SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

September 2021, Vol. 64 192201:1–192201:12
https://doi.org/10.1007/s11432-020-2979-0

PID control of uncertain nonlinear stochastic systems with state observer

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Received 9 February 2020/Revised 7 April 2020/Accepted 29 May 2020/Published online 23 August 2021

Abstract The classical proportional-integral-derivative (PID) controller is ubiquitous in engineering systems that are typically nonlinear with various uncertainties, including random noise. However, most of the literature on PID control focused on linear deterministic systems. Thus, a theory that explains the rationale of the linear PID when dealing with nonlinear uncertain stochastic systems and a concrete design method that can provide explicit formulas for PID parameters are required. Recently, we have demonstrated that the PID controller can globally stabilize a class of second-order nonlinear uncertain stochastic systems, where the derivative of the system output is assumed to be obtainable, which is generally unrealistic in practical applications. This has motivated us to present some theoretical results on PID control with a state observer for nonlinear uncertain stochastic systems. Specifically, a five-dimensional parameter manifold can be explicitly constructed, within which the three PID parameters and two observer gain parameters can be arbitrarily selected to globally stabilize nonlinear uncertain stochastic systems, as long as some knowledge about the unknown nonlinear drift and diffusion terms is available.

Keywords PID controller, system uncertainty, nonlinear stochastic systems, global stability, state observer

Citation Cong X R, Zhao C. PID control of uncertain nonlinear stochastic systems with state observer. Sci China Inf Sci, 2021, 64(9): 192201, https://doi.org/10.1007/s11432-020-2979-0

1 Introduction

Over the past 60 years, remarkable progress in modern control theory has been made; however, the classical proportional-integral-derivative (PID) controller still plays a dominating role in various engineering systems [1, 2].

Thus, it is natural and necessary to ask why the linear PID controller is so effective in practice. Some common answers include the following: (1) it has a simple controller structure and does not require precise mathematical models of controlled dynamical systems; (2) it can reduce the influence of various uncertainties, including internal structure uncertainties and external disturbances through the linear feedback mechanism; (3) it can eliminate steady state offsets through the integral action and predict the future tendency through the derivative action.

A critical issue for engineers in PID controller implementation is the selection of the three PID parameters. The PID controller has been extensively investigated by various engineers and scientists; however, most existing studies have focused on linear deterministic dynamic systems [3–8]. To justify the remarkable effectiveness of the PID controller in real-world systems and understand its rationale when dealing with nonlinearity, uncertainty, and randomness, we have to face with uncertain nonlinear stochastic dynamical systems [9, 10].

Recently, Zhao and Guo [11, 12] presented a mathematical theory of PID control for a class of second-order uncertain nonlinear deterministic systems. They demonstrated that a simple three-dimensional manifold can be constructed such that whenever the three PID parameters are selected from this manifold, the closed-loop system will be globally stable, and the system's regulation error will asymptotically

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converge to zero. Then, Cong and Guo [13] extended this theory to stochastic uncertain nonlinear systems because nearly all practical systems are subject to random influences, e.g., external fluctuations, internal agitation, and fluctuating initial conditions. Note that the abovementioned results assume the availability of the derivative of the regulation error, which is unrealistic in practice. These facts have inspired us to investigate the rationale of PID control for stochastic uncertain nonlinear dynamical systems and construct a state observer to replace the derivative terms in the PID controller. Therefore, we need to construct a state observer for nonlinear uncertain stochastic systems and determine whether a state observer-based PID controller can globally stabilize and regulate the system. These are important and challenging issues for both control practitioners and theorists.

In this paper, we consider a basic class of uncertain nonlinear stochastic systems under state observer-based PID control. We demonstrate that a concrete five-dimensional manifold can be constructed, from which the three PID parameters and two observer gain parameters can be selected arbitrarily to globally stabilize the considered systems, under the condition that the upper bounds of the Lipschitz constants of both the nonlinear drift and diffusion terms are known. In addition, the second moment of the regulation error converges to zero asymptotically simultaneously.

The remainder of this paper is organized as follows. Section 2 states the problem formulation. Section 3 presents the main results with mathematical proofs, and Section 4 concludes the paper. Auxiliary results are presented in Appendixes A and B.

2 Problem formulation

Stochastic differential equations are generally appropriate models of randomly influenced systems [14–19].

Assume that $\{B(t)\}_{t\geqslant 0}$ is standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with natural filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (Definition A1 in Appendix A). We consider the following basic class of nonlinear uncertain stochastic systems:

$$\begin{cases} \mathrm{d}x_1 = x_2\mathrm{d}t, \\ \mathrm{d}x_2 = f(x_1, x_2, t)\mathrm{d}t + u(t)\mathrm{d}t + \sigma(x_1, x_2, t)\mathrm{d}B(t), \quad x_1(t) \in \mathbb{R}^d, \\ y(t) = x_1(t), \end{cases} \tag{1}$$

where $u(t)$ and $y(t)$ denote the input and output signals, respectively, and $f(x_1, x_2, t)$ and $\sigma(x_1, x_2, t)$ are uncertain nonlinear functions.

The control objective is to design a PID feedback controller such that for any initial state $(x_1(0), x_2(0)) \in \mathbb{R}^{2d}$, the output $y(t)$ converges to a given setpoint $y^* \in \mathbb{R}^d$.

The classical PID controller has the following standard form:

$$u(t) = k_1 e(t) + k_0 \int_0^t e(s)\mathrm{d}s + k_2 \dot{e}(t),\ e(t) = y(t) - y^*,$$

where $e(t)$ is the regulation error, and k_0, k_1, k_2 are the three PID parameters.

From the definition of the classical PID control, we know that PID implementation requires the derivative of the regulation error. However, in most practical situations, the derivatives may not be available directly. Therefore, we need to construct a state observer to obtain an online estimation of the derivative of the regulation error.

Thus, a natural question is how to design the state observers for uncertain stochastic systems (1). Can a state observer-based PID controller globally stabilize the system and achieve the desired control objective?

In this paper, we construct the following state observer to estimate the state (x_1, x_2):

$$\begin{cases} \mathrm{d}\hat{x}_1 = \hat{x}_2\mathrm{d}t + \beta_1(y(t)\mathrm{d}t - \hat{x}_1\mathrm{d}t), \\ \mathrm{d}\hat{x}_2 = \beta_2(y(t)\mathrm{d}t - \hat{x}_1\mathrm{d}t) + \hat{u}(t)\mathrm{d}t, \end{cases} \tag{2}$$

where β_1, β_2 are observer gain parameters, and

$$\hat{u}(t) = k_1 e(t) + k_2 \hat{x}_2(t). \tag{3}$$

Note that $\dot{e}(t) = x_2(t)$, and $x_2(t)$ is not obtainable for feedback. Thus, we use $\hat{x}_2(t)$ to replace the D-term $\dot{e}(t)$. Then, the state observer-based PID controller $u(t)$ has the following form:

$$u(t) = k_1 e(t) + k_0 \int_0^t e(s)\mathrm{d}s + k_2 \hat{x}_2(t). \tag{4}$$

The goal of this paper is to demonstrate that the ubiquitous PID can stabilize and regulate the above stochastic uncertain nonlinear dynamical systems globally under suitable conditions on the system's unknown functions. In addition, a quantitative design method for the PID and observer gain parameters is also provided.

