_1)\beta\lambda^2 I_{2n\times 2n} + 2(1+\alpha_2)\beta\lambda^2\bar{I} - \alpha_3 P_k < 0, \tag{22}$$

where $\bar{I}$ is a diagonal matrix with the first n diagonal elements being 1 and the others being 0, then the state and fault estimator (20) and controllers (4) can guarantee the nonlinear discrete time-varying stochastic system (19) to be globally UESMS.

Proof. Referring to the proof of Theorem 1, the design of $K_{k,1,2}$ and $K_{k,2,2}$ as in (8) results in that

$$\begin{aligned} e_{x_{k+1}} =&[A_k - \hat{L}_k H_k - (B_{k,2} - \hat{L}_k M_k)(M_k' M_k)^{-1} M_k' H_k] e_{x_k} + F_{k,1}(x_k) - F_{k,1}(\hat{x}_k) \\ &+ [(-D_{k,1} K_{k,2,1} - D_{k,2}(M_k' M_k)^{-1} M_k' H_k) e_{x_k} + (D_{k,1} K_{k,2,1} + C_k) x_k + F_{k,2}(x_k)] w_k \end{aligned}$$

and

$$\tilde{x}_{k+1} = G_k(\tilde{x}_k, w_k) := \bar{A}_k \tilde{x}_k + \begin{bmatrix} F_{k,1}(x_k) \\ F_{k,1}(x_k) - F_{k,1}(\hat{x_k}) \end{bmatrix} + \left(\bar{C}_k \tilde{x}_k + \begin{bmatrix} F_{k,2}(x_k) \\ F_{k,2}(x_k) \end{bmatrix} \right) w_k. \tag{23}$$

According to Theorem 2 in [36], if a positive definite radially unbounded time-varying Lyapunov function $V_k(\tilde{x})$ satisfying $\Delta V_k(\tilde{x}) < 0$ can be found, then stochastic system (19) can be globally uniformly exponentially stabilizable in mean square. Therefore, in what follows, we just need to look for such a $V_k(\tilde{x}_k)$. Choose $V_k(\tilde{x}) = \tilde{x}' P_k \tilde{x}$ with $0 < P_k < \beta I$. By the definition of $\Delta V_k(\tilde{x})$ [36], we have

$$\begin{aligned} \Delta V_k(\tilde{x}) =& E(G_k(\tilde{x}, w_k)' P_{k+1} G_k(\tilde{x}, w_k)) - \tilde{x}' P_k \tilde{x} \\ =& \left(\bar{A}_k \tilde{x} + \begin{bmatrix} F_{k,1}(x) \\ F_{k,1}(x) - F_{k,1}(\hat{x}) \end{bmatrix} \right)' P_{k+1} \left(\bar{A}_k \tilde{x} + \begin{bmatrix} F_{k,1}(x) \\ F_{k,1}(x) - F_{k,1}(\hat{x}) \end{bmatrix} \right) \\ &+ \left(\bar{C}_k \tilde{x} + \begin{bmatrix} F_{k,2}(x) \\ F_{k,2}(x) \end{bmatrix} \right)' P_{k+1} \left(\bar{C}_k \tilde{x} + \begin{bmatrix} F_{k,2}(x) \\ F_{k,2}(x) \end{bmatrix} \right) - \tilde{x}' P_k \tilde{x}. \end{aligned}$$

Based on Young's inequality, we have

$$\begin{aligned} 2(\bar{A}_k \tilde{x})' P_{k+1} \begin{bmatrix} F_{k,1}(x) \\ F_{k,1}(x) - F_{k,1}(\hat{x}) \end{bmatrix} \leqslant& \alpha_1^{-1} (\bar{A}_k \tilde{x})' P_{k+1} \bar{A}_k \tilde{x} \\ &+ \alpha_1 \begin{bmatrix} F_{k,1}(x) \\ F_{k,1}(x) - F_{k,1}(\hat{x}) \end{bmatrix}' P_{k+1} \begin{bmatrix} F_{k,1}(x) \\ F_{k,1}(x) - F_{k,1}(\hat{x}) \end{bmatrix}, \\ 2(\bar{C}_k \tilde{x})' P_{k+1} \begin{bmatrix} F_{k,2}(x) \\ F_{k,2}(x) \end{bmatrix} \leqslant& \alpha_2^{-1} (\bar{C}_k \tilde{x})' P_{k+1} \bar{C}_k \tilde{x} + \alpha_2 \begin{bmatrix} F_{k,2}(x) \\ F_{k,2}(x) \end{bmatrix}' P_{k+1} \begin{bmatrix} F_{k,2}(x) \\ F_{k,2}(x) \end{bmatrix}, \end{aligned}$$

