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Global adaptive stabilization for planar nonlinear systems with unknown input powers

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Appendix A The explicit construction of $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$

Choose $\lambda_1(y,z)=z\big(1+y^2\big)$ and $\lambda_2(y,z)=z\big(1+y^2\big)^z\big(1+\exp\big(1+y^2\big)\big)$. Clearly, $\lambda_1(\cdot)$ satisfies (2) and $\lambda_2(\cdot)$ satisfies the first relation in (3), and the last two relations in (3) can be verified as follows:

$$\begin{cases} \lim\limits_{z\to+\infty}\inf\limits_{y\in\mathbf{R}\setminus\{0\}}\dfrac{z\big(1+y^2\big)^z\big(1+\exp\big(1+y^2\big)\big)}{|y|^z} \\ =\lim\limits_{z\to+\infty}\inf\limits_{y\in\mathbf{R}\setminus\{0\}}z\Big(\dfrac{1}{|y|}+|y|\Big)^z\Big(1+\exp\big(1+y^2\big)\Big) \;\geqslant\; \lim\limits_{z\to+\infty}z \;=\; +\infty, \\ \lim\limits_{|y|\to+\infty}\inf\limits_{z\geqslant 1}\dfrac{z\big(1+y^2\big)^z\big(1+\exp\big(1+y^2\big)\big)}{|y|^c} \\ =\lim\limits_{|y|\to+\infty}\dfrac{\big(1+y^2\big)\big(1+\exp\big(1+y^2\big)\big)}{|y|^c} \;=\; +\infty, \quad \forall c\geqslant 1. \end{cases}$$

Appendix B Proof of Proposition 1

We first introduce the following lemmas, which would be frequently used in the proof.

Lemma 1. If $p_1>0$, $p_2>0$ and $c>0$, then there holds for $\forall x,\,y\in\mathbf{R}$,

$$|x|^{p_1}|y|^{p_2} \;\leqslant\; c\frac{p_1}{p_1+p_2}|x|^{p_1+p_2}+c^{-\frac{p_1}{p_2}}\frac{p_2}{p_1+p_2}|y|^{p_1+p_2}.$$

Lemma 2. If $p\geqslant 1$, then there holds for $\forall x_i\in\mathbf{R}$, $i=1,\dots,m$,

$$\sum_{i=1}^m|x_i|^p \;\leqslant\; \left(\sum_{i=1}^m|x_i|\right)^p \;\leqslant\; m^{p-1}\sum_{i=1}^m|x_i|^p.$$

Lemma 3. If $\underline{p}<p<\bar{p}$, then there holds for $\forall x\in\mathbf{R}$,

$$|x|^p \;\leqslant\; |x|^{\underline{p}}+|x|^{\bar{p}}.$$

Lemma 4. If $p\geqslant 1$, then there holds for $\forall x,\,y\in\mathbf{R}$,

$$\big|[x]^p-[y]^p\big|\leqslant|x-y|^p+p|x-y|\big(|x|^{p-1}+|y|^{p-1}\big),$$

with $[\cdot]^p=\mathbf{sign}(\cdot)|\cdot|^p$.

The next proof is divided into the following two steps.

Step 1: Let $V_1(x_1)=\frac{1}{2}x_1^2$. Along the trajectories of system (1) and by Assumption 3, we have

$$\dot{V}_1 \;=\; g_1(x)x_1[x_2]^{p_1(t)}+x_1f_1(x) \;\leqslant\; g_1(x)x_1[x_2]^{p_1(t)}+\theta h_1(x_1)|x_1|^{1+\bar{p}}. \tag{B1}$$

Then, by (4), (6) and Assumption 2, and noting $\lambda_2(\cdot)\geqslant 1$ and $h_1(x_1)\geqslant 1$, we have

$$\begin{aligned}\dot{V}_1 &\leqslant g_1(x)x_1[x_2]^{p_1(t)}+\theta h_1(x_1)|x_1|^{1+\bar{p}}-g_1(x)x_1[\alpha_1]^{p_1(t)} \\ &\quad +g_1(x)x_1\left[-\left(\underline{g}_1^{-1}(x_1)+\underline{g}_1^{-\frac{1}{\lambda_1(x_1,K)}}(x_1)\right)\lambda_2(x_1,K)h_1(x_1)x_1\right]^{p_1(t)} \\ &\leqslant g_1(x)x_1\left([x_2]^{p_1(t)}-[\alpha_1]^{p_1(t)}\right)+\theta h_1(x_1)|x_1|^{1+\bar{p}}\end{aligned}$$