3 The main results

Note that the unknown functions $f(x_1, x_2, t)$ and $\sigma(x_1, x_2, t)$ are defined on $\mathbb{R}^{2d} \times \mathbb{R}^+$ and take values in $\mathbb{R}^d$. We define two function spaces as follows:

$$\mathcal{F}_M = \left\{ f \in H \middle| \|f(x_1, x_2, t) - f(y_1, y_2, t)\| \leqslant M\sqrt{\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2},\ \forall x_1, x_2, y_1, y_2 \in \mathbb{R}^d, \forall t \in \mathbb{R}^+ \right\},$$
$$\mathcal{D}_N = \left\{ \sigma \in H \middle| \|\sigma(x_1, x_2, t) - \sigma(y_1, y_2, t)\| \leqslant N\sqrt{\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2},\ \forall x_1, x_2, y_1, y_2 \in \mathbb{R}^d, \forall t \in \mathbb{R}^+ \right\},$$

where M and N are positive constants that measure the size of the uncertainty quantitatively, $\|\cdot\|$ is the standard Euclidean norm, and H denotes the space of functions from $\mathbb{R}^{2d} \times \mathbb{R}^+$ to $\mathbb{R}^d$ which are piecewise continuous in t.

The performance of the system (1) under the PID controller (4) can be presented by Theorem 1.

Theorem 1. Consider the nonlinear uncertain stochastic system (1), where $u(t)$ is the state observer-based PID controller (4). Assume that $f(y^*, 0, t) = f(y^*, 0, 0)$ and $\sigma(y^*, 0, t) = 0$, $\forall t \in \mathbb{R}^+$. Then, for any $M > 0$ and $N > 0$, there exists an unbounded open set $\Omega_{k,\beta} \subset \mathbb{R}^5$ such that whenever the controller parameters $(k_0, k_1, k_2, \beta_1, \beta_2) \in \Omega_{k,\beta}$, the closed-loop system will be globally stable and asymptotically optimal in the sense that

$$\sup_{t \geqslant 0} \mathbb{E}[\|x_1(t)\|^2 + \|x_2(t)\|^2 + \|u(t)\|^2] < \infty,$$

and

$$\lim_{t\to\infty} \mathbb{E}\|y^* - x_1(t)\|^2 = 0$$

for any initial value $x(0) = (x_1(0), x_2(0)) \in \mathbb{R}^{2d}$ and $\hat{x}(0) = (\hat{x}_1(0), \hat{x}_2(0)) \in \mathbb{R}^{2d}$, and for any $f \in \mathcal{F}_M$, $\sigma \in \mathcal{D}_N$. Simultaneously, the estimate error satisfies the following:

$$\lim_{t\to\infty} \mathbb{E}\|x_i(t) - \hat{x}_i(t)\|^2 = 0,\ i = 1, 2.$$

Remark 1. From the proof of Theorem 1, the concrete construction of $\Omega_{k,\beta}$ in $\mathbb{R}^5$ can be taken as the following parameterized form:

$$\Omega_{k,\beta} = \left\{ \begin{bmatrix} k_0 \\ k_1 \\ k_2 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \lambda_0\lambda_1\lambda_2 \\ -(\lambda_0\lambda_1 + \lambda_0\lambda_2 + \lambda_1\lambda_2) \\ \lambda_0 + \lambda_1 + \lambda_2 \\ -(\mu_1 + \mu_2) \\ \mu_1\mu_2 \end{bmatrix} \middle| \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \mu_1 \\ \mu_2 \end{bmatrix} \in \Delta_{\lambda,\mu} \right\}$$

with

$$\begin{aligned} \Delta_{\lambda,\mu} = \big\{ (\lambda, \mu) \big| \lambda_i < 0, \lambda_i \neq \lambda_j, i \neq j; \mu_i < 0, \mu_1 \neq \mu_2; \\ 4\mu_0[\lambda_0\lambda_1\lambda_2(1 - M\phi(\lambda)\varphi(\lambda)) + \Phi(\mu, \lambda)N^2\varphi^2(\lambda)] > \Theta(\lambda, \mu) \big\}, \end{aligned}$$

where

$$\lambda \triangleq (\lambda_0, \lambda_1, \lambda_2), \quad \mu \triangleq (\mu_1, \mu_2), \quad \mu_0 \triangleq \max\{\mu_1, \mu_2\},$$

$$\begin{aligned}
&\varphi(\lambda)=\sqrt{3+\lambda_0^2+\lambda_1^2+\lambda_2^{-2}},\quad \phi(\lambda)=\sqrt{\frac{1}{b_0^2}+\frac{1}{b_1^2}+\frac{\lambda_2^2}{b_2^2}},\\
&a_i=\frac{\lambda_i}{b_i},\ i=0,1,\quad a_2=\frac{\lambda_2^2}{b_2},\quad b_j\triangleq\prod_{i\in\{0,1,2\}\backslash\{j\}}(\lambda_j-\lambda_i),\ j=0,1,2,\\
&\Psi(\lambda,\mu)=\sqrt{\frac{\mu_1^2+\mu_2^2}{(\mu_2-\mu_1)^2}}\left(M\varphi(\lambda)+\sqrt{\lambda_1^2\lambda_2^2+\lambda_0^2\lambda_2^2+\frac{\lambda_0^2\lambda_1^2}{\lambda_2^2}}\right),\\
&\Phi(\lambda,\mu)=\frac{1}{2}\left(\frac{\mu_2^2+\mu_1^2}{(\mu_2-\mu_1)^2}+a_0^2\lambda_1\lambda_2+a_1^2\lambda_0\lambda_2+a_2^2\lambda_0\lambda_1\right),\\
&\Theta(\lambda,\mu)=(\Psi(\lambda,\mu)+\sqrt{2}\lambda_0\lambda_1\lambda_2\phi(\lambda)(\lambda_0+\lambda_1+\lambda_2))^2.
\end{aligned}$$

Remark 2. Theorem 1 is a global convergence result, for which the upper bounds of the Lipschitz constants of the nonlinear drift function f and diffusion function σ play a key role in designing the parameters. It is worth noting that the selection of the controller parameters does not rely on the initial values or the setpoint y^*.

Remark 3. We remark that the parameter set $\Omega_{k,\beta}$ is open and unbounded in $\mathbb{R}^5$, and the function space $\mathcal{F}_M$ and $\mathcal{D}_N$ are "relatively large". Thus, we conclude that the PID controller (4) has good robustness with respect to the system uncertainty, the selection of controller parameters, and the system randomness.

The following is a proof of Theorem 1.

Step 1. First, we derive the closed-loop system equation.

