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An exact null controllability of stochastic singular systems

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Dear editor,

The stochastic singular system is also called the stochastic differential algebraic system, stochastic descriptor system, generalized stochastic system, and stochastic degenerate system (e.g., [1–4]). This type of system is found in numerous fields of application, which include fluid dynamics, the modeling of multi-body mechanisms, finance, input-output economics and the problem of protein folding. The theory of stochastic singular systems, in which a special class of linear stochastic singular systems is considered, has been developed recently [1]. The form of the initial function is given, so the corresponding initial value problem is uniquely solvable by applying the theory of generalized stochastic processes. Using white noise and fractional white noise, two illustrative applications are presented in a previously conducted study [2]. The basic question of solvability has been formulated and considered [5, 6]. Moreover, they propose a normalization procedure, and they completely solve the problem of exact controllability for a class of linear stochastic singular systems. The existence and uniqueness of the impulse solution for a type of stochastic singular systems were discussed by applying Laplace transform [7]. However, theorem 5.2.1 of [8] was incorrectly applied, leading to inappropriate conclusion regarding impulse solution. The solution and exact controllability of the degenerate Sobolev equation have been discussed [4]. The impulse terms may exist in the solution for a stochastic singular system. In addition, the exact null controllability gives some internal properties of the system. Consequently, this study considers the concepts of impulse solution and exact null controllability for a type of stochastic singular systems. In particular, it considers the following Ito singular system:

$$A\mathrm{d}x(t) = (Bx(t) + Cu(t))\mathrm{d}t + (Dx(t) + Gu(t))\mathrm{d}w(t), \quad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector, $u(t) \in \mathbb{R}^m$ denotes the control vector, $w(t)$ denotes a one-dimensional standard Wiener process, $x_0 \in \mathbb{R}^n$ denotes the initial condition, and $A, B, D \in \mathbb{R}^{n\times n}$, $C, G \in \mathbb{R}^{n\times m}$ denote the deterministic and constant matrices with $\mathrm{rank}A \leqslant n$. First, we give the conditions for the existence and uniqueness of the impulse solution to (1), and we explain the reason why Lemma 1 of [7] is incorrect. Second, we obtain the necessary and sufficient conditions for the exact null controllability of (1) using the matrix theory. Finally, an example is presented to illustrate the effectiveness of the obtained theoretical results.

Notations. $(\Omega, F, \{F_t\}, P)$ denotes a complete probability space with filtration $\{F_t\}$ satisfying the usual condition (i.e., the filtration contains all P-null sets and is right continuous); $w(t)$ is defined on $(\Omega, F, \{F_t\}, P)$; E denotes the mathematical expectation; $I_n \in \mathbb{R}^{n\times n}$ denotes the identical matrix; T denotes the transpose of a vector or a matrix; $\|\cdot\|$ denotes the Euclidean norm of a vector; $L^2(\Omega, F_t, P, \mathbb{R}^n)$ denotes the set of all random variables $\eta \in \mathbb{R}^n$ such that η is F_t-measurable and the mean square norm $\|\eta\|_2 = (E(\|\eta\|^2))^{1/2} < +\infty$; $L^2(J, \Omega, \mathbb{R}^n)$ denotes the set of all processes $x(t) \in \mathbb{R}^n$ such that $\|x(t)\|_2 < +\infty, \forall t \in J$, where $J = [0, T]$ or $[0, +\infty)$; all processes are F_t-adapted; $L^2([0, T], F, \mathbb{R}^n)$ denotes the set of all processes $x(t) \in \mathbb{R}^n$ such that $E(\int_0^T \|x(t)\|^2 \,\mathrm{d}t) < +\infty$; $C^k(J, \Omega, \mathbb{R}^n)$ denotes the set of all k times continuously differentiable processes $x(t) \in \mathbb{R}^n$ such that $x^{(i)}(t) \in L^2(J, \Omega, \mathbb{R}^n)$ $(i = 0, 1, \ldots, k)$; the definitions of continuity, differentiability, and integrability are in the sense of mean square; for instance, we say that a process $x(t) \in L^2([0, +\infty), \Omega, \mathbb{R}^n)$ is locally integrable in $[0, +\infty)$ if, for all finite interval $[t_1, t_2] \subset [0, +\infty)$, the integral $\int_{t_1}^{t_2} \|x(t)\|_2 \,\mathrm{d}t < +\infty$.