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$$-a_1\left(\underline{g}_1^{1-p_1(t)}(x_1)+\underline{g}_1^{1-\frac{p_1(t)}{\lambda_1(x_1,K)}}(x_1)\right)\lambda_2(x_1,K)h_1(x_1)|x_1|^{1+p_1(t)}. \tag{B2}$$

Step 2: Along the trajectories of system (1), and by (4) and $V(\cdot)=V_1(\cdot)+\frac{1}{2}\xi_2^2$, we have

$$\dot V \;=\; \dot V_1+g_2(x)\xi_2[u]^{p_2(t)}+\xi_2f_2(x)-\xi_2\frac{\partial\alpha_1}{\partial x_1}\big(g_1(x)[x_2]^{p_1(t)}+f_1(x)\big)-\xi_2\frac{\partial\alpha_1}{\partial K}\dot K. \tag{B3}$$

We next give the appropriate estimations for the last three terms (denoted by ①, ②, ③, respectively) on the right-hand side of (B3).

By Assumption 3, (4), and Lemmas 1 and 2, we have

$$\begin{aligned}
① \;=\; \xi_2f_2(x) \;&\leqslant\; \theta h_2(x_{[2]})\big(|x_1|^{\bar p}+|x_2|^{\bar p}\big)|\xi_2|\\
&\leqslant\; \theta h_2(x_{[2]})\Big(|x_1|^{\bar p}+2^{\bar p-1}\left(|\xi_2|^{\bar p}+\beta_1^{\bar p}|x_1|^{\bar p}\right)\Big)|\xi_2|\\
&\leqslant\; \theta 2^{\bar p-1}h_2(x_{[2]})|\xi_2|^{1+\bar p}+\theta 2^{\bar p-1}h_2(x_{[2]})\Big(1+\big(1+\beta_1\big)^{\bar p}\Big)|x_1|^{\bar p}|\xi_2|\\
&\leqslant\; |x_1|^{1+\bar p}+\Theta_1H_1(x,K)|\xi_2|^{1+\bar p},
\end{aligned} \tag{B4}$$

where $\Theta_1 \;=\; \theta 2^{\bar p-1}+2^{\bar p^2-1}\theta^{1+\bar p}$ is an unknown positive constant, and

$$H_1(\cdot)\;=\;h_2(x)+h_2^{1+\bar p}(x)\Big(1+\big(1+\beta_1\big)^{\bar p}\Big)^{1+\bar p}$$

is a positive smooth function. Remark that $H_1(\cdot)\geqslant 1$ is strictly increasing on the unknown positive constant $\bar p$. Then, to show the dependence on $\bar p$ intuitively, we let $H_1(x,K)=H_1(x,K,\bar p)$ with some abuse of notation.

By (1), (4), Lemmas 1 and 2, and noting $p_1(t)\leqslant\bar p$, we have

$$\begin{aligned}
② \;&=\; -\xi_2\frac{\partial\alpha_1}{\partial x_1}\big(g_1(x)[x_2]^{p_1(t)}+f_1(x)\big)\\
&\leqslant\; |\xi_2|\cdot\Big|\frac{\partial\alpha_1}{\partial x_1}\Big|\Big(b_1\bar g_1(x)2^{p_1(t)-1}\Big(|\xi_2|^{p_1(t)}+\beta_1^{p_1(t)}|x_1|^{p_1(t)}\Big)+\theta h_1(x_1)|x_1|^{\bar p}\Big)\\
&\leqslant\; 2^{p_1(t)-1}b_1\bar g_1(x)\Big|\frac{\partial\alpha_1}{\partial x_1}\Big|\cdot|\xi_2|^{1+p_1(t)}+2^{p_1(t)-1}b_1\bar g_1(x)\Big|\frac{\partial\alpha_1}{\partial x_1}\Big|\big(1+\beta_1\big)^{p_1(t)}|x_1|^{p_1(t)}|\xi_2|\\
&\quad+\theta h_1(x_1)\Big|\frac{\partial\alpha_1}{\partial x_1}\Big|\cdot|x_1|^{\bar p}|\xi_2|\\
&\leqslant\; |x_1|^{1+p_1(t)}+\Theta_2H_2(x,K)|\xi_2|^{1+p_1(t)}+|x_1|^{1+\bar p}+\Theta_3H_3(x,K)|\xi_2|^{1+\bar p},
\end{aligned} \tag{B5}$$