For this, we introduce some notations. Let ξ_i denote the estimation error, i.e.,

$$\xi_i(t)=x_i(t)-\hat{x}_i(t),\ i=1,2,$$

and

$$e_0(t)=\int_0^t e(s)\mathrm{d}s+\frac{f(y^*,0,0)}{k_0},\quad e_1(t)=x_1(t)-y^*,\quad e_2(t)=x_2(t).$$

We also define the following two functions: $g_1(e_1,e_2,t)\triangleq f(e_1+y^*,e_2,t)-f(y^*,0,t)$, $g_2(e_1,e_2,t)\triangleq\sigma(e_1+y^*,e_2,t)-\sigma(y^*,0,t)$.

Then, by (1)–(4) and the fact that $f(y^*,0,t)=f(y^*,0,0)$ and $\sigma(y^*,0,t)=0$, it is easy to derive the following closed-loop equation:

$$\begin{cases}
\mathrm{d}\xi_1(t)=\xi_2(t)\mathrm{d}t-\beta_1\xi_1(t)\mathrm{d}t,\\
\mathrm{d}\xi_2(t)=-\beta_2\xi_1(t)\mathrm{d}t+[g_1(e_1,e_2,t)+k_0e_0]\mathrm{d}t+g_2(e_1,e_2,t)\mathrm{d}B(t),\\
\mathrm{d}e_0(t)=e_1(t)\mathrm{d}t,\\
\mathrm{d}e_1(t)=e_2(t)\mathrm{d}t,\\
\mathrm{d}e_2(t)=(k_0e_0+k_1e_1+k_2e_2)\mathrm{d}t+[g_1(e_1,e_2,t)-k_2\xi_2]\mathrm{d}t+g_2(e_1,e_2,t)\mathrm{d}B(t).
\end{cases}\tag{5}$$

In addition, we define $\xi\triangleq(\xi_1^{\mathrm{T}},\xi_2^{\mathrm{T}})^{\mathrm{T}}$, $E\triangleq(e_0^{\mathrm{T}},e_1^{\mathrm{T}},e_2^{\mathrm{T}})^{\mathrm{T}}$, $E'\triangleq(e_1^{\mathrm{T}},e_2^{\mathrm{T}})^{\mathrm{T}}$,

$$A_0=\begin{bmatrix}-\beta_1 I & I\\ -\beta_2 I & 0\end{bmatrix},\quad A_1=\begin{bmatrix}0 & I & 0\\ 0 & 0 & I\\ k_0I & k_1I & k_2I\end{bmatrix},$$

where I is a $d\times d$ unit matrix. Then, Eq. (5) can be rewritten in the following matrix equation form:

$$\begin{cases}
\mathrm{d}\xi(t)=A_0\xi(t)\mathrm{d}t+\begin{bmatrix}0\\ g_1(e_1,e_2,t)+k_0e_0\end{bmatrix}\mathrm{d}t+\begin{bmatrix}0\\ g_2(e_1,e_2,t)\end{bmatrix}\mathrm{d}B(t),\\
\mathrm{d}E(t)=A_1E(t)\mathrm{d}t+\begin{bmatrix}0\\ 0\\ g_1(e_1,e_2,t)-k_2\xi_2\end{bmatrix}\mathrm{d}t+\begin{bmatrix}0\\ 0\\ g_2(e_1,e_2,t)\end{bmatrix}\mathrm{d}B(t).
\end{cases}\tag{6}$$

By the definitions of $g_i(e_1, e_2, t)$, $i = 1, 2$, it is easy to obtain $g_i(0, 0, t) = 0$, which implies that $0 \in \mathbb{R}^5$ is an equilibrium of (6). Notice that $f(x_1, x_2, t) \in \mathcal{F}_M$ and $\sigma(x_1, x_2, t) \in \mathcal{D}_N$, hence the two uncertain functions $g_i(e_1, e_2, t)$ are both of linear growth:

$$\|g_1(e_1, e_2, t)\| \leqslant M\|(e_1^{\mathrm{T}}, e_2^{\mathrm{T}})\|, \quad \|g_2(e_1, e_2, t)\| \leqslant N\|(e_1^{\mathrm{T}}, e_2^{\mathrm{T}})\|. \tag{7}$$

Step 2. We deduce some useful properties of matrices A_0 and A_1.

First, we calculate the characteristic polynomial of A_0 and A_1. Note that

$$A_0 = \begin{bmatrix} -\beta_1 & 1 \\ -\beta_2 & 0 \end{bmatrix} \otimes I \triangleq A_{01} \otimes I, \quad A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k_0 & k_1 & k_2 \end{bmatrix} \otimes I \triangleq A_{11} \otimes I,$$

where " $\otimes$ " denotes the Kronecker product. It is not difficult to obtain

$$\det(\mu I - A_0) = \det(\mu I - A_{01})^d = (\mu^2 + \mu\beta_1 + \beta_2)^d,$$
$$\det(\lambda I - A_1) = \det(\lambda I - A_{11})^d = (\lambda^3 - k_2\lambda^2 - k_1\lambda - k_0)^d.$$

Assume we take β_1, β_2, k_0, k_1, k_2 such that the quadratic equation $\mu^2 + \mu\beta_1 + \beta_2$ has two distinct negative roots μ_1, μ_2 and the cubic equation $\lambda^3 - k_2\lambda^2 - k_1\lambda - k_0$ has three distinct negative roots λ_0, λ_1, λ_2.

Next, we define the following matrices:

$$Q = \begin{bmatrix} \frac{1}{\mu_2} & \frac{1}{\mu_1} \\ 1 & 1 \end{bmatrix} \otimes I = \begin{bmatrix} \frac{I}{\mu_2} & \frac{I}{\mu_1} \\ I & I \end{bmatrix}, \quad P = \begin{bmatrix} \lambda_0^{-1} & \lambda_1^{-1} & \lambda_2^{-2} \\ 1 & 1 & \lambda_2^{-1} \\ \lambda_0 & \lambda_1 & 1 \end{bmatrix} \otimes I = \begin{bmatrix} \lambda_0^{-1} I & \lambda_1^{-1} I & \lambda_2^{-2} I \\ I & I & \lambda_2^{-1} I \\ \lambda_0 I & \lambda_1 I & I \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & \lambda_2^{-1} \\ \lambda_0 & \lambda_1 & 1 \end{bmatrix} \otimes I = \begin{bmatrix} I & I & \lambda_2^{-1} I \\ \lambda_0 I & \lambda_1 I & I \end{bmatrix}, \quad C' = \begin{bmatrix} \lambda_0^{-1} & \lambda_1^{-1} & \lambda_2^{-2} \end{bmatrix} \otimes I = \begin{bmatrix} \lambda_0^{-1} I & \lambda_1^{-1} I & \lambda_2^{-2} I \end{bmatrix}.$$

Then, it is easy to obtain

$$Q^{-1} = \frac{\mu_1\mu_2}{\mu_1 - \mu_2} \begin{bmatrix} I & -\frac{I}{\mu_1} \\ -I & \frac{I}{\mu_2} \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} * & * & a_0 I \\ * & * & a_1 I \\ * & * & a_2 I \end{bmatrix},$$

where $a_i = \frac{\lambda_i}{b_i}$, $i = 0, 1$, $a_2 = \frac{\lambda_2^2}{b_2}$, $b_j = \prod_{i \in \{0,1,2\} \setminus \{j\}} (\lambda_j - \lambda_i)$, $j = 0, 1, 2$, and the "$*$" in the elements of P^{-1} means that we do not care what it is in our proof of Theorem 1.

