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An exact null controllability of stochastic singular systems

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Appendix A Proof of Theorem 1

In order to prove Theorem 1, we need the following preparatory knowledge.
Lemma A1 Let $f \in C^k([0,+\infty),\Omega,\mathbb{R}^n), f^{(i)} \in H_n (i = 0,1,\cdots,k)$. Then

$$L[f^{(k)}](s) = s^k F(s) - \sum_{j=0}^{k-1} s^j f^{(k-1-j)}(0). \tag{A1}$$

Proof. For $k \geqslant 1$,from definition (3) and using the fundamental theorem of mean square calculus (p.104, [1]), we get

$$L[f^{(k)}](s) = \int_0^{+\infty} f^{(k)}(t)e^{-st}dt = f^{(k-1)}e^{-st}|_0^{+\infty} + s\int_0^{+\infty} f^{(k-1)}(t)e^{-st}dt = f^{(k-1)}e^{-st}|_0^{+\infty} + sL[f^{(k-1)}](s). \tag{A2}$$

Now by applying condition (2) to $f^{(k-1)}(t)$ and take $\text{Re}s > a$, it is verified

$$\|f^{(k-1)}(t)e^{-st}\|_2 = \|f^{(k-1)}(t)\|_2 e^{-t\text{Re}s} \leqslant M_0 e^{-t(\text{Re}s-a)} \to 0 (t \to +\infty). \tag{A3}$$

Therefore, from (A2) and (A3), we obtain $L[f^{(k)}](s) = sL[f^{(k-1)}](s) - f^{(k-1)}(0)$. Then the result follows from the above computation and mathematical induction method. The proof is complete.

Let K denote the space of infinite times continuously differentiable functions from $[0,+\infty)$ to $\mathbb{R}$ with compact support. The Dirac function $\delta(t)$ is the continuous linear mapping on K defined by $\int_0^{+\infty}\delta(t)\phi(t)dt = \phi(0)$ for every $\phi \in K$.Its ith derivative $\delta^{(i)}(t)$is the continuous linear mapping on K defined by $\int_0^{+\infty}\delta^{(i)}(t)\phi(t)dt = (-1)^i\phi^{(i)}(0)$ for every $\phi \in K$. For the Laplace transform of $\delta(t)$ and $\delta^{(i)}(t)$, we use the formula

$$L[\delta^{(i)}](s) = s^i, i = 0,1,2.\cdots. \tag{A4}$$

For the more details, see [2].

The next operational rule is the convolution for $f, g \in H_n$, denoted by $f \star g$ and defined by the mean square integral

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau, t \geqslant 0. \tag{A5}$$

As it occurs for the deterministic case, see p.259 of [3] by writing $L[f * g]$as a double mean square integral and reversing the order of integration, we obtain a stochastic convolution formula for the stochastic Laplace transform

$$L[f * g](s) = L[f](s)L[g](s) = F(s)G(s), f, g \in H_n. \tag{A6}$$

By (A1) and (A6), we obtain the following lemma.
Lemma A2 Let $f \in C^k([0,+\infty),\Omega,\mathbb{R}^n), f^{(i)} \in H_n (i = 0,1,\cdots,k)$. Then

$$(\delta^{(k)} * f)(t) = f^{(k)}(t) + \sum_{j=0}^{k-1}\delta^{(j)}(t)f^{(k-1-j)}(0). \tag{A7}$$

Proof of Theorem 1. Since (6) admits the stochastic Laplace transform, equation (7) is true. By (7), we obtain

$$X_2(s) = (sN - I_{n_2})^{-1}Nx_{20} + (sN - I_{n_2})^{-1}C_2U(s). \tag{A8}$$

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Since $(sN-I_{n_2})^{-1}=-\sum\limits_{i=0}^{h-1}N^is^i$, we have, from (A8),

$$X_2(s)=-\sum_{i=0}^{h-1}N^{i+1}s^ix_{20}-\sum_{i=0}^{h-1}N^is^iU(s). \tag{A9}$$

