SCIENCE CHINA
Information Sciences

• LETTER •

September 2021, Vol. 64 199203:1–199203:3
https://doi.org/10.1007/s11432-018-9853-9

Finite-time distributed projection scheme for intersections of convex sets

Zhengquan YANG[1,3*], Qing ZHANG[1] & Zengqiang CHEN[2,1]

[1]*College of Science, Civil Aviation University of China, Tianjin* 300300, *China;*
[2]*College of Computer and Control Engineering, Nankai University, Tianjin* 300350, *China;*
[3]*Air Traffic Management Research Base, Civil Aviation University of China, Tianjin* 300300, *China*

Received 3 December 2018/Revised 13 February 2019/Accepted 29 March 2019/Published online 15 June 2020

Citation Yang Z Q, Zhang Q, Chen Z Q. Finite-time distributed projection scheme for intersections of convex sets. Sci China Inf Sci, 2021, 64(9): 199203, https://doi.org/10.1007/s11432-018-9853-9

Dear editor,

Recently, finite-time consensus of multi-agent systems has attracted extensive attention of many researchers due to its wide application in physics, biology, and control engineering. Cortés [1] provided a signed gradient descent algorithm that can be used to solve the consensus of multi-agent systems with undirected connected topology in finite time. Sundaram et al. [2] gave a linear iteration scheme for solving consensus in distributed systems in a finite number of time steps. In order to control all agents for achieving finite-time consensus, Wang et al. [3] proposed two simple algorithms and provided a good framework to prove the effectiveness of these algorithms. Wang et al. [4] studied multi-agent systems with switching topology and first-order dynamics, and constructed several distributed finite-time consensus rules for solving the x-consensus problem. Lin et al. [5] studied the distributed constrained optimization problem. The controller designed by Lin enables multi-agent systems to achieve finite-time consensus, but minimizes the sum objective function as $t \to \infty$.

In this study, a finite-time convergence algorithm is proposed to solve the intersection of convex sets. The convex intersection computation problem is important to pose and solve physics and engineering problems. There are many algorithms for solving this problem; some of them are based on alternating projection algorithms which are executed by iteratively projecting onto each set, whereas others are based on non-affine transformations. Necoara et al. [6] introduced a general random projection algorithmic framework, which is used for solving the general convex feasibility problem and extends the existing projection algorithms to random cases. In recent years, some distributed algorithms have been used to compute the intersections of convex sets. Shah et al. [7] proposed a stochastic distributed algorithm for solving the intersection of a finite family of convex sets. Shi et al. [8] solved the distributed optimization problem by computing the intersections of convex sets.

The motivation of this study is to provide a novel algorithm for finding the intersections of convex sets. The algorithm has a consensus part and a projection part. The Lyapunov method is used to prove that the control algorithm can achieve finite-time convergence. The algorithms of [5, 8] can also be used to solve the intersections of convex sets, but the convergence rate of these algorithms is asymptotic. However, our algorithm can be used to solve the problem in finite time. Moreover, compared to the distributed optimization with a common constraint [5], the control law described in this study can solve this type of problem having local constraints.

Problem formulation and main result. The multi-agent system studied in this study consists of N ($N \geqslant 2$) agents. Each agent is described by the following dynamic equation:

$$\dot{x}_i(t) = u_i(t), \quad i \in \mathcal{N}, \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$ is the state of agent i, and $u_i(t) \in \mathbb{R}^n$ is the control protocol to be designed.

We equip each agent i with a constraint set X_i. The global constraint is described as the intersection of all X_i, expressed by X:

$$X = \bigcap_{i=1}^{N} X_i.$$

We assume that, for all i, X_i is convex and compact. Further, global constraint X has a non-empty interior.

Then, we consider the following convex feasibility problem:

$$\text{Find} \;\; x \in X = \bigcap_{i=1}^{N} X_i. \tag{2}$$

An undirected network denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is considered. The network $\mathcal{G}$ consists of a vertex set $\mathcal{V} = \{1, 2, \ldots, N\}$, an edge set $\mathcal{E}$, and an adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$. The adjacency matrix $\mathcal{A}$ is defined as $a_{ii} = 0$, $a_{ij} = a_{ji} = 1$ if $(i, j) \in \mathcal{E}$, or otherwise $a_{ij} = 0$. Let the set $N_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ represent the neighbors of vertex i. If there is a path from any vertex to any other one in $\mathcal{G}$, then $\mathcal{G}$ is connected. Here we only consider using static networks for the research.