It is also easy to verify that $A_0 = QJ_0Q^{-1}$ and $A_1 = PJ_1P^{-1}$, where

$$J_0 = \begin{bmatrix} \mu_1 I & 0 \\ 0 & \mu_2 I \end{bmatrix}, \quad J_1 = \begin{bmatrix} \lambda_0 I & 0 & 0 \\ 0 & \lambda_1 I & 0 \\ 0 & 0 & \lambda_2 I \end{bmatrix}.$$

Step 3. In this step, we introduce invertible linear transformations $\xi = Q\eta$, $E = P\omega$ and derive the closed-loop system equation under the new coordinate (η, ω).

Here, we define $\eta = (\eta_1^{\mathrm{T}}, \eta_2^{\mathrm{T}})^{\mathrm{T}}$, $\omega = (\omega_0^{\mathrm{T}}, \omega_1^{\mathrm{T}}, \omega_2^{\mathrm{T}})^{\mathrm{T}}$. Using the relationship $A_0 = QJ_0Q^{-1}$ and $A_1 = PJ_1P^{-1}$, Eq. (6) can be rewritten as follows:

$$\begin{cases} \mathrm{d}\eta_1 = \left[\mu_1\eta_1 + \dfrac{\mu_2}{\mu_2 - \mu_1}(g_1(e_1, e_2, t) + k_0 e_0)\right] \mathrm{d}t + \dfrac{\mu_2}{\mu_2 - \mu_1} g_2(e_1, e_2, t) \mathrm{d}B(t), \\ \mathrm{d}\eta_2 = \left[\mu_2\eta_2 - \dfrac{\mu_1}{\mu_2 - \mu_1}(g_1(e_1, e_2, t) + k_0 e_0)\right] \mathrm{d}t - \dfrac{\mu_1}{\mu_2 - \mu_1} g_2(e_1, e_2, t) \mathrm{d}B(t), \\ \mathrm{d}\omega_0 = [\lambda_0\omega_0 + a_0(g_1(e_1, e_2, t) - k_2\xi_2)]\mathrm{d}t + a_0 g_2(e_1, e_2, t)\mathrm{d}B(t), \\ \mathrm{d}\omega_1 = [\lambda_1\omega_1 + a_1(g_1(e_1, e_2, t) - k_2\xi_2)]\mathrm{d}t + a_1 g_2(e_1, e_2, t)\mathrm{d}B(t), \\ \mathrm{d}\omega_2 = [\lambda_2\omega_2 + a_2(g_1(e_1, e_2, t) - k_2\xi_2)]\mathrm{d}t + a_2 g_2(e_1, e_2, t)\mathrm{d}B(t). \end{cases} \tag{8}$$

Step 4. Now, we use the following Lyapunov function:

$$V(\eta,\omega)=\frac{1}{2}(\|\eta_1\|^2+\|\eta_2\|^2+\lambda_1\lambda_2\|\omega_0\|^2+\lambda_0\lambda_2\|\omega_1\|^2+\lambda_0\lambda_1\|\omega_2\|^2), \tag{9}$$

and calculate the differential operator L (Appendix A) associated with (8).

Let

$$F(\eta,\omega)\triangleq\begin{bmatrix}\mu_1\eta_1+\dfrac{\mu_2}{\mu_2-\mu_1}(g_1(e_1,e_2,t)+k_0e_0)\\ \mu_2\eta_2-\dfrac{\mu_1}{\mu_2-\mu_1}(g_1(e_1,e_2,t)+k_0e_0)\\ \lambda_0\omega_0+a_0(g_1(e_1,e_2,t)-k_2\xi_2)\\ \lambda_1\omega_1+a_1(g_1(e_1,e_2,t)-k_2\xi_2)\\ \lambda_2\omega_2+a_2(g_1(e_1,e_2,t)-k_2\xi_2)\end{bmatrix},\quad G(\eta,\omega)\triangleq\begin{bmatrix}\dfrac{\mu_2}{\mu_2-\mu_1}g_2(e_1,e_2,t)\\ -\dfrac{\mu_1}{\mu_2-\mu_1}g_2(e_1,e_2,t)\\ a_0g_2(e_1,e_2,t)\\ a_1g_2(e_1,e_2,t)\\ a_2g_2(e_1,e_2,t)\end{bmatrix}.$$

Now, we can calculate the differential operator L (Appendix A) associated with (8):

$$LV(\eta,\omega)=\frac{\partial V}{\partial t}+F^{\mathrm{T}}(\eta,\omega)\nabla V+\frac{1}{2}\mathrm{Tr}[G(\eta,\omega)G^{\mathrm{T}}(\eta,\omega)H(V)]. \tag{10}$$

Obviously, the first term on the right-hand side (RHS) of (10) is zero, and the second term of (10) is

$$\begin{aligned}&\mu_1\eta_1^{\mathrm{T}}\eta_1+\frac{\mu_2}{\mu_2-\mu_1}(g_1(e_1,e_2,t)+k_0e_0)\eta_1+\mu_2\eta_2^{\mathrm{T}}\eta_2-\frac{\mu_1}{\mu_2-\mu_1}(g_1(e_1,e_2,t)+k_0e_0)\eta_2\\&\quad+\lambda_0\lambda_1\lambda_2\omega_0^{\mathrm{T}}\omega_0+a_0(g_1(e_1,e_2,t)-k_2\xi_2)\omega_0\lambda_1\lambda_2+\lambda_0\lambda_1\lambda_2\omega_1^{\mathrm{T}}\omega_1+a_1(g_1(e_1,e_2,t)-k_2\xi_2)\omega_1\lambda_0\lambda_2\\&\quad+\lambda_0\lambda_1\lambda_2\omega_2^{\mathrm{T}}\omega_2+a_2(g_1(e_1,e_2,t)-k_2\xi_2)\omega_2\lambda_0\lambda_1\\&=\mu_1\|\eta_1\|^2+\mu_2\|\eta_2\|^2+\frac{\mu_2\eta_1-\mu_1\eta_2}{\mu_2-\mu_1}(g_1(C\omega,t)+k_0C'\omega)\\&\quad+\lambda_0\lambda_1\lambda_2\left(\|\omega\|^2+\left(\frac{1}{b_0}\omega_0+\frac{1}{b_1}\omega_1+\frac{\lambda_2}{b_2}\omega_2\right)(g_1(C\omega,t)-k_2(\eta_1+\eta_2))\right).\end{aligned} \tag{11}$$