Taking the inverse stochastic Laplace transform on both sides of (A9), by (A6) and (A7), and noting $N^h=0$ gives

$$L^{-1}[X_2](t)=x_2(t), \tag{A10}$$

$$L^{-1}[-\sum_{i=0}^{h-1}N^{i+1}s^ix_{20}]=-\sum_{i=1}^{h-1}N^i\delta^{(i-1)}(t)x_{20}, \tag{A11}$$

$$L^{-1}[-\sum_{i=0}^{h-1}N^is^iC_2U(s)]=-\sum_{i=0}^{h-1}N^i\int_0^t\delta^{(i)}(t-\tau)C_2u(\tau)d\tau=-\sum_{i=1}^{h-1}N^i[\sum_{j=0}^{i-1}\delta^{(j)}(t)C_2u^{(i-1-j)}(0)]-\sum_{i=0}^{h-1}N^iC_2u^{(i)}(t). \tag{A12}$$

Furthermore, exchanging the order of the double sum and noting that $N^h=0$ in (A12), similar to the proof of Theorem 2 of [4], we have

$$\sum_{i=1}^{h-1}N^i[\sum_{j=0}^{i-1}\delta^{(j)}(t)C_2u^{(i-1-j)}(0)]=\sum_{i=1}^{h-1}\delta^{(i-1)}(t)N^i[\sum_{j=0}^{h-1}N^jC_2u^{(j)}(0)] \tag{A13}$$

Combing (A10), (A11), (A12) and (A13), we get that (10) is true. The proof is complete.

Appendix B Explanation

In the following, we explain the reason why Lemma 1 of [5] is improper, which corresponds to [7] in Letter part.

In Lemma 1 of [5], the condition (4) is

$$PAQ=\begin{bmatrix}I_{n_1}&0\\0&N\end{bmatrix},PBQ=\begin{bmatrix}B_1&0\\0&I_{n_2}\end{bmatrix},PC=\begin{bmatrix}0\\0\end{bmatrix},PDQ=\begin{bmatrix}D_1&D_2\\0&0\end{bmatrix},PG=\begin{bmatrix}0\\0\end{bmatrix}. \tag{B1}$$

Under the condition (B1), system (1) is equivalent to

$$dx_1(t)=B_1x_1(t)dt+(D_1x_1(t)+D_2x_2(t))dw(t),x_1(0)=x_{10},t\geqslant0 \tag{B2}$$

$$Ndx_2(t)=x_2(t)dt,x_2(0)=x_{20},t\geqslant0. \tag{B3}$$

The impulse solution of (B3) is

$$x_2(t)=-\sum_{i=1}^{h-1}\delta^{(i-1)}(t)N^ix_{20}. \tag{B4}$$

Substituting (B4) into (B2), we obtain

$$dx_1(t)=B_1x_1(t)dt+(D_1x_1(t)-D_2\sum_{i=1}^{h-1}\delta^{(i-1)}(t)N^ix_{20})dw(t),x_1(0)=x_{10},t\geqslant0. \tag{B5}$$

In the proof of Lemma 1 (i) of [5], they said that "by Theorem 5.2.1 of [6], the solution of (B5) exists and is unique, so does (1)." We point that such a proof is not correct for the existence and uniqueness of the solution to system (1) and (B5), because there exist impulse terms in $\sigma(t,x_1)=D_1x_1(t)-D_2\sum\limits_{i=1}^{h-1}\delta^{(i-1)}(t)N^ix_{20},b(t,x_1)=B_1x_1$ and $\sigma(t,x_1)$ in (B5) do not satisfy the conditions of Theorem 5.2.1 of [6]. In this case, the Lemma 1 (i) of [5] is improper. In fact, we have that $b(t,x_1)$ and $\sigma(t,x_1)$ satisfy the condition of Theorem 5.2.1 of [6] if and only if $D_2N^i=0(i=1,2,\cdots,h-1)$. In this case, $PDQ=\begin{bmatrix}D_1&D_2\\0&0\end{bmatrix}$ can be corrected as$PDQ=\begin{bmatrix}D_1&0\\0&0\end{bmatrix}$ in (B1).