* Corresponding author (email: zquanyang@163.com)

Then the following distributed control law is proposed:

$$u_i(t) = -\sum_{j\in N_i}\operatorname{sgn}(x_i(t)-x_j(t)) -\alpha\frac{x_i(t)-P_{X_i}(x_i(t))}{\|x_i(t)-P_{X_i}(x_i(t))\|}, \quad (3)$$

where $\alpha > 0$ is a constant. $P_{X_i}(x_i(t))$ denotes a projection of a point $x_i(t)$ to the set $X_i(t)$. It is well known that the projection of $x_i(t)$ on the set $X_i(t)$ exists and is unique.

Theorem 1. Suppose the graph $\mathcal{G}$ is undirected and connected for t. For a system (1) with the control algorithm (3), if $\alpha > N$, all agents will reach a consensus and the problem (2) is solved in finite time.

Proof. Note that the control protocol (3) is discontinuous. Because the signum function is measurable and locally essentially bounded, the Filippov solution for system (1) with the control algorithm (3) exists [9].

Define the nonnegative function as follows:

$$V_i(t) = \frac{1}{2}\|x_i(t)-P_{X_i}(x_i(t))\|^2 \quad (4)$$

for all i. Taking the time derivative of $V_i(t)$, we get

$$\begin{aligned}\dot{V}_i(t) &= (x_i(t)-P_{X_i}(x_i(t)))^{\mathrm{T}}\dot{x}_i(t)\\ &= -\sum_{j\in N_i}(x_i(t)-P_{X_i}(x_i(t)))^{\mathrm{T}}\operatorname{sgn}(x_i(t)-x_j(t))\\ &\quad -\alpha(x_i(t)-P_{X_i}(x_i(t)))^{\mathrm{T}}\frac{x_i(t)-P_{X_i}(x_i(t))}{\|x_i(t)-P_{X_i}(x_i(t))\|}\\ &\leqslant (N-\alpha)\|x_i(t)-P_{X_i}(x_i(t))\|\\ &= -(\alpha-N)\sqrt{2V_i(t)}.\end{aligned} \quad (5)$$

It follows that

$$\frac{\dot{V}_i(t)}{\sqrt{2V_i(t)}} \leqslant -(\alpha-N).$$

Integrating both sides of the above inequality, one can get

$$\sqrt{2V_i(t)}-\sqrt{2V_i(t_0)} \leqslant -(\alpha-N)(t-t_0).$$

If $\alpha > N$, $V_i(t)$ can converge to zero in finite time. In other words, there is a constant $t_1 > t_0$ so that for all $t > t_1$, $\|x_i(t)-P_{X_i}(x_i(t))\| = 0$. Then for all $t > t_1$, the state $x_i(t)$ of a system (1) will converge to X_i. That is, if $t > t_1$, $x_i(t) \in X_i$ under the control (3).

From the above results, if $t > t_1$, we also get

$$\dot{x}_i(t) = -\sum_{j\in N_i}\operatorname{sgn}(x_i(t)-x_j(t)), \quad i\in\mathcal{N}. \quad (6)$$

Because X_i is a closed convex set, there is a constant $s > 0$, which makes $\|x_i(t)\| \leqslant s$, $\forall i\in\mathcal{N}$. Thus, $\|x_i(t)-x_j(t)\| \leqslant \|x_i(t)\|+\|x_j(t)\| \leqslant 2s$. That is, $\|x_i(t)-x_j(t)\|$ is bounded for the system (1) with the algorithm (3).

Next, let us prove that all agents can attain consensus in finite time. Consider the nonnegative function as follows:

$$V(t) = \frac{1}{2}\sum_{i=1}^N\left\|x_i(t)-\frac{1}{N}\sum_{k=1}^N x_k(t)\right\|^2 \quad (7)$$

for all $t > t_1$. Taking the time derivative of $V(t)$, we get

$$\begin{aligned}\dot{V}(t) &= -\sum_{i=1}^N\sum_{j\in N_i}\left[x_i(t)-\frac{1}{N}\sum_{k=1}^N x_k(t)\right]^{\mathrm{T}}\\ &\quad \cdot\operatorname{sgn}(x_i(t)-x_j(t))\\ &= -\frac{1}{2}\sum_{i=1}^N\sum_{j\in N_i}(x_i(t)-x_j(t))^{\mathrm{T}}\operatorname{sgn}(x_i(t)-x_j(t))\\ &= -\frac{1}{2}\sum_{i=1}^N\sum_{j\in N_i}\|x_i(t)-x_j(t)\|_1\\ &\leqslant -\frac{1}{2}\sum_{i=1}^N\sum_{j\in N_i}\|x_i(t)-x_j(t)\|\\ &\leqslant 0.\end{aligned} \quad (8)$$