The third term of (10) is expressed as

$$\begin{aligned}&\frac{1}{2}\left[\left(\frac{\mu_2}{\mu_2-\mu_1}\right)^2+\left(\frac{\mu_1}{\mu_2-\mu_1}\right)^2+\lambda_0\lambda_1\lambda_2\left(\frac{\lambda_0}{b_0^2}+\frac{\lambda_1}{b_1^2}+\frac{\lambda_2^3}{b_2^2}\right)\right]g_2^2(C\omega,t)\\&\quad=\frac{1}{2}\left[\left(\frac{\mu_2}{\mu_2-\mu_1}\right)^2+\left(\frac{\mu_1}{\mu_2-\mu_1}\right)^2+a_0^2\lambda_1\lambda_2+a_1^2\lambda_0\lambda_2++a_2^2\lambda_0\lambda_1\right]g_2^2(C\omega,t).\end{aligned} \tag{12}$$

Then, according to (11) and (12), we obtain

$$\begin{aligned}LV(\eta,\omega)=&\,\mu_1\|\eta_1\|^2+\mu_2\|\eta_2\|^2+\frac{\mu_2\eta_1-\mu_1\eta_2}{\mu_2-\mu_1}(g_1(C\omega,t)+k_0C'\omega)\\&+\lambda_0\lambda_1\lambda_2\left(\|\omega\|^2+\left(\frac{1}{b_0}\omega_0+\frac{1}{b_1}\omega_1+\frac{\lambda_2}{b_2}\omega_2\right)(g_1(C\omega,t)-k_2(\eta_1+\eta_2))\right)\\&+\frac{1}{2}\left[\left(\frac{\mu_2}{\mu_2-\mu_1}\right)^2+\left(\frac{\mu_1}{\mu_2-\mu_1}\right)^2+\lambda_0\lambda_1\lambda_2\left(\frac{\lambda_0}{b_0^2}+\frac{\lambda_1}{b_1^2}+\frac{\lambda_2^3}{b_2^2}\right)\right]g_2^2(C\omega,t).\end{aligned} \tag{13}$$

Step 5. We proceed to estimate the upper bound of (13) and use the Lyapunov stability theory to analyze the behavior of stochastic differential equation (8).

Let $\mu_0=\max\{\mu_1,\mu_2\}$, and then it is easy to get

$$\mu_1\|\eta_1\|^2+\mu_2\|\eta_2\|^2\leqslant\mu_0\|\eta\|^2. \tag{14}$$

Using (7), we can obtain

$$\|g_1(C\omega,t)\|\leqslant M\|C\|\|\omega\|. \tag{15}$$

In addition, we have

$$\|k_0 C'\omega\| \leqslant |k_0|\|C'\|\|\omega\|. \tag{16}$$

Next, we estimate the upper bounds of $\|C\|$ and $\|C'\|$, where the matrix norm $\|\cdot\|$ is the operator norm induced by the Euclidean norm:

$$\|C\| = \sup_{\|\nu\|=1} \|C\nu\|.$$

For any $\nu = (\nu_1^{\mathrm{T}}, \nu_2^{\mathrm{T}}, \nu_3^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{3d}$ with $\|\nu\| = 1$, where $\nu_i \in \mathbb{R}^d,\ i = 1, 2, 3$. Then, by the definition of C, it is easy to obtain

$$C\nu = \begin{bmatrix} I & I & \lambda_2^{-1} I \\ \lambda_0 I & \lambda_1 I & I \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} = \begin{bmatrix} \nu_1 + \nu_2 + \dfrac{1}{\lambda_2}\nu_3 \\ \lambda_0\nu_1 + \lambda_1\nu_2 + \nu_3 \end{bmatrix}.$$

Using the Cauchy inequality, we have

$$\begin{aligned} \|C\nu\|^2 &= \left\|\nu_1 + \nu_2 + \frac{1}{\lambda_2}\nu_3\right\|^2 + \|\lambda_0\nu_1 + \lambda_1\nu_2 + \nu_3\|^2 \\ &\leqslant \left(1 + 1 + \frac{1}{\lambda_2^2}\right)(\|\nu_1\|^2 + \|\nu_2\|^2 + \|\nu_3\|^2) + (\lambda_0^2 + \lambda_1^2 + 1)(\|\nu_1\|^2 + \|\nu_2\|^2 + \|\nu_3\|^2) \\ &= 3 + \lambda_0^2 + \lambda_1^2 + \frac{1}{\lambda_2^2}. \end{aligned}$$

Thus, we obtain

$$\|C\| \leqslant \sqrt{3 + \lambda_0^2 + \lambda_1^2 + \frac{1}{\lambda_2^2}} \triangleq \varphi(\lambda). \tag{17}$$

Similarly, it can be seen that

$$\|C'\| \leqslant \sqrt{\frac{1}{\lambda_0^2} + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^4}}. \tag{18}$$

Using the Cauchy inequality, we get

$$\frac{\mu_2\eta_1 - \mu_1\eta_2}{\mu_2 - \mu_1} \leqslant \sqrt{\left[\left(\frac{\mu_2}{\mu_2 - \mu_1}\right)^2 + \left(\frac{\mu_1}{\mu_2 - \mu_1}\right)^2\right](\|\eta_1\|^2 + \|\eta_2\|^2)} = \sqrt{\frac{\mu_2^2 + \mu_1^2}{(\mu_2 - \mu_1)^2}}\|\eta\|. \tag{19}$$

According to (15)–(19), we can get

$$\begin{aligned} &\frac{\mu_2\eta_1 - \mu_1\eta_2}{\mu_2 - \mu_1}(g_1(C\omega, t) + k_0 C'\omega) \\ &\leqslant \sqrt{\frac{\mu_2^2 + \mu_1^2}{(\mu_2 - \mu_1)^2}\|\eta\|^2}\,(M\|C\|\|\omega\| + |k_0|\|C'\|\|\omega\|) \\ &\leqslant \sqrt{\frac{\mu_2^2 + \mu_1^2}{(\mu_2 - \mu_1)^2}}\,(M\|C\| + |k_0|\|C'\|)\|\eta\|\|\omega\| \\ &= \sqrt{\frac{\mu_2^2 + \mu_1^2}{(\mu_2 - \mu_1)^2}}\left(M\varphi(\lambda) + \sqrt{\lambda_1^2\lambda_2^2 + \lambda_0^2\lambda_2^2 + \frac{\lambda_0^2\lambda_1^2}{\lambda_2^2}}\right)\|\eta\|\|\omega\| \\ &\triangleq \Psi(\lambda, \mu)\|\eta\|\|\omega\|. \end{aligned} \tag{20}$$

On the other hand, it is not difficult to obtain

$$\begin{aligned} &\left(\frac{1}{b_0}\omega_0 + \frac{1}{b_1}\omega_1 + \frac{\lambda_2}{b_2}\omega_2\right) g_1(C\omega, t) \\ &\leqslant \sqrt{\left(\frac{1}{b_0^2} + \frac{1}{b_1^2} + \frac{\lambda_2^2}{b_2^2}\right)(\|\omega_0\|^2 + \|\omega_1\|^2 + \|\omega_2\|^2)}M\|C\|\|\omega\| \end{aligned}$$

$$\triangleq M\phi(\lambda)\varphi(\lambda)\|\omega\|^2. \tag{21}$$

According to (21), we have

$$\lambda_0\lambda_1\lambda_2\|\omega\|^2+\lambda_0\lambda_1\lambda_2\left(\frac{1}{b_0}\omega_0+\frac{1}{b_1}\omega_1+\frac{\lambda_2}{b_2}\omega_2\right)g_1(C\omega,t)\leqslant(1-M\phi(\lambda)\varphi(\lambda))\|\omega\|^2\lambda_0\lambda_1\lambda_2. \tag{22}$$

