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On-line quantum state estimation using continuous weak measurement and compressed sensing

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Dear editor,

Quantum state estimation (QSE) is the most important work in quantum information processing and quantum feedback control, which is usually formulated by means of strong measurements of an informationally complete set of measurement operators and corresponding observables. However, strong measurements collapse the original quantum state, the ensemble must be reprepared, and the measurement apparatus has to be reconfigured at each step. Weak measurements (WM) [1] offer an alternative in acquiring quantum measurements and estimating quantum states. In the measuring process, by using continuous weak measurements (CWM) it is possible to gain the target state information without disturbing it substantially, and the value recovered in CWM can be obtained by computing the ensemble averaging. Compressed sensing (CS) [2] has been brought into the quantum domain in the context of reducing the number of measurements required for QSE [3,4]. However, whether a unified efficient scheme for on-line quantum state estimation using CWM and partial measurements is feasible, remains unknown.

In this study, we propose a new information acquisition and processing for on-line quantum state estimation on the basis of continuous weak-measurements with the help of compressive sensing and the optimization algorithm. Our key idea is to make a weak measurement on the complete measurement operators in an ensemble system by coupling the ensemble to some probe which can be measured. At each instant time, we obtain the records of the expectation values corresponding to some measurement operators using the indirect results of continuous weak measurements, and the estimated state is obtained by solving an on-line optimization problem with physical constraints. CS is used to reduce the number of the measurements needed and to improve the efficiency of the estimation. The existing techniques usually perform continuous weak measuring on-line yet estimate the state off-line, and the estimated state is a fixed state. While during the on-line state estimation in this study, the states estimated on-line are dynamic system states.

In a quantum weak measurement, a probe P is coupled with the estimated system S, and they become a joint coupled system $S\otimes P$. For one qubit density matrix ρ, suppose the initial state of the probe P is $|\phi\rangle$, and the initial state of the system S is $\rho_0 = |\psi\rangle\langle\psi|$. H_S and H_P are the Hamiltonians of systems S and P, respectively, and $H = H_P \otimes H_S$ is the Hamiltonian of the joint system. The initial state of the coupled system is $|\Psi\rangle$: $|\Psi\rangle = |\phi\rangle \otimes |\psi\rangle$. After the joint evolution of S and P for time Δt, the state $|\Psi\rangle$ becomes $|\Psi(\Delta t)\rangle = U(\Delta t)|\Psi\rangle$, where $U(\Delta t)$ is the joint evolution operator $U(\Delta t) = \exp(-\mathrm{i}\xi\Delta t H/\hbar)$, and ξ represents the interaction strength between systems S and P. At time Δt, a projective measurement is performed on P with the measurement operator $X = \sum I \otimes |k\rangle\langle k|$, where $|k\rangle$ is the eigenstate of the system P: $|0\rangle$ or $|1\rangle$. The output is the eigenvalue corresponding to $|k\rangle$. The state of the joint system after the weak measurement becomes

$$|\psi_k(\Delta t)\rangle=\langle k|\otimes I\cdot U(\Delta t)|\phi\rangle\otimes|\psi\rangle/\Theta_k, \tag{1}$$

where $\Theta_k=\sqrt{\langle\Psi(\Delta t)|\Pi_k|\Psi(\Delta t)\rangle}$.

We define the weak measurement operator M_k as

$$M_k = \langle k|\otimes I\cdot U(\Delta t)\cdot|\phi\rangle\otimes I, \tag{2}$$

which is a Kraus operator and satisfies $\sum_k M_k^\dagger M_k = 1$. In this case, Θ_k becomes $\Theta_k=\sqrt{\langle\psi|M_k^\dagger M_k|\psi\rangle}$. Finally, we can get the relationship between the state of the system S before and after the whole measurement process as

$$|\psi_k(\Delta t)\rangle=\frac{M_k}{\sqrt{\langle\psi|M_k^\dagger M_k|\psi\rangle}}|\psi\rangle. \tag{3}$$

In such a way, we obtain the weak measurement operator M_k in (2) on the system S.

On-line state estimation makes the measurement operators be no longer a constant matrix group, and they become a set of time varying measurement operators $M_k(t)$. We need to deduce the time varying measurement operators used in the on-line state estimation.