where α_1 and α_2 are any positive numbers. Therefore,

$$\begin{aligned} \Delta V_k(\tilde{x}) \leqslant& (1 + \alpha_1^{-1})(\bar{A}_k \tilde{x})' P_{k+1} (\bar{A}_k \tilde{x}) + (1 + \alpha_1) \beta \lambda^2 \tilde{x}' \tilde{x} + (1 + \alpha_2^{-1})(\bar{C}_k \tilde{x})' P_{k+1} (\bar{C}_k \tilde{x}) \\ &+ 2(1 + \alpha_2) \beta \lambda^2 \tilde{x}' \bar{I} \tilde{x} - \tilde{x}' P_k \tilde{x}. \end{aligned}$$

Note that inequality (22) leads to $\Delta V_k(\tilde{x}) \leqslant -(1 - \alpha_3) V_k(\tilde{x})$. So, as long as control gain matrix sequence $\{K_{k,1,1}, K_{k,2,1}\}$ and observer parameter sequence $\{\hat{L}_k\}$ can guarantee (22), the system (19) is globally UESMS.

Remark 10. Although the condition (22) is not a strict LMI for each $k \in \mathcal{N}$, we can still adopt similar design steps as in Algorithm 1 to solve the inequality (22) for the periodic coefficients of (19) with pregiven parameters $0 < \alpha_1, \alpha_2, \alpha_3 < 1$, which can be transformed into solving a new convex optimization problem:

$$\min_{X_{k,i+1}, P_{k,i+1}} \sum_{k=0}^{T-1} \{\mathrm{trace}(P_{k,i} X_{k,i+1}) + \mathrm{trace}(X_{k,i} P_{k,i+1})\}$$

subject to LMIs

$$\begin{bmatrix} -\alpha_3 P_{k,i+1} + (1 + \alpha_1) \beta \lambda^2 + 2(1 + \alpha_2) \beta \lambda^2 \bar{I} & \sqrt{(1 + \alpha_1^{-1})} \bar{A}_k' & \sqrt{(1 + \alpha_2^{-1})} \bar{C}_k' \\ * & -X_{k+1,i+1} & 0 \\ * & * & -X_{k+1,i+1} \end{bmatrix} < 0, \ k \in \mathcal{N}_{T-1},$$

and

$$\begin{bmatrix} P_{k,i+1} & I \\ I & X_{k,i+1} \end{bmatrix} \geqslant 0,\ k \in \mathcal{N}_{T-1}.$$

5 Simulation

In this section, two examples are provided to illustrate the effectiveness of our main results.

Example 1. Consider a linear periodic stochastic system (1) with the period $T = 2$ and the system parameters given by

$$A_0 = \begin{bmatrix} -0.52 & 0.70 \\ 0.10 & -0.18 \end{bmatrix},\ A_1 = \begin{bmatrix} 0.43 & -0.50 \\ -0.13 & 0.14 \end{bmatrix},\ C_0 = \begin{bmatrix} -0.19 & -0.32 \\ 0.54 & 0.06 \end{bmatrix},\ C_1 = \begin{bmatrix} -0.46 & -0.18 \\ -0.22 & 0.12 \end{bmatrix},$$

$$B_{0,1} = \begin{bmatrix} -0.75 & -0.03 \\ 1.52 & 1.64 \end{bmatrix},\ B_{1,1} = \begin{bmatrix} -0.43 & -0.06 \\ 0.59 & -2.02 \end{bmatrix},\ B_{0,2} = \begin{bmatrix} -0.98 & -0.05 \\ 0.61 & -1.12 \end{bmatrix},\ B_{1,2} = \begin{bmatrix} -0.63 & -0.99 \\ 0.25 & 0.98 \end{bmatrix},$$