By Cauchy inequality, we have

$$\begin{aligned}-\lambda_0\lambda_1\lambda_2\left(\frac{1}{b_0}\omega_0+\frac{1}{b_1}\omega_1+\frac{\lambda_2}{b_2}\omega_2\right)k_2(\eta_1+\eta_2) &\leqslant -\lambda_0\lambda_1\lambda_2\phi(\lambda)\|\omega\||k_2|\sqrt{2(\|\eta_1\|^2+\|\eta_2\|^2)}\\ &\triangleq \sqrt{2}(-\lambda_0\lambda_1\lambda_2)\phi(\lambda)|k_2|\|\eta\|\|\omega\|.\end{aligned} \tag{23}$$

From (7), we obtain

$$\|g_2(C\omega,t)\|=\|g_2(C\omega,t)\|\leqslant N\|C\|\|\omega\|,$$

which in turn gives

$$\|g_2(C\omega,t)\|^2\leqslant N^2\|C\|^2\|\omega\|^2\triangleq N^2\varphi^2(\lambda)\|\omega\|^2. \tag{24}$$

Therefore, we have

$$\begin{aligned}&\frac{1}{2}\left[\frac{\mu_2^2+\mu_1^2}{(\mu_2-\mu_1)^2}+\lambda_0\lambda_1\lambda_2\left(\frac{\lambda_0}{b_0^2}+\frac{\lambda_1}{b_1^2}+\frac{\lambda_2^3}{b_2^2}\right)\right]g_2^2(C\omega,t)\\ &\leqslant\frac{1}{2}\left(\frac{\mu_2^2+\mu_1^2}{(\mu_2-\mu_1)^2}+a_0^2\lambda_1\lambda_2+a_1^2\lambda_0\lambda_2+a_2^2\lambda_0\lambda_1\right)N^2\varphi^2(\lambda)\|\omega\|^2\\ &\triangleq\Phi(\mu,\lambda)N^2\varphi^2(\lambda)\|\omega\|^2.\end{aligned} \tag{25}$$

According to Eq. (13) and by using inequalities (14), (20), (22), (23), and (25), we can estimate $LV(\eta,\omega)$ as follows:

$$\begin{aligned}LV(\eta,\omega)&=\mu_1\|\eta_1\|^2+\mu_2\|\eta_2\|^2+\frac{\mu_2\eta_1-\mu_1\eta_2}{\mu_2-\mu_1}(g_1(C\omega,t)+k_0C'\omega)\\ &\quad+\lambda_0\lambda_1\lambda_2\left(\|\omega_1\|^2+\left(\frac{1}{b_0}\omega_0+\frac{1}{b_1}\omega_1+\frac{\lambda_2}{b_2}\omega_2\right)(g_1(C\omega,t)-k_2(\eta_1+\eta_2))\right)\\ &\quad+\frac{1}{2}\left[\left(\frac{\mu_2}{\mu_2-\mu_1}\right)^2+\left(\frac{\mu_1}{\mu_2-\mu_1}\right)^2+\lambda_0\lambda_1\lambda_2\left(\frac{\lambda_0}{b_0^2}+\frac{\lambda_1}{b_1^2}+\frac{\lambda_2^3}{b_2^2}\right)\right]g_2^2(C\omega,t)\\ &\leqslant\mu_0\|\eta\|^2+\Psi(\mu,\lambda)\|\eta\|\|\omega\|+(1-M\phi(\lambda)\varphi(\lambda))\|\omega\|^2\lambda_0\lambda_1\lambda_2\\ &\quad+\sqrt{2}(-\lambda_0\lambda_1\lambda_2)\phi(\lambda)|k_2|\|\eta\|\|\omega\|+\Phi(\mu,\lambda)N^2\varphi^2(\lambda)\|\omega\|^2\\ &=\mu_0\|\eta\|^2+(\Psi(\mu,\lambda)+\sqrt{2}(-\lambda_0\lambda_1\lambda_2)\phi(\lambda)|k_2|)\|\eta\|\|\omega\|\\ &\quad+((1-M\phi(\lambda)\varphi(\lambda))\lambda_0\lambda_1\lambda_2+\Phi(\mu,\lambda)N^2\varphi^2(\lambda))\|\omega\|^2.\end{aligned} \tag{26}$$

Now, suppose that the parameters k_0, k_1, k_2, β_1, β_2 are selected from $\Omega_{k,\beta}$, and then the corresponding parameters λ, μ should belong to $\Delta_{\lambda,\mu}$. Thus, we have

$$4\mu_0[\lambda_0\lambda_1\lambda_2(1-M\phi(\lambda)\varphi(\lambda))+\Phi(\mu,\lambda)N^2\varphi^2(\lambda)]>\Theta(\lambda,\mu),$$

where $\Theta(\lambda,\mu)=(\Psi(\mu,\lambda)-\sqrt{2}\lambda_0\lambda_1\lambda_2\phi(\lambda)|k_2|)^2$. This means that LV is a negative definite function of (η,ω):

$$LV(\eta,\omega)\leqslant-\delta(\|\eta\|^2+\|\omega\|^2) \tag{27}$$

for some positive δ.

There, by the Itô's formula (Appendix A), we have

$$\mathrm{d}V(\eta,\omega)=LV(\eta,\omega)\mathrm{d}t+G_1(\eta,\omega)\mathrm{d}B(t), \tag{28}$$

where

$$G_1(\eta,\omega)=\left[\frac{\mu_2\eta_1-\mu_1\eta_2}{\mu_2-\mu_1}+\lambda_0\lambda_1\lambda_2\left(\frac{1}{b_0}\omega_0+\frac{1}{b_1}\omega_1+\frac{\lambda_2}{b_2}\omega_2\right)\right]g_2(C\omega,t).$$

From (28), we have the following equality for any $T > 0$,

$$V(\eta(T), \omega(T)) = V(\eta_0, \omega_0) + \int_0^T LV(\eta, \omega)\mathrm{d}t + \int_0^T G_1(\eta, \omega)\mathrm{d}B(t). \tag{29}$$

We now prove that

$$\mathbb{E}\int_0^T G_1(\eta(t), \omega(t))\mathrm{d}B(t) = 0. \tag{30}$$

To obtain (30), we need to prove that

$$\mathbb{E}\int_0^T \|G_1(\eta(t), \omega(t))\|^2\mathrm{d}t < \infty. \tag{31}$$

Using the Cauchy inequality, we obtain the following:

$$\begin{aligned}\left(\frac{1}{b_0}\omega_0 + \frac{1}{b_1}\omega_1 + \frac{\lambda_2}{b_2}\omega_2\right) g_2(C\omega, t) &\leqslant \sqrt{\left(\frac{1}{b_0^2} + \frac{1}{b_1^2} + \frac{\lambda_2^2}{b_2^2}\right)(\|\omega_0\|^2 + \|\omega_1\|^2 + \|\omega_2\|^2)} N\|C\|\|\omega\| \\ &\triangleq N\phi(\lambda)\varphi(\lambda)\|\omega\|^2. \end{aligned} \tag{32}$$

With the help of (19) and (32), and by the expression of $G_1(\eta, \omega)$, we know that

$$\|G_1(\eta, \omega)\|^2 = O(\|\eta\|^4 + \|\omega\|^4).$$

Thus, we obtain (31) by taking $p = 4$ in Theorem A1 (Appendix A).