Appendix C Example

Consider the singular stochastic system.

$$dx_1(t)=u(t)dt+x_1(t)dw(t),x_1(0)=x_{10},t\geqslant0 \tag{C1}$$

$$\begin{bmatrix}a&a\\-a&-a\end{bmatrix}\begin{bmatrix}dx_2(t)\\dx_3(t)\end{bmatrix}=\begin{bmatrix}x_2(t)\\x_3(t)\end{bmatrix}dt+\begin{bmatrix}1\\1\end{bmatrix}u(t)dt,\begin{bmatrix}x_2(0)\\x_3(0)\end{bmatrix}=\begin{bmatrix}x_{20}\\x_{30}\end{bmatrix}, \tag{C2}$$

where $N=\begin{bmatrix}a&a\\-a&-a\end{bmatrix}$ is a nilpotent matrix. If (C1)-(C2) is exactly null controllable, then $N=0$.

Proof. The solution of (C1)-(C2) is

$$x_1(t)=x_{10}\exp[-\frac{1}{2}t+w(t)]+\exp[-\frac{1}{2}t+w(t)]\int_0^t\exp[\frac{1}{2}\tau-w(\tau)]u(\tau)d\tau,\begin{bmatrix}x_2(t)\\x_3(t)\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}u(t)+\begin{bmatrix}2a\\-2a\end{bmatrix}u'(t),t>0.$$

If (C1)-(C2) is exactly null controllable, for $x_{10} = -1$ and $\begin{bmatrix} x_{20} \\ x_{30} \end{bmatrix} \in \mathbb{R}^2$, there exists $u \in L^2([0,T], F, \mathbb{R})$, such that

$$1 = \int_0^T e^{-w(t)+\frac{1}{2}t} u(t)dt, 0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(T) + \begin{bmatrix} 2a \\ -2a \end{bmatrix} u'(T), t > 0. \tag{C3}$$

Since $L^2([0,T], F, \mathbb{R})$ is a Hilbert space with inner product $< x(\cdot), y(\cdot) >_{L^2([0,T],F,R)} = E(\int_0^T x^{\mathrm{T}}(t)y(t)dt)$, we have that $u(t) = c_0 e^{\frac{1}{2}t-w(t)} + u_1(t)$, where $e^{\frac{1}{2}t-w(t)}$ and $u_1(t)$ are orthogonal. Therefore $u'(t)$ does not exist. Hence, if (C3) is true, the necessary condition is $a = 0$, i.e., $N = 0$. The proof is complete.

Appendix D Proof of Theorem 3

In order to prove Theorem 3, first of all we prove the following lemma.

Lemma D1 If $G_1 = 0$, the necessary condition for (5) to be exactly null controllable on $[0,T]$ is that

$$E(\int_0^T \Phi^{-1}(t)C_1(\Phi^{-1}(t)C_1)^{\mathrm{T}} dt)$$

is invertible.

Proof. If (5) is exactly null controllable, by the definition of exact null controllability, $\forall x_{10} \in \mathbb{R}^{n_1}, \exists u(\cdot) \in L^2([0,T], F, \mathbb{R}^m)$, such that $x_1(T) = 0$ in (9), i.e.,

$$x_1(0) = -\int_0^T \Phi^{-1}(t)C_1 u(\tau)d\tau = E(x_{10}) = -E(\int_0^T \Phi^{-1}(t)C_1 u(\tau)d\tau) \tag{D1}$$

Let Gram$=E(\int_0^T \Phi^{-1}(t)C_1(\Phi^{-1}(t)C_1)^{\mathrm{T}} dt)$. Using reduction ad absurdum, if Gram is not invertible, then there exists a vector $\xi \in \mathbb{R}^{n_1}$ which is not zero such that $E(\int_0^T \xi^{\mathrm{T}}\Phi^{-1}(t)C_1(\Phi^{-1}(t)C_1)^{\mathrm{T}}\xi dt) = 0$, that is to say, $\xi^{\mathrm{T}}\Phi^{-1}(t)C_1 = 0$, a.e. on $[0,T]$. For any given initial value $x_{10} \in \mathbb{R}^{n_1}$, from (D1), we get $< \xi, x_{10} >_{L^2(\Omega,F_t,P,\mathbb{R}^n)} = -E(\int_0^T \xi^{\mathrm{T}}\Phi^{-1}(t)C_1 u(\tau)d\tau) = 0$, i.e., $\xi = 0$. This contradict with the fact $\xi \neq 0$. So Gram is invertible. The proof is complete.