$$D_{0,1} = \begin{bmatrix} 0.94 & -0.37 \\ 0.30 & 0.82 \end{bmatrix},\ D_{1,1} = \begin{bmatrix} 0.80 & 0.57 \\ 0.12 & 0.41 \end{bmatrix},\ D_{0,2} = \begin{bmatrix} -0.99 & -0.66 \\ 0.76 & -0.60 \end{bmatrix},\ D_{1,2} = \begin{bmatrix} 0.18 & -0.13 \\ -0.31 & 0.60 \end{bmatrix},$$

$$H_0 = \begin{bmatrix} 1.05 & 0.33 \\ -0.20 & -0.24 \end{bmatrix},\ H_1 = \begin{bmatrix} 0.23 & -0.62 \\ 0.44 & 0.27 \end{bmatrix},\ M_0 = \begin{bmatrix} 0.60 & 1.73 \\ 0.09 & -0.61 \end{bmatrix},\ M_1 = \begin{bmatrix} -0.74 & 0.91 \\ -1.75 & 0.87 \end{bmatrix}.$$

We assume that the fault signal $f_k = [\sin(0.1k\pi), \cos(0.1k\pi)]'$ is also a periodic failure. From Theorem 3, we know that a fault estimator as (2) and a control strategy as (4) can guarantee the convergence of system state and fault estimation errors. Substituting the system parameters into Algorithm 1, we can obtain a set of feasible solutions as

$$P_0 = \begin{bmatrix} 0.67 & 0.34 & -0.20 & 0.23 \\ 0.34 & 1.63 & 0.36 & 0.61 \\ -0.20 & 0.36 & 0.66 & 0.47 \\ 0.23 & 0.61 & 0.47 & 1.44 \end{bmatrix},\ P_1 = \begin{bmatrix} 0.74 & -0.27 & -0.24 & -0.41 \\ -0.27 & 1.69 & -0.31 & 0.95 \\ -0.24 & -0.31 & 0.81 & -0.69 \\ -0.41 & 0.95 & -0.69 & 3.96 \end{bmatrix},$$

$$X_0 = \begin{bmatrix} 2.44 & -0.60 & 1.47 & -0.62 \\ -0.60 & 0.91 & -0.60 & -0.09 \\ 1.47 & -0.60 & 2.91 & -0.93 \\ -0.62 & -0.09 & -0.93 & 1.14 \end{bmatrix},\ X_1 = \begin{bmatrix} 1.96 & 0.31 & 0.96 & 0.27 \\ 0.31 & 0.76 & 0.29 & -0.09 \\ 0.96 & 0.29 & 1.95 & 0.35 \\ 0.27 & -0.09 & 0.35 & 0.37 \end{bmatrix},$$

and the observer and controller parameters can be designed as

$$K_{0,1,1} = \begin{bmatrix} -0.68 & 0.92 \\ 0.67 & -0.54 \end{bmatrix}, K_{1,1,1} = \begin{bmatrix} 0.75 & 0.29 \\ 0.18 & -0.30 \end{bmatrix}, K_{0,1,2} = \begin{bmatrix} -1.34 & -0.11 \\ 0.87 & 0.78 \end{bmatrix}, K_{1,1,2} = \begin{bmatrix} -1.43 & -2.31 \\ -0.29 & -0.19 \end{bmatrix},$$

$$K_{0,2,1} = \begin{bmatrix} 0.44 & 0.18 \\ -0.57 & 0.28 \end{bmatrix}, K_{1,2,1} = \begin{bmatrix} 0.30 & -0.56 \\ 0.04 & 0.68 \end{bmatrix}, K_{1,2,1} = \begin{bmatrix} 0.59 & 0.86 \\ -1.15 & 0.42 \end{bmatrix}, K_{1,2,2} = \begin{bmatrix} -0.95 & 1.51 \\ 1.02 & -1.88 \end{bmatrix},$$

$$\hat{L}_0 = \begin{bmatrix} -0.01 & 0.02 \\ 0.09 & 0.03 \end{bmatrix}, \hat{L}_1 = \begin{bmatrix} 0.01 & 0.01 \\ 0.04 & 0.28 \end{bmatrix}.$$

We use Matlab to perform 50 simulations about the state trajectories of system (1) under the proposed estimator and controllers and initial vale $x_0 = [2\ \ 1]'$ shown in Figure 2. Meanwhile, the estimation effect of the fault f_k is presented in Figure 3. It is obvious that the estimation error e_{f_k} is convergent. The mean square values $E\|e_{x_k}\|^2$ and $E\|e_{f_k}\|^2$ are exhibited in Figure 4.