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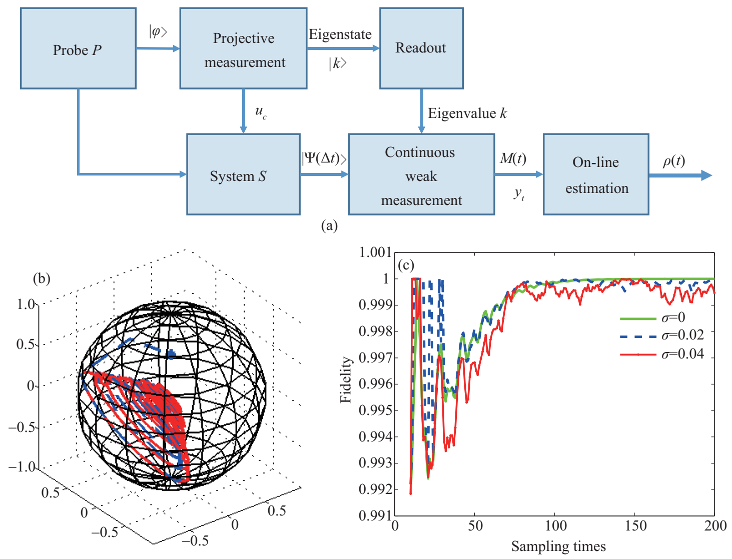

Figure 1 (Color online) (a) On-line estimation of the quantum state based on continuous weak measurements; (b) evolution trajectories of the actual state $\rho(t)$ (red line) and the on-line estimation state $\tilde{\rho}(t)$ (blue line) in the Bloch sphere; (c) fidelity of sampling times with three different noise amplitudes.

The process of on-line estimation of quantum states based on continuous weak measurements is shown in Figure 1(a). Assume $\hbar = 1$. We can get the Taylor expansion of U and neglect more than three orders of magnitude as $U(\Delta t) \approx I \otimes I - \mathrm{i}\xi\Delta t H - (\xi\Delta t)^2 H^2/2$, which is substituted into (2). We can obtain the expression of the weak measurement operator as $M_k(\Delta t) \approx I \langle k|\ \phi\rangle - \mathrm{i}\xi\Delta t H_S \langle k|\ H_P\,|\phi\rangle - (\xi\Delta t)^2 H_S^2 \langle k|\ H_P^2\,|\phi\rangle /2$. Let $r_k = (\xi\Delta t) H_S^2 \langle k|\ H_P^2\,|\phi\rangle /2, k = 1, 2, \ldots, d$, and the general form of the weak measure operator is $M_k(\Delta t) = I \langle k|\ \phi\rangle - [r_k\lambda/2 + \mathrm{i}\lambda H_S \langle k|\ H_P\,|\phi\rangle]$, where $\lambda = \xi\Delta t$ denotes the weak measurement strength and it tends to zero in the case of $\Delta t \to 0$. Supposing $\langle j\mid\phi\rangle = 1$ when $k = j$, we can obtain $M_i(t)$ as $M_j(\Delta t) = I - (\xi r_{k=j}/2 + \mathrm{i}\xi H_S)\Delta t$, and all the other measurement operators of $k \neq j$ can be combined as one operator as $M_{k\neq j}(\Delta t) = M_{j\perp}(\Delta t) = \sqrt{r_{k\neq j}\Delta t}$, where $M_{j\perp}$ and M_j are orthogonal and satisfy $(M_{j\perp})^2 + (M_j)^2 = I$. For the continuous weak measurements of a two-level quantum system, the measurement operator group only contains two operators: $M_0(\Delta t)$ and $M_1(\Delta t)$, which can be constructed, respectively, as

$$\begin{aligned} M_0(\Delta t) &= M_j - \mathrm{i}(1-\xi) H_S \Delta t \\ &= I - (\xi r_k/2 + \mathrm{i}H(t))\,\Delta t \\ &= I - \left(L^\dagger L\Big/2 + \mathrm{i}H(t)\right)\Delta t, \\ M_1(\Delta t) &= M_{k\neq j} = L\cdot\sqrt{\Delta t}, \end{aligned} \tag{4}$$

in which $L^\dagger L = \xi r_k$.

The stochastic master equation (SME) of the open quantum system can be written as

$$\begin{aligned} &\rho(t+\mathrm{d}t) - \rho(t) = -\frac{\mathrm{i}}{\hbar}[H(t), \rho(t)]\mathrm{d}t \\ &+ \sum\left[L\rho(t)L^\dagger - \left(\frac{1}{2}L^\dagger L\rho(t) + \frac{1}{2}\rho(t)L^\dagger L\right)\right]\mathrm{d}t \\ &+ \sqrt{\eta}\sum\left[L\rho(t) + \rho(t)L^\dagger\right]\mathrm{d}W, \\ &\rho_0 = \rho(0), \end{aligned} \tag{5}$$

where $\rho(t)$ is the density matrix. $H(t) = H_S + H_P + u(t)H_c$, where $H(t)$ is the whole Hamiltonian, H_S is the measured system Hamiltonian, H_P is the Hamiltonian of probe system, and H_c is the control Hamiltonian. $u(t)$ is the external regulate value. η is the measure efficiency and satisfies $0 < \eta \leqslant 1$. The discrete-time dynamic evolution equation of the stochastic open quantum system S can be written as

$$\rho(k+1) = A_0\rho(k)A_0^\dagger + A_1\rho(k)A_1^\dagger, \tag{6}$$

where $\mathrm{d}t = \Delta t$ represents the very short time interval required for the weak measurement, $A_0 = M_0(\mathrm{d}t) + \sqrt{\eta}L\cdot\mathrm{d}W$, $A_1 = M_1(\mathrm{d}t) + \sqrt{\eta}L\cdot\mathrm{d}W$, $L\cdot\mathrm{d}W$ denotes the noise caused by the continuous weak measurements, and $\mathrm{d}W$ denotes Gaussian white noise.