Consequently, by taking the expectation on both sides of (29) and using (27), we obtain

$$\mathbb{E}(V(\eta(T), \omega(T))) \leqslant V(\eta_0, \omega_0).$$

By the positive property of $V(\eta, \omega)$, there exists some $\alpha > 0$ such that for all $T > 0$,

$$\mathbb{E}(\|\eta(T)\|^2 + \|\omega(T)\|^2) \leqslant \alpha V(\eta_0, \omega_0),$$

and therefore

$$\sup_{t\geqslant 0} \mathbb{E}[\|\eta(t)\|^2 + \|\omega(t)\|^2] < \infty.$$

In addition, by invertible linear transformations $\xi = Q\eta$ and $E = P\omega$, we obtain

$$\sup_{t\geqslant 0} \mathbb{E}[\|\xi(t)\|^2 + \|E(t)\|^2] < \infty,$$

which implies global stability:

$$\sup_{t\geqslant 0} \mathbb{E}[\|x_1(t)\|^2 + \|x_2(t)\|^2 + \|u(t)\|^2] < \infty.$$

To prove the optimality of tracking, using (30) and (27), and by taking the expectation of (29), we can see that

$$\int_0^T \mathbb{E}(\|\eta(t)\|^2 + \|\omega(t)\|^2)\mathrm{d}t \leqslant \alpha V(\eta_0, \omega_0). \tag{33}$$

Notice that T is arbitrary in (33). Hence, we have

$$\int_0^\infty \mathbb{E}(\|\eta(t)\|^2 + \|\omega(t)\|^2)\mathrm{d}t \leqslant \alpha V(\eta_0, \omega_0) < \infty.$$

Then, by Lemma B1 (Appendix B), we conclude that $\mathbb{E}(\|\eta(t)\|^2 + \|\omega(t)\|^2)$ is uniformly continuous of t. From the Barbalat Lemma [20], we obtain the following:

$$\lim_{t\to\infty} \mathbb{E}(\|\eta(t)\|^2) = 0, \quad \lim_{t\to\infty} \mathbb{E}(\|\omega(t)\|^2) = 0.$$

Recall that $\xi = Q\eta$, $E = P\omega$. Thus, we can obtain

$$\lim_{t\to\infty} \mathbb{E}(\|\xi(t)\|^2) = 0, \quad \lim_{t\to\infty} \mathbb{E}(\|E(t)\|^2) = 0.$$

Consequently, we have

$$\lim_{t\to\infty} \mathbb{E}\|x_i(t) - \hat{x}_i(t)\|^2 = 0, \;\; i = 1, 2,$$
$$\lim_{t\to\infty} \mathbb{E}\|y^* - x_1(t)\|^2 = 0.$$

Step 6. Finally, we demonstrate that $\Delta_{\lambda,\mu}$ is open and unbounded.

Let us select two distinct negative numbers λ_0, λ_1 arbitrarily. By the definitions of φ and ϕ, we easily obtain that $\varphi(\lambda) = O(1)$ and $\phi(\lambda) = O(\frac{1}{|\lambda_2|})$ as λ_2 tends to $-\infty$.

Take $\mu_2 = 2\mu_1 = -2|\lambda_2|^{1+\varepsilon}$, where ε is any given positive number. Using the fact $k_2 = \lambda_0 + \lambda_1 + \lambda_2$, we obtain $\Psi(\lambda, \mu) = O(|\lambda_2|)$, $\Phi(\lambda, \mu) = O(|1|)$, and $\Theta(\lambda, \mu) = (\Psi(\mu, \lambda) + \sqrt{2}\lambda_0\lambda_1\lambda_2\phi(\lambda)(\lambda_0 + \lambda_1 + \lambda_2))^2 = O(\lambda_2^2)$ as λ_2 tends to $-\infty$. Therefore, we obtain the following:

$$\liminf_{\lambda_2\to-\infty} \frac{4\mu_0[\lambda_0\lambda_1\lambda_2(1 - M\phi(\lambda)\varphi(\lambda)) + \Phi(\lambda, \mu)N^2\varphi^2(\lambda)]}{|\lambda_2|^{2+\varepsilon}} \geqslant m > 0.$$

This implies that the following inequality always holds when $\lambda_2 \to -\infty$:

$$4\mu_0[\lambda_0\lambda_1\lambda_2(1 - M\phi(\lambda)\varphi(\lambda)) + \Phi(\mu, \lambda)N^2\varphi^2(\lambda)] > \Theta(\lambda, \mu).$$

Thus, $(\lambda, \mu) \in \Delta_{\lambda,\mu}$ when $\lambda_2 \to -\infty$, which means that $\Delta_{\lambda,\mu}$ is nonempty and unbounded. The proof of the openness of $\Omega_{k,\beta}$ is similar to [11]; thus, we omit it. This completes the proof of Theorem 1.

4 Conclusion

In this paper, we have considered a basic class of nonlinear uncertain stochastic systems and proposed a state observer-based PID controller. A five-dimensional parameter space was constructed, within which the three PID and two observer parameters can be selected arbitrarily to guarantee the global stability of closed-loop stochastic control systems. Our theory and design methods demonstrate that the PID controller is quite robust to the design parameters, nonlinear uncertainties, and system randomness. Of course, many interesting problems still remain open. For example, it may be interesting to provide a new parameter formula for the design of a PID controller, which is derived from the inherent (but rarely noticed) relationship between PID and active disturbance rejection control [21]. It would also be interesting to consider more complicated situations such as saturation, dead zone, time-delayed inputs, sampled-data PID controllers under a prescribed sampling rate.

Acknowledgements This work was supported by the Youth Scholars Fund of Beijing Technology and Business University (Grant No. PXM2018_014213_000033) and National Natural Science Foundation of China (Grant No. 61973329). The authors would like to thank Professor Lei GUO from Academy of Mathematics and Systems Science, Chinese Academy of Sciences, for valuable discussion on PID control of nonlinear stochastic systems.

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Appendix A

Definition A1 ([17])**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space. A filtration is a family of σ-algebra $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying $\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$ for all $0 \leqslant t < s < \infty$. The filtration is considered to be right continuous if $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for all $t \geqslant 0$. When the probability space is complete, the filtration is considered to satisfy the usual conditions if it is right continuous and $\mathcal{F}_0$ contains all P-null sets.