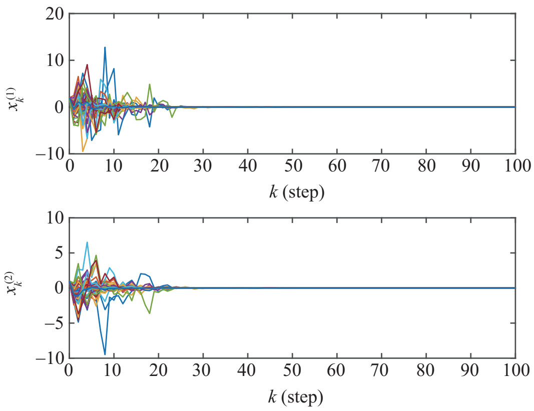

Figure 2 (Color online) The state x_k of system (1) under f_k.

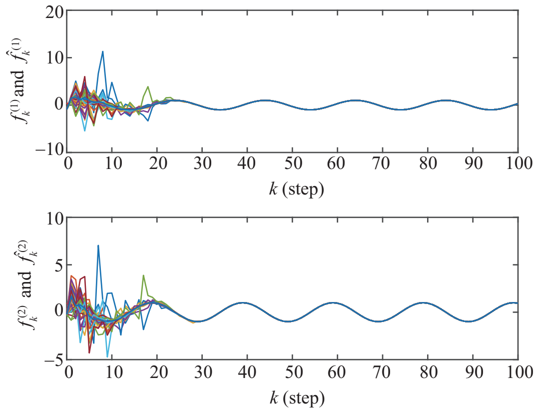

Figure 3 (Color online) The fault f_k and fault estimation $\hat{f}_k$.

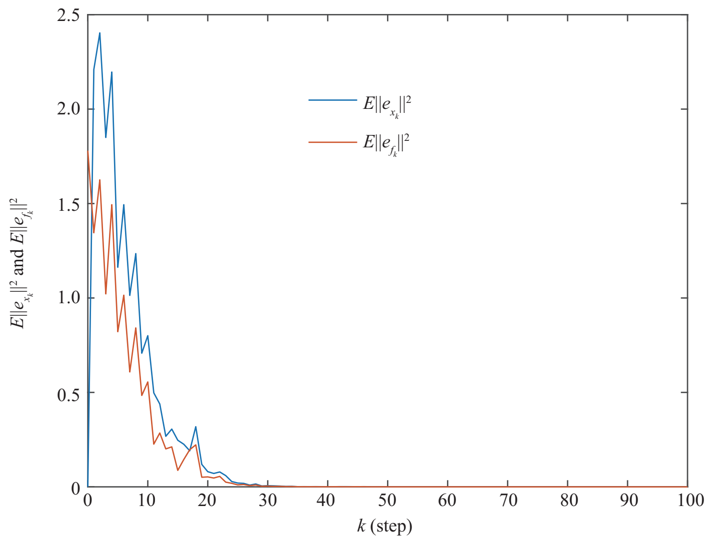

Figure 4 (Color online) The mean square values $E\|e_{x_k}\|^2$ and $E\|e_{f_k}\|^2$.

Furthermore, even the fault signal has some random fluctuation properties, our designed method is also applicable. For example, if the elements in $f_k = \bar{f}_k$ are independent white noise processes, by doing 4 random experiments and drawing the fault estimations and state trajectories in Figures 5–7, we can find that the controlled system is still stable.