where $\Theta_2=2^{\bar p-1}b_1+2^{\bar p^2-1}b_1^{1+\bar p}$ and $\Theta_3=\theta^{1+\bar p}$ are unknown positive constants, and

$$\begin{cases}
H_2(\cdot) \;=\; \Big(1+\bar g_1^2(x)\big(\frac{\partial\alpha_1}{\partial x_1}\big)^2\Big)^{\frac{1+p_1(t)}{2}}\Big(1+\big(1+\beta_1\big)^{p_1(t)(1+p_1(t))}\Big),\\
H_3(\cdot) \;=\; \Big(1+\big(\frac{\partial\alpha_1}{\partial x_1}\big)^2\Big)^{\frac{1+\bar p}{2}}h_1^{1+\bar p}(x_1).
\end{cases}$$

Similar to the discussions on $H_1(\cdot)$, we see that $H_2(\cdot)$ and $H_3(\cdot)$ are strictly increasing respectively on the unknown positive parameters $p_1(t)$ and $\bar p$, and let $H_2(x,K)=H_2(x,K,p_1(t))$ and $H_3(x,K)=H_3(x,K,\bar p)$.

By (5) and Lemma 1, we have

$$③ \;=\; -\xi_2\frac{\partial\alpha_1}{\partial K}\big(|x_1|^K+|\xi_2|^K\big)\;\leqslant\;|x_1|^{1+K}+H_4(x_1,K)|\xi_2|^{1+K}, \tag{B6}$$

where

$$H_4(\cdot)=\left(1+\Big(\frac{\partial\alpha_1}{\partial K}\Big)^2\right)^{\frac{1+K}{2}}+\left(1+\Big(\frac{\partial\alpha_1}{\partial K}\Big)^2\right)^{\frac{1}{2}}.$$

Moreover, by (4) and Lemmas 1, 2 and 4, we have

$$\begin{aligned}
&g_1(x)x_1\left([x_2]^{p_1(t)}-[\alpha_1]^{p_1(t)}\right)\;\leqslant\; b_1\bar g_1(x)|x_1|\left(|\xi_2|^{p_1(t)}+p_1(t)|\xi_2|\big(|x_2|^{p_1(t)-1}+|\alpha_1|^{p_1(t)-1}\big)\right)\\
&\leqslant\; b_1\bar g_1(x)|x_1|\left(|\xi_2|^{p_1(t)}+p_1(t)|\xi_2|\Big(2^{p_1(t)-1}\Big(|\xi_2|^{p_1(t)-1}+\beta_1^{p_1(t)-1}|x_1|^{p_1(t)-1}\Big)+|\beta_1x_1|^{p_1(t)-1}\Big)\right)\\
&\leqslant\; |x_1|^{1+p_1(t)}+\Theta_4H_5(x,K)|\xi_2|^{1+p_1(t)},
\end{aligned} \tag{B7}$$

where $\Theta_4=b_1^2\big(4+2^{\bar p}\bar p^2\big)+4^{\bar p}(b_1\bar p)^{1+\bar p}\big(1+2^{\bar p^2-1}\big)$ is an unknown positive constant, and

$$H_5(\cdot)=\big(1+\bar g_1(x)\big)^2+\big(1+\beta_1\bar g_1(x)\big)^{1+p_1(t)}+\big(1+\bar g_1(x)\big)^{1+p_1(t)}\big(1+\beta_1\big)^{p_1^2(t)-1}$$

is a positive smooth function, and strictly increasing on $p_1(t)$. Similarly, we let $H_5(x,K)=H_5(x,K,p_1(t))$.