Laplace transform. Let us introduce the class H_n of all processes $f(t) \in L^2([0, +\infty), \Omega, \mathbb{R}^n)$ such that

(i) $f(t)$ is mean square locally integrable, and

(ii) There exist constants $a \geqslant 0$ and $M_0 > 0$ such that

$$\|f(t)\|_2 \leqslant M_0 \mathrm{e}^{at}, \quad t \geqslant 0. \tag{2}$$

Then, the stochastic Laplace transform of $f(t)$ is defined by the mean square integral

$$F(s) = L[f](s) = \int_0^{+\infty} f(t)\mathrm{e}^{-st}\mathrm{d}t, \quad \mathrm{Re}s > a, \tag{3}$$

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where s denotes the complex number. From inequality (2), we know that the integral (3) is well-defined in the half-plane $\mathrm{Re}s > a$.

Here we assume that there are a pair of nonsingular deterministic and constant matrices $P, Q \in \mathbb{R}^{n\times n}$ such that the following condition is satisfied:

$$PAQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \quad PBQ = \begin{bmatrix} B_1 & 0 \\ 0 & I_{n_2} \end{bmatrix},$$

$$PC = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \; PDQ = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}, \; PG = \begin{bmatrix} G_1 \\ 0 \end{bmatrix}, \qquad (4)$$

where $N \in \mathbb{R}^{n_2\times n_2}$ denotes a nilpotent matrix with order h, i.e., $h = \min\{k : k \geqslant 1, N^k = 0\}$; $B_1, D_1 \in \mathbb{R}^{n_1\times n_1}, C_1, G_1 \in \mathbb{R}^{n_1\times m}, C_2 \in \mathbb{R}^{n_2\times m}$, and $n_1 + n_2 = n$. Let $[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}] = Q^{-1}x$, and then system (1) is equivalent to

$$\begin{aligned} \mathrm{d}x_1(t) = (B_1x_1(t) + C_1u(t))\mathrm{d}t + (D_1x_1(t) \\ + G_1u(t))\mathrm{d}w(t), x_1(0) = x_{10}, \end{aligned} \qquad (5)$$

$$N\mathrm{d}x_2(t) = x_2(t)\mathrm{d}t + C_2u(t)\mathrm{d}t, \; x_2(0) = x_{20}. \qquad (6)$$

Now, we consider the initial value problem (6). In the following, assume that the solution to (5) is the strong solution in the sense of [8] and Eq. (6) admits the stochastic Laplace transform. Applying the stochastic Laplace transform to (6), we have

$$(sN - I_{n_2})X_2(s) = Nx_{20} + C_2U(s). \qquad (7)$$

Impulse solution. Suppose that $x_2(t)$ is the inverse stochastic Laplace transform of $X_2(s)$ obtained from (7). Then, $x_2(t)$ is the impulse solution to (6) in the sense of the stochastic Laplace transform, or simply, the impulse solution to (6). In this case, if $x_1(t)$ denotes the solution to (5), then $x(t) = Q[\begin{smallmatrix} x_1(t) \\ x_2(t) \end{smallmatrix}]$ is called the impulse solution of (1).

Let $\Phi(t)$ be the solution of system

$$\mathrm{d}\Phi(t) = (B_1\mathrm{d}t + D_1\mathrm{d}w(t))\Phi(t), \; \Phi(0) = I_{n_1}. \qquad (8)$$

By applying theorem 5.2.1 of [8] and Ito's formular, the following proposition can be obtained.

Proposition 1. Let $u : [0, T] \to \mathbb{R}^m$ be a bounded Borel measurable function; then Eq. (5) has a unique solution on $[0, T]$ with any initial condition $x_{10} \in L^2(\Omega, F_0, P, \mathbb{R}^{n_1})$, and the solution is given by the following stochastic process:

$$\begin{aligned} x_1(t) &= \Phi(t)x_{10} \\ &+ \Phi(t)\int_0^t \Phi^{-1}(\tau)(C_1 - D_1G_1)u(\tau)\mathrm{d}\tau \\ &+ \Phi(t)\int_0^t \Phi^{-1}(\tau)G_1u(\tau)\mathrm{d}w(\tau), \; 0 \leqslant t \leqslant T. \end{aligned} \qquad (9)$$

For the solution of (6), we have the following theorem by applying Laplace transform and [9].

Theorem 1. For any $x_{20} \in L^2(\Omega, F_0, P, \mathbb{R}^{n_2}), u \in C^{h-1}([0, \infty), \Omega, \mathbb{R}^m)$ and $u^{(i)} \in H_m(i = 0, 1, \ldots, h - 1)$, if $h \geqslant 2$, then subsystem (6) has a unique impulse solution, which is given by

$$x_2(t) = -\sum_{i=1}^{h-1} \delta^{(i-1)}(t)\Bigg[N^ix_{20} + \sum_{k=i}^{h-1} N^kC_2u^{(k-i)}(0)\Bigg] - \sum_{i=0}^{h-1} N^iC_2u^{(i)}(t), \qquad (10)$$

where $\delta(t)$ denotes the Dirac function and $\delta^{(i)}(t)$ denotes the ith derivative of $\delta(t)$ (for proof, see Appendix A).