We consider the following stochastic system:

$$\mathrm{d}x(t) = f(x(t), t)\mathrm{d}t + \sigma(x(t), t)\mathrm{d}B(t), \tag{A1}$$

where $x \in \mathbb{R}^n$ is the state of the system, $f \in \mathbb{R}^n$, $\sigma \in \mathbb{R}^n$, and $B(t)$ is (standard one-dimensional) Brownian motion. The f term is referred to as a drift or vector field, and the noise term $\sigma \mathrm{d}B(t)$ is an uncertainty model. The uncertainty of this model could be caused by external random influences or by fluctuating coefficients and parameters in a mathematical model. The σ function is referred to as a diffusion coefficient.

Itô's formula [17]**.** We denote $C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^+)$ as the space of all nonnegative functions $V(x,t)$ defined on $\mathbb{R}^n \times \mathbb{R}^+$ that are continuously twice differentiable in x and once in t. We define the differential operator L associated with (A1) as follows:

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i(x,t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} [\sigma(x,t)\sigma^{\mathrm{T}}(x,t)]_{ij}\frac{\partial^2}{\partial x_i \partial x_j}.$$

If L acts on function $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^+)$, then

$$LV(x,t) = \frac{\partial V}{\partial t}(x,t) + f^{\mathrm{T}}\nabla V(x,t) + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^{\mathrm{T}} H(V)](x,t).$$

By Itô's formula, we can obtain

$$\mathrm{d}V(x(t),t) = LV(x,t)\mathrm{d}t + (\nabla V(x(t),t))^{\mathrm{T}}\sigma(x(t),t)\mathrm{d}B(t),$$

where ∇V is the gradient of V, $H(V) = V_{x_i x_j}$ is the $n \times n$ Hessian matrix of V, and $\mathrm{Tr}(A)$ denotes the trace of a matrix A.

Lemma A1 (Barbalat)**.** Assume that function $f : \mathbb{R}^+ \to \mathbb{R}$ is uniformly continuous and $\lim_{t\to\infty}\int_0^t f(\tau)\mathrm{d}\tau$ exists and is finite. Then

$$\lim_{t\to\infty} f(t) = 0.$$

See Lemma A.6 in [20] for a detailed discussion.

Theorem A1. Let $p \geqslant 2$ and $x_0 \in L^p(\Omega; \mathbb{R}^d)$. Assume there exists a constant $\alpha > 0$ such that, for all $(x,t) \in \mathbb{R}^d \times [t_0, T]$:

$$x^{\mathrm{T}} f(x,t) + \frac{p-1}{2}|\sigma(x,t)|^2 \leqslant \alpha(1 + |x|^2).$$

Then, for the solution of (A1) on $t \in [t_0, T]$, we obtain the following:

$$\mathbb{E}|x(t)|^p \leqslant 2^{\frac{p-2}{2}}(1 + \mathbb{E}|x_0|^p)\mathrm{e}^{p\alpha(t-t_0)}.$$

See Theorem 4.1 in [16] for a detailed discussion.

Appendix B

Lemma B1. Let the following linear growth condition hold for all $(x,t) \in \mathbb{R}^d \times [t_0, \infty)$:

$$\|f(x,t)\| \vee \|\sigma(x,t)\| \leqslant K\|x\|, \tag{B1}$$

and let $x(t)$ be a solution to SDE (A1) on $[t_0, \infty)$. Let $h(t) = E\|x(t)\|^2$, and assume that $\sup_{t\geqslant t_0} h(t) < \infty$. Then, $h(t)$ is a uniformly continuous function of t in $[t_0, \infty)$.

Proof. Let $C = \sup_{t\geqslant t_0} h(t)$, and assume that $t_0 \leqslant t_1 < t_2$. Then, we obtain the inequality $|h(t_2) - h(t_1)| \leqslant \mathbb{E}|\,\|x(t_2)\|^2 - \|x(t_1)\|^2|$.

According to the Schwarz inequality, we obtain

$$\mathbb{E}\big|\|x(t_2)\|^2 - \|x(t_1)\|^2\big| \leqslant \sqrt{\mathbb{E}(\|x(t_2)\| + \|x(t_1)\|)^2}\sqrt{\mathbb{E}(\|x(t_2)\| - \|x(t_1)\|)^2}.$$

Therefore, it is easy to see that

$$\mathbb{E}(\|x(t_2)\| + \|x(t_1)\|)^2 \leqslant 2(\mathbb{E}\|x(t_2)\|^2 + \mathbb{E}\|x(t_1)\|^2) \leqslant 4\sup_{t\geqslant t_0} h(t) = 4C, \tag{B2}$$

where the RHS of (B2) is a constant that is independent of t_1, t_2.

In addition, we can obtain the following:

$$\begin{aligned}\mathbb{E}(\|x(t_2)\| - \|x(t_1)\|)^2 \leqslant \mathbb{E}(\|x(t_2) - x(t_1)\|^2) &= \mathbb{E}\left\|\int_{t_1}^{t_2} f(x(t),t)\mathrm{d}t + \int_{t_1}^{t_2} \sigma(x(t),t)\mathrm{d}B(t)\right\|^2 \\ &\leqslant 2\left(\mathbb{E}\left\|\int_{t_1}^{t_2} f(x(t),t)\mathrm{d}t\right\|^2 + \mathbb{E}\left\|\int_{t_1}^{t_2} \sigma(x(t),t)\mathrm{d}B(t)\right\|^2\right).\end{aligned}$$

From (B1), it is easy to obtain the following:

$$\begin{aligned}\mathbb{E}\left\|\int_{t_1}^{t_2} f(x(t),t)\mathrm{d}t\right\|^2 &\leqslant \mathbb{E}\left[\int_{t_1}^{t_2} K\|x(t)\|\mathrm{d}t\right]^2 \leqslant K^2(t_2 - t_1)\mathbb{E}\left[\int_{t_1}^{t_2} \|x(t)\|^2\mathrm{d}t\right] \\ &= K^2(t_2 - t_1)\int_{t_1}^{t_2} h(t)\mathrm{d}t \leqslant K^2C(t_2 - t_1)^2.\end{aligned}$$

From Itô's isometry, we obtain

$$\mathbb{E}\left\|\int_{t_1}^{t_2} \sigma(x(t),t)\mathrm{d}B(t)\right\|^2 = \mathbb{E}\left[\int_{t_1}^{t_2} \|\sigma(x(t),t)\|^2\mathrm{d}t\right] \leqslant K^2\mathbb{E}\left[\int_{t_1}^{t_2} \|x(t)\|^2\mathrm{d}t\right] = K^2\int_{t_1}^{t_2} h(t)\mathrm{d}t \leqslant K^2C(t_2 - t_1).$$

Therefore, we conclude that

$$|h(t_2) - h(t_1)| \leqslant \sqrt{4C}\sqrt{2K^2C[(t_2 - t_1)^2 + (t_2 - t_1)]} < 4KC\sqrt{(t_2 - t_1)^2 + (t_2 - t_1)},$$

which implies $h(t)$ is uniformly continuous.