Example 2. An electromechanical servo system with faults and internal noises can be described by a linear time-invariant discrete stochastic model [35], that is

$$\begin{cases} x_{k+1} = \begin{bmatrix} 1.01 & -0.93 \\ 0.26 & 0.02 \end{bmatrix} x_k + \begin{bmatrix} -1.90 & 0.57 \\ -0.37 & -1.95 \end{bmatrix} u_k + \begin{bmatrix} -0.22 & 0.98 \\ -0.71 & -1.21 \end{bmatrix} f_k + \begin{bmatrix} -0.14 & 0.03 \\ -0.20 & -0.18 \end{bmatrix} x_k w_k, \\ y_k = \begin{bmatrix} 1.23 & -1.70 \\ 0.78 & 0.77 \end{bmatrix} x_k + \begin{bmatrix} -0.88 & 1.6 \\ 1.20 & 2.07 \end{bmatrix} f_k, \ k \in \mathcal{N}, \end{cases} \tag{24}$$

where $x_k = [\begin{smallmatrix} x_k^{(1)} \\ x_k^{(2)} \end{smallmatrix}]$, $x_k^{(1)}$ represents the load angular position, $x_k^{(2)}$ denotes the shaft speed, u_k stands for the input voltage. By Corollary 3, we can find that the minimum value of λ is 0.8407. Meanwhile,

$$u_k = \begin{bmatrix} 0.3148 & -0.4550 \\ -0.2646 & -0.1061 \end{bmatrix} \hat{x}_k + \begin{bmatrix} -0.2150 & 0.3101 \\ -0.3232 & -0.6817 \end{bmatrix} \hat{f}_k,$$

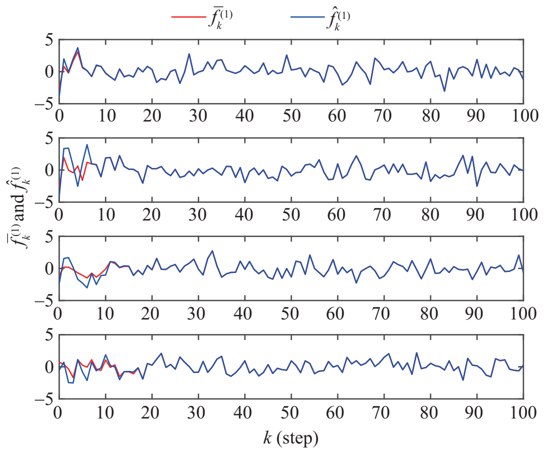

Figure 5 (Color online) The fault $\bar{f}_k^{(1)}$ and fault estimation $\hat{f}_k^{(1)}$.

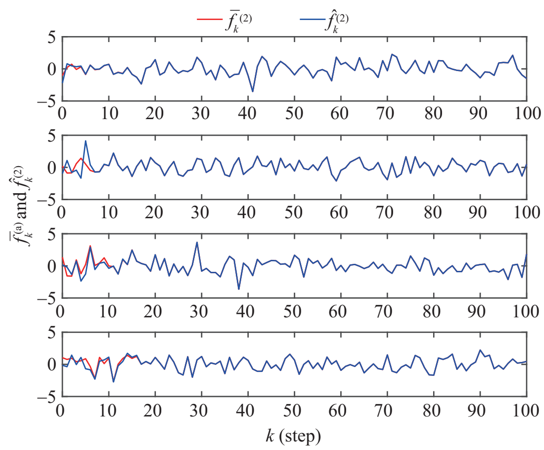

Figure 6 (Color online) The fault $\bar{f}_k^{(2)}$ and fault estimation $\hat{f}_k^{(2)}$.