The evolution equation of the measurement operator $M_i(t)$ is $\dot{M}_k(t) = \frac{\mathrm{i}}{\hbar}[H(t), M_k(t)] + LM_k(t)L^\dagger - \frac{1}{2}(L^\dagger LM_k(t) + M_k(t)L^\dagger L)$. The corresponding discrete-time evolution equation of continuous weak measurement operators is

$$M_k(k+1) = M_0^\dagger M_k(k)M_0 + M_1^\dagger M_k(k)M_1. \tag{7}$$

According to the theory of CS, the density matrix of the quantum state can be reconstructed with only $O\,(rd\ln d)$ measurements' numbers of random measurement operators by solving an optimization problem, where r and d are the dimension and rank of the density matrix ρ, respectively, and $r \ll d$. One can estimate the quantum state on-line with a small amount of time-evolving measurement operators $\{M_{k_l}\}$, $l = 1, 2, \ldots, m$ and corresponding measure records $y(t_l)$ by solving the optimization problem:

$$\begin{aligned} &\arg\min \|\boldsymbol{A}\ \cdot \mathrm{vec}(\hat{\rho}) - y\|_2 \\ &\text{s.t.}\quad \hat{\rho} \geqslant 0,\ \mathrm{tr}(\hat{\rho}) = 1, \end{aligned} \tag{8}$$

where $\mathrm{vec}(\cdot)$ represents the transformation from a matrix to a vector by stacking the matrix's columns in order on the top of one another. The sampling matrix $\boldsymbol{A}$ is the matrix form of the all the sampled measurement operators $M_{k_l}(t_l)$; M_{k_l}, $l = 1, 2, \ldots, m$ is an arbitrary measurement operator in the l-th or the t_l-th measurement. For the sake of simplicity, we let $M_{k_l} = M_{k_l}(t_l)$.

The vector y and matrix $\boldsymbol{A}$ can be expressed according to the current measurement configurations as $y(t_l) = (\langle M_{k_1} \rangle, \langle M_{k_2} \rangle, \ldots, \langle M_{k_l} \rangle)^{\mathrm{T}}, l = 1, 2, \ldots, m$, and $\boldsymbol{A}(t_l) = (\,\mathrm{vec}(M_{k_1})^{\mathrm{T}}\ \mathrm{vec}(M_{k_2})^{\mathrm{T}}\ \cdots\ \mathrm{vec}(M_{k_l})^{\mathrm{T}}\,), l = 1, 2, \ldots, m$, where $\langle M_{k_l} \rangle$ is the corresponding measurement value in the l-th measurement. The sampling vector y is the vector form of the corresponding observation values $\langle M_{k_l} \rangle, l = 1, 2, \ldots, m$.

Experiments. Consider a 1/2 spin particle ensemble $\rho(t)$ as the system for on-line state estimation. The Hamiltonian of system is $H = H_0 + u_x H_x$, where $H_0 = -(\hbar/2)\,\omega_0 \sigma_z$ is the free Hamiltonian, and $u_x \in \mathbb{R}^+$ is the time-independent control amplitude. We use the least-square algorithm to solve the optimization problem (8), and the on-line estimated solution $\hat{\rho}(t)$ is the estimation of $\rho(t)$. In the experiments, the fidelity $f(t)$ is used to represent the performance of state estimation: $f(t) = \mathrm{Tr}\sqrt{\hat{\rho}(t)^{1/2}\rho(t)\hat{\rho}(t)^{1/2}}$. The initial state of the 1/2 spin system is $\rho(0) = [\,3/4\ -\sqrt{3}\big/4;\ -\sqrt{3}\big/4\ 1/4\,]$, and the Bloch sphere coordinate of $\rho(0)$ is $(\sqrt{3}\big/2,\ 0, 1/2)$. The interval time between two weak measurements is $\Delta t = 0.1$ atomic unit, the measure efficiency is set as $\eta = 0.5$, and $\mathrm{d}W = \sigma \cdot \mathrm{randn}(2, 2)$. Figure 1(b) shows the experimental results, where $u_x = 2$, $\xi_1 = 0.3$, $L = \xi_1 \sigma_z$, and the variance of noise $\sigma = 0.02$. Figure 1(c) shows the fidelity of sampling times with σ being 0, 0.02 and 0.04, respectively. From Figure 1(c) one can see that on-line estimations of the quantum state can achieve more than 95% accuracy of fidelity after two times measurements.

Conclusion. We proposed an on-line quantum state estimation method in this study and provided an implementable method for more complex application of high accurate closed-loop quantum feedback control [5]. This is particular interest for microscopic systems.

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