Thus, substituting (B2) and (B4)–(B7) into (B3) yields

$$\begin{aligned}
\dot V \;\leqslant\; &-a_1\left(\underline{g}_1^{1-p_1(t)}(x_1)+\underline{g}_1^{1-\frac{p_1(t)}{\lambda_1(x_1,K)}}(x_1)\right)\lambda_2(x_1,K)h_1(x_1)|x_1|^{1+p_1(t)}+\big(\theta h_1(x_1)+2\big)|x_1|^{1+\bar p}\\
&+2|x_1|^{1+p_1(t)}+|x_1|^{1+K}+\big(\Theta_1H_1(x,K,\bar p)+\Theta_3H_3(x,K,\bar p)\big)|\xi_2|^{1+\bar p}\\
&+\big(\Theta_2H_2(x,K,p_1(t))+\Theta_4H_5(x,K,p_1(t))\big)|\xi_2|^{1+p_1(t)}+H_4(x_1,K)|\xi_2|^{1+K}+g_2(x)\xi_2[u]^{p_2(t)}.
\end{aligned}$$

By this, (4), (6) and noting (by Lemma 3)

$$\begin{cases} H_2(x,K,p_1(t)) \ \leqslant \ H_2(x,K,\bar{p}), \\ H_5(x,K,p_1(t)) \ \leqslant \ H_5(x,K,\bar{p}), \\ |\xi_2|^{1+\bar{p}} \ \leqslant \ \left(1+\xi_2^2\right)^{\frac{\bar{p}}{2}} |\xi_2|^{1+p_1(t)}, \end{cases}$$

we obtain (7) with $\Theta = 1+\sum_{i=1}^{4}\Theta_i$, and

$$H(\cdot) = \Big(H_1(x,K,\bar{p}) + H_3(x,K,\bar{p})\Big)\left(1+\xi_2^2\right)^{\frac{\bar{p}}{2}} + H_2(x,K,\bar{p}) + H_5(x,K,\bar{p}) + H_4(x_1,K).$$

Clearly, it is not difficult to see that $H(\cdot)$ is a positive smooth function and strictly increasing on $\bar{p}$.

Appendix C Proof of (8)

From (2), (3), and $\lim_{t\to T_f} K(t) = +\infty$, there always exists t_1 such that for $\forall t \in [t_1,\, T_f)$

$$\begin{cases} K(t) \ \geqslant \ \bar{p}+2, \\ \lambda_1(x_1(t),K(t)) \ \geqslant \ \bar{p}, \\ \lambda_1\left(x_1^2(t)+\xi_2^2(t),K(t)\right) \ \geqslant \ \bar{p}, \\ \frac{1}{2}a_1\lambda_2(x_1(t),K(t)) \ \geqslant \ (1+\theta)\left(4|x_1(t)|^{K(t)}+6\right), \\ \frac{1}{2}a_2\lambda_2\left(x_1^2(t)+\xi_2^2(t),K(t)\right) \ \geqslant \ (1+\Theta)\left(2+2|\xi_2(t)|^{K(t)}\right). \end{cases} \tag{C1}$$

Actually, the first three relations can be seen directly from the first relation of (2) and the fact $\lim_{t\to T_f} K(t) = +\infty$, and the last two relations can be shown by the first two relations of (3) and $\lim_{t\to T_f} K(t) = +\infty$.

From the first three relations in (C1), and Lemma 3 in Appendix B, it follows that for $\forall t \in [t_1,\, T_f)$

$$\begin{cases} \underline{g}_1^{1-p_1(t)}(x_1(t)) + \underline{g}_1^{1-\frac{p_1(t)}{\lambda_1(x_1(t),K(t))}}(x_1(t)) \ \geqslant \ 1, \\ \underline{g}_2^{1-p_2(t)}(x(t)) + \underline{g}_2^{1-\frac{p_2(t)}{\lambda_1\left(x_1^2(t)+\xi_2^2(t),K(t)\right)}}(x(t)) \ \geqslant \ 1, \\ |\xi_2(t)|^{1+p_1(t)} + |\xi_2(t)|^{1+K(t)} \ \leqslant \ \left(2+2|\xi_2(t)|^{K(t)}\right)|\xi_2(t)|^{1+\frac{p_2(t)}{\lambda_1\left(x_1^2(t)+\xi_2^2(t),K(t)\right)}}. \end{cases} \tag{C2}$$