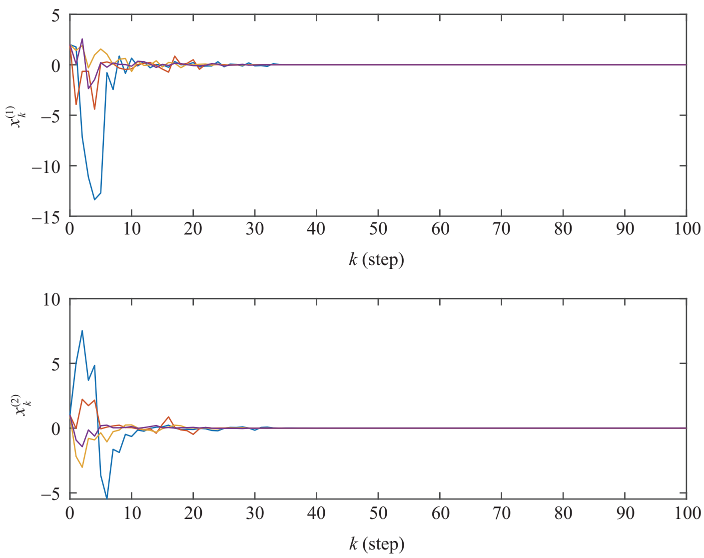

Figure 7 (Color online) The state x_k of system (1) under $\bar{f}_k$.

and

$$\hat{L} = \begin{bmatrix} -0.2195 & -0.1109 \\ -1.2338 & 0.4091 \end{bmatrix}.$$

Then, the state responses of the closed-loop system (24) are depicted in Figure 8, where the initial condition $x_0 = [2, 1]'$ and fault information is given by Figure 9. From the simulation results, one can see that the proposed state transition matrix method effectively overcomes the presence of randomly occurring fault while ensuring the stability of the system.

6 Conclusion

In this paper, a novel simultaneous state and fault estimator has been proposed for the sake of fault compensation problems in linear discrete time-varying stochastic systems. The stability of the closed-loop system is analyzed through the Lyapunov function and the state transition matrix method, respectively. Finally, through a numerical simulation and an application in electromechanical servo systems, we have demonstrated that the proposed fault compensation scheme yields expected performance. The fault diagnosis of systems with external disturbances and model uncertainties is an important research direction [37–39], which is worth further exploration. The fault-tolerant control method adopted in this paper

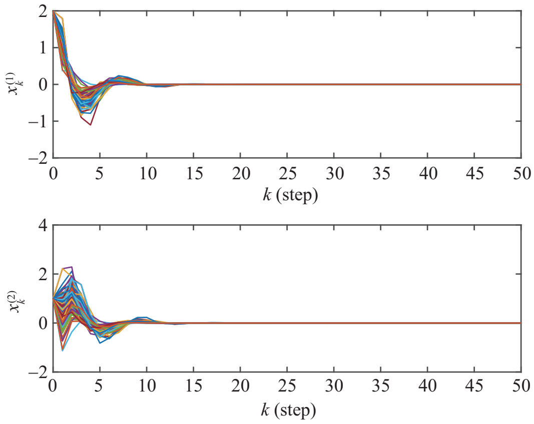

Figure 8 (Color online) The state responses of the closed-loop system (24).

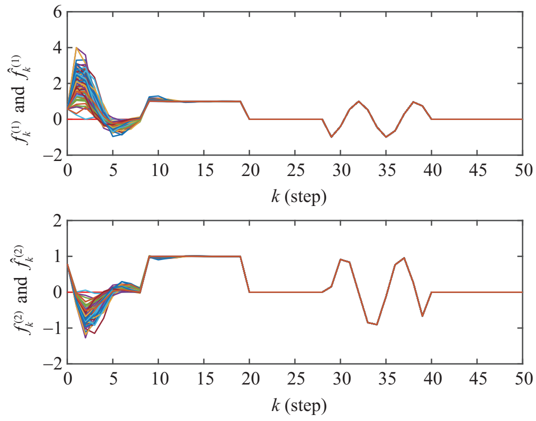

Figure 9 (Color online) The fault f and estimated value $\hat{f}$ in system (24).

can be applied to some uncertain stochastic systems with norm-bounded uncertainty and measurement disturbances. If the system under consideration has unknown disturbances in the dynamic equations, we believe that the fault observation and fault-tolerant control methods proposed in this paper can still be used.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 62073144, 61733008, 61873099, 61803108), National Science Foundation of Guangdong Province (Grant No. 2020A1515010441), and Guangzhou Science and Technology Planning Project (Grant Nos. 202002030158, 202002030389), Key-area Research and Development Program of Guangdong Province (Grant No. 2020B0909020001), Science and Technology Research Project of Chongqing Education Commission (Grant Nos. KJZD-M201900801, KJQN201900831), and Chongqing Natural Science Foundation (Grant No. cstc2020jcyj-msxmX0077).

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