Based on Proposition 1 and Theorem 1, we obtain the following theorem.

Theorem 2. Assume that Eq. (1) is equivalent to (5) and (6), $u : [0, T] \to \mathbb{R}^m$ is a bounded Borel measurable function, $u \in C^{h-1}([0, \infty), \Omega, \mathbb{R}^m)$, and $u^{(i)} \in H_m$ $(i = 0, 1, \ldots, h - 1)$. If $h \geqslant 2$, then Eq. (1) has a unique impulse solution, for any $x_0 \in L^2(\Omega, F_0, P, \mathbb{R}^n)$, which is given by $x(t) = Q[\begin{smallmatrix} x_1(t) \\ x_2(t) \end{smallmatrix}]$, where $x_1(t)$ and $x_2(t)$ are given by systems (9) and (10), respectively.

We remark that Lemma 1 of [7] is incorrect. The reason is explained in Appendix B. Now, we introduce the concept of exact null controllability.

Exact null controllability. System (5) and (6) is said to be exactly null controllable on $[0, T]$ if for any $[\begin{smallmatrix} x_{10} \\ x_{20} \end{smallmatrix}] \in \mathbb{R}^n$, there exists $u \in L^2([0, T], F, \mathbb{R}^m)$, such that system (5) and (6) has a unique solution $[\begin{smallmatrix} x_1(t) \\ x_2(t) \end{smallmatrix}]$ satisfying the initial condition $[\begin{smallmatrix} x_1(0) \\ x_2(0) \end{smallmatrix}] = [\begin{smallmatrix} x_{10} \\ x_{20} \end{smallmatrix}]$ in addition to the terminal condition $[\begin{smallmatrix} x_1(T) \\ x_2(T) \end{smallmatrix}] = 0$.

It is obvious that if system (5) and (6) is exactly null controllable, so are the subsystems (5) and (6) respectively. Using the example in Appendix C, we obtain that if $N \neq 0$, then system (5) and (6) is not necessarily exactly null controllable. Consequently, we assume that $N = 0$ in the following.

Theorem 3. If $G_1 = 0$, then the necessary condition for (5) to be exactly null controllable on $[0, T]$ is that $E(\int_0^T f^2(t)\Phi^{-1}(t)C_1(\Phi^{-1}(t)C_1)^{\mathrm{T}}\mathrm{d}t)$ is invertible for any real valued polynomial $f(t)$ not identical zero (see Appendix D for a proof).

Let $\mathrm{rank}G_1 = n_1$, $u(t) = M[\begin{smallmatrix} 0 \\ v(t) \end{smallmatrix}]$, and $z(t) = D_1x_1(t)$, where M denotes an $m \times m$ matrix, which satisfies $G_1M = [I_{n_1} \;\; 0]$, and $v(t)$ denotes an $(m - n_1)$-dimension vector.

For the above $u(t)$, system (5) and (6) is equivalent to

$$\begin{aligned} -\mathrm{d}x_1(t) = (F_1x_1(t) + F_2z(t) + F_3v(t))\mathrm{d}t \\ - z(t)\mathrm{d}w(t), \;\; x_1(0) = x_{10}, \end{aligned} \qquad (11)$$

$$x_2(t) = -C_2M\begin{bmatrix} 0 \\ v(t) \end{bmatrix}, \;\; t > 0, \qquad (12)$$

where $F_1 = D_1 - B_1, F_2 = -I_{n_1}$, $F_3v(t) = -C_1M[\begin{smallmatrix} 0 \\ v(t) \end{smallmatrix}]$.

Let $\Psi(t)$ denote the solution of system $\mathrm{d}\Psi(t) = \Psi(t)(F_1\mathrm{d}t + F_2\mathrm{d}w(t)), \Psi(0) = I_{n_1}$.

Theorem 4. System (11) and (12) is exactly null controllable on $[0, T]$ if and only if

$$E\left(\int_0^T f^2(t)\Psi^{-1}(t)F_3(\Psi^{-1}(t)F_3)^{\mathrm{T}}\mathrm{d}t\right) \qquad (13)$$

is invertible for any real valued polynomial $f(t)$ not identical to zero (see Appendix E for a proof).

Conclusion. We provided the conditions for the existence and uniqueness of the impulse solution, and we also proposed the necessary and sufficient conditions for the exact null controllability. An illustrative example is given in Appendix F, which validates the effectiveness of Theorem 4.

The exact observability of singular stochastic systems will be discussed in the future.

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Supporting information Appendixes A–F. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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