Again by using the first relation in (C1), and Lemma 3 in Appendix B, we have for $\forall t \in [t_1,\, T_f)$

$$\begin{aligned} &\left(\theta h_1(x_1(t))+2\right)|x_1(t)|^{1+\bar{p}} + 2|x_1(t)|^{1+p_1(t)} + |x_1(t)|^{1+K(t)} \\ &\leqslant (1+\theta)h_1(x_1(t))\Big(3|x_1(t)|^{\bar{p}-p_1(t)} + 2 + |x_1(t)|^{K(t)-p_1(t)}\Big)|x_1(t)|^{1+p_1(t)} \\ &\leqslant (1+\theta)h_1(x_1(t))\left(4|x_1(t)|^{K(t)}+6\right)|x_1(t)|^{1+p_1(t)}. \end{aligned} \tag{C3}$$

Then, from (7), (C2), (C3), and noting $h_1(x_1) \geqslant 1$ and

$$H\big(x(t),K(t),\lambda_1(x_1^2(t)+\xi_2^2(t),K(t))\big) \ \geqslant \ H(x(t),K(t),\bar{p}) \ \geqslant \ 1,$$

it follows that for $\forall t \in [t_1,\, T_f)$

$$\dot{V}(t) \ \leqslant \ -\frac{1}{2}a_1\lambda_2(x_1(t),K(t))|x_1(t)|^{1+p_1(t)} - \frac{1}{2}a_2\lambda_2\left(x_1^2(t)+\xi_2^2(t),K(t)\right)|\xi_2|^{1+\frac{p_2(t)}{\lambda_1\left(x_1^2(t)+\xi_2^2(t),K(t)\right)}}.$$

By this and noting that the last two relations of (C1) directly imply (by Lemma 3 in Appendix B)

$$\begin{cases} \frac{1}{2}a_1\lambda_2(x_1(t),K(t)) \ \geqslant \ 1+|x_1(t)|^{K(t)} \ \geqslant \ |x_1(t)|^{K(t)-p_1(t)-1}, \\ \frac{1}{2}a_2\lambda_2\left(x_1^2(t)+\xi_2^2(t),K(t)\right) \ \geqslant \ 1+|\xi_2(t)|^{K(t)} \ \geqslant \ |\xi_2(t)|^{K(t)-1-\frac{p_2(t)}{\lambda_1\left(x_1^2(t)+\xi_2^2(t),K(t)\right)}}, \end{cases}$$

we thus obtain (8).

Appendix D Proof of (10)

Noting the boundedness of $K(t)$ on $[t_0,\, T_f)$, we have $K(t) \leqslant M,\ t\in[t_0,\, T_f)$ with M being some positive constant. From $\lim_{t\to T_f}\|x(t)\| = +\infty$ with $T_f \leqslant +\infty$, the next proof is proceeded in the following two cases.

Case I: $\lim_{t\to T_f}|x_1(t)| = +\infty$**.** From (2) and (3), and the boundedness of $K(t)$, there exists t_3, such that for $\forall t\in[t_3,\, T_f)$,

$$\begin{cases} |x_1(t)| \ \geqslant \ 1, \\ \lambda_1(x_1(t),K(t)) \ \geqslant \ \bar{p}, \\ \lambda_1\left(x_1^2(t)+\xi_2^2(t),K(t)\right) \ \geqslant \ \bar{p}, \\ a_1\lambda_2(x_1(t),K(t)) \ \geqslant \ (1+\theta)\Big(3+4|x_1(t)|^{1+\bar{p}}+|x_1(t)|^{1+M}\Big), \\ a_2\lambda_2(x_1^2(t)+\xi_2^2(t),K(t)) \ \geqslant \ \Theta\left(2+|\xi_2(t)|^{\bar{p}}+|\xi_2(t)|^{M}\right). \end{cases} \tag{D1}$$

Actually, the first three relations can be obtained directly from the second relation of (2) and the fact $\lim_{t\to T_f}|x_1(t)|=+\infty$, and the last two relations can be shown by the last relation of (3) and $\lim_{t\to T_f}|x_1(t)|=+\infty$.