Because $\|x_i(t)-x_j(t)\|$ is bounded, we can let $\|x_{i_0}(t)-x_{j_0}(t)\| = \max_{i,j\in\mathcal{N}}\|x_i(t)-x_j(t)\|$. We get

$$\sum_{i=1}^N\sum_{j\in N_i}\|x_i(t)-x_j(t)\| \geqslant \|x_{i_0}(t)-x_{j_0}(t)\|,$$

and

$$\begin{aligned}\sqrt{V(t)} &= \sqrt{\frac{1}{2}\sum_{i=1}^N\left\|x_i(t)-\frac{1}{N}\sum_{k=1}^N x_k(t)\right\|^2}\\ &\leqslant \frac{\sqrt{2}}{2}\sum_{i=1}^N\left\|x_i(t)-\frac{1}{N}\sum_{k=1}^N x_k(t)\right\|\\ &\leqslant \frac{\sqrt{2}}{2}\sum_{i=1}^N\frac{1}{N}\sum_{k=1}^N\|x_i(t)-x_k(t)\|\\ &\leqslant \frac{\sqrt{2}}{2}\sum_{i=1}^N\|x_{i_0}(t)-x_{j_0}(t)\|\\ &= \frac{\sqrt{2}N}{2}\|x_{i_0}(t)-x_{j_0}(t)\|.\end{aligned} \quad (9)$$

From (8) and (9), it follows that

$$\begin{aligned}\frac{\dot{V}(t)}{\sqrt{V(t)}} &\leqslant \frac{-\frac{1}{2}\sum_{i=1}^N\sum_{j\in N_i}\|x_i(t)-x_j(t)\|}{\frac{\sqrt{2}N}{2}\|x_{i_0}(t)-x_{j_0}(t)\|}\\ &\leqslant \frac{-\frac{1}{2}\|x_{i_0}(t)-x_{j_0}(t)\|}{\frac{\sqrt{2}N}{2}\|x_{i_0}(t)-x_{j_0}(t)\|}\\ &\leqslant -\frac{1}{\sqrt{2}N} < 0.\end{aligned}$$

We integrate both sides of the inequality and get the following inequality:

$$2\sqrt{V(t)}-2\sqrt{V(t_0)} \leqslant -\frac{1}{\sqrt{2}N}(t-t_0).$$

Thus, $V(t)$ converges to zero in finite time. That is, there is a constant $t_2 > t_1$ such that for all $t > t_2$, $x_1(t) = x_2(t) = \cdots = x_N(t) \in X_i$. There exists a state $x*$, such that for all $t > t_2$, $x_1(t) = x_2(t) = \cdots = x_N(t) = x* \in X = \bigcap_{i=1}^N X_i$. Therefore, for system (1), if $\alpha > N$, all agents will reach a consensus and the problem (2) is solved in finite time under the control algorithm (3).

Remark 1. Our algorithm can be used to solve the problems of [5, 8] in finite time. Compared to the distributed optimization with a common constraint [5], we extended the control law with local constraints.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 61573199, 61571441) and Basic Research Projects of High Education (Grant No. 3122015C025).

References

1 Cortés J. Finite-time convergent gradient flows with applications to network consensus. Automatica, 2006, 42: 1993–2000
2 Sundaram S, Hadjicostis C N. Finite-time distributed consensus in graphs with time-invariant topologies. In: Proceedings of American Control Conference, New York, 2007. 711–716
3 Wang L, Xiao F. Finite-time consensus problems for networks of dynamic agents. IEEE Trans Autom Control, 2010, 55: 950–955
4 Wang X L, Hong Y G. Distributed finite-time χ-consensus algorithms for multi-agent systems with variable coupling topology. J Syst Sci Complex, 2010, 23: 209–218
5 Lin P, Ren W, Farrell J A. Distributed continuous-time optimization: nonuniform gradient gains, finite-time convergence, and convex constraint set. IEEE Trans Autom Control, 2017, 62: 2239–2253
6 Necoara I, Patrascu A. Randomized projection methods for convex feasibility problems. 2018. ArXiv:1801.04873v1
7 Shah S M, Borkar V S. Distributed stochastic approximation with local projections. 2017. ArXiv:1708.08246v1
8 Shi G D, Johansson K H, Hong Y G. Reaching an optimal consensus: dynamical systems that compute intersections of convex sets. IEEE Trans Autom Control, 2013, 58: 610–622
9 Paden B, Sastry S. A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators. IEEE Trans Circ Syst, 1987, 34: 73–82