From the second and third relations in (D1), Lemma 3 in Appendix B, and $K(t)\leqslant M,\ t\in[t_0,\,T_f)$, it follows that for $\forall t\in[t_3,\,T_f)$

$$\begin{cases}\underline{g}_1^{1-p_1(t)}(x_1(t))+\underline{g}_1^{1-\frac{p_1(t)}{\lambda_1(x_1(t),K(t))}}(x_1(t))\ \geqslant\ 1,\\ \underline{g}_2^{1-p_2(t)}(x(t))+\underline{g}_2^{1-\frac{p_2(t)}{\lambda_1\left(x_1^2(t)+\xi_2^2(t),K(t)\right)}}(x(t))\ \geqslant\ 1,\\ |\xi_2(t)|^{1+p_1(t)}+|\xi_2(t)|^{1+K(t)}\ \leqslant\ \big(2+|\xi_2(t)|^{\bar p}+|\xi_2(t)|^M\big)|\xi_2(t)|^{1+\frac{p_2(t)}{\lambda_1\left(x_1^2(t)+\xi_2^2(t),K(t)\right)}}.\end{cases}\tag{D2}$$

Then, by (7), and noting for $\forall t\in[t_3,\,T_f)$

$$\begin{cases}\big(\theta h_1(x_1(t))+2\big)|x_1(t)|^{1+\bar p}+2|x_1(t)|^{1+p_1(t)}+|x_1(t)|^{1+K(t)}\\ \quad\leqslant\ (1+\theta)h_1(x_1(t))\Big(3+4|x_1(t)|^{1+\bar p}+|x_1(t)|^{1+M}\Big),\\ H\big(x(t),K(t),\lambda_1(x_1^2(t)+\xi_2^2(t),K(t))\big)\ \geqslant\ H(x(t),K(t),\bar p)\ \geqslant\ 1,\end{cases}$$

we obtain

$$\dot V(t)\ \leqslant\ 0,\quad \forall t\in[t_3,\,T_f).$$

Case II: $\sup_{t\in[t_0,\,T_f)}|x_1(t)|<+\infty$ **and** $\lim_{t\to T_f}|x_2(t)|=+\infty$**.** From (4) and the boundedness of $K(t)$ on $[t_0,\,T_f)$, it follows that

$$\begin{cases}\lim_{t\to T_f}|\xi_2(t)|\ =\ +\infty,\\ \big(\theta h_1(x_1(t))+2\big)|x_1(t)|^{1+\bar p}+2|x_1(t)|^{1+p_1(t)}+|x_1(t)|^{1+K(t)}\ \leqslant\ \bar M,\ \forall t\in[t_0,\,T_f),\end{cases}$$

where $\bar M\geqslant 1$ is some constant. Then, there exists t_4, such that for $\forall t\in[t_4,\,T_f)$,

$$\begin{cases}|\xi_2(t)|\ \geqslant\ \bar M,\\ \lambda_1\big(x_1^2(t)+\xi_2^2(t),K(t)\big)\ \geqslant\ \bar p,\\ \frac12 a_2\lambda_2(x_1^2(t)+\xi_2^2(t),K(t))\ \geqslant\ \Theta\big(2+|\xi_2(t)|^{\bar p}+|\xi_2(t)|^M\big)+1.\end{cases}\tag{D3}$$

Actually, the first two relations can be obtained directly from the second relation of (2) and the fact $\lim_{t\to T_f}|\xi_2(t)|=+\infty$, and the last relation can be shown by the last relation of (3) and $\lim_{t\to T_f}|\xi_2(t)|=+\infty$.

By (D3) and Lemma 3 in Appendix B, we obtain that for $\forall t\in[t_4,\,T_f)$,

$$\begin{cases}\underline{g}_2^{1-p_2(t)}(x(t))+\underline{g}_2^{1-\frac{p_2(t)}{\lambda_1\left(x_1^2(t)+\xi_2^2(t),K(t)\right)}}(x(t))\ \geqslant\ 1,\\ |\xi_2(t)|^{1+p_1(t)}+|\xi_2(t)|^{1+K(t)}\ \leqslant\ \big(2+|\xi_2(t)|^{\bar p}+|\xi_2(t)|^M\big)|\xi_2(t)|^{1+\frac{p_2(t)}{\lambda_1\left(x_1^2(t)+\xi_2^2(t),K(t)\right)}},\\ H\big(x(t),K(t),\lambda_1(x_1^2(t)+\xi_2^2(t),K(t))\big)\ \geqslant\ H(x(t),K(t),\bar p)\ \geqslant\ 1,\end{cases}$$

which together with (7) implies

$$\dot V(t)\ \leqslant\ 0,\quad \forall t\in[t_4,\,T_f).$$

Thus, combining the above two cases and letting $t_2=\max\{t_3,\,t_4\}$, we obtain (10).