Proof of Theorem 3. Using reduction ad absurdum, if $E(\int_0^T f^2(t)\Phi^{-1}(t)C_1(\Phi^{-1}(t)C_1)^{\mathrm{T}} dt)$ is not invertible, then there exists a vector $\xi \in \mathbb{R}^{n_1}$ which is not zero such that $E(\int_0^T \xi^{\mathrm{T}} f(t)\Phi^{-1}(t)C_1(\Phi^{-1}(t)C_1)^{\mathrm{T}} f(t)\xi dt) = 0$, that is to say, $\xi^{\mathrm{T}} f(t)\Phi^{-1}(t)C_1 = 0$, a.e. on $[0,T]$. Since $f(t)$ can have only finite many zero on $[0,T]$, we get $\xi^{\mathrm{T}}\Phi^{-1}(t)C_1 = 0$, a.e. on $[0,T]$. Therefore $E(\int_0^T \xi^{\mathrm{T}}\Phi^{-1}(t)C_1(\Phi^{-1}(t)C_1)^{\mathrm{T}}\xi dt) = 0$. By Lemma D1, $E(\int_0^T \Phi^{-1}(t)C_1(\Phi^{-1}(t)C_1)^{\mathrm{T}} dt)$ is invertible. Therefore $\xi = 0$. This contradict with the fact $\xi \neq 0$. Hence $E(\int_0^T f^2(t)\Phi^{-1}(t)C_1(\Phi^{-1}(t)C_1)^{\mathrm{T}} dt)$ is invertible. The proof is complete.

Appendix E Proof of Theorem 4

Proof. If (11)-(12) is exactly null controllable, then system (11) is exactly null controllable. Proof similar to Theorem 3, we have that $E(\int_0^T f^2(t)\Psi(t)F_3(\Psi(t)F_3)^{\mathrm{T}} dt)$ is invertible for any real valued polynomial $f(t)$ not identical zero.

By the proof of Theorem 2 of [7], if $E(\int_0^T f^2(t)\Psi(t)F_3(\Psi(t)F_3)^{\mathrm{T}} dt)$ is invertible for any real valued polynomial $f(t)$ not identical zero, system (11) is exactly null controllable on $[0,T]$ if and only if

$$x_1(0) = E(\int_0^T \Psi(t)F_3 v(t)dt). \tag{E1}$$

For any initial $x_1(0)$, we construct controller

$$v(t) = (t-T)^2(\Psi(t)F_3)^{\mathrm{T}} E(\int_0^T (t-T)^2\Psi(t)F_3(\Psi(t)F_3)^{\mathrm{T}} dt)x_1(0). \tag{E2}$$

Taking (E2) into (E1), it is obvious that (E1) holds true. Therefore $x_1(T) = 0, x_2(T) = -C_2 M \begin{bmatrix} 0 \\ v(T) \end{bmatrix} = 0$, and $v \in L^2([0,T], F, R^{m-n_1})$. Hence, system (11)-(12) is exactly null controllable. The proof is complete.

Appendix F An illustrative example

In the following, an illustrative example is given, which shows the effectiveness of Theorem 4.

Consider the fundamental dynamics Leontief model of economic system ([8]), which is given as follows:

$$x(t) = Fx(t) + A\frac{dx(t)}{dt} + s(t), \tag{F1}$$

where $x(t)$ is the $n-$dimensional production vector of n sectors; $F \in \mathbb{R}^{n\times n}$ is an input-output (or production) matrix; $Fx(t)$ stands for the fraction of production required as input for the current production;$A \in \mathbb{R}^{n\times n}$ is the capital coefficient matrix, and $A\frac{dx(t)}{dt}$ is the amount for capacity expansion, which often appears in the form of capital; $s(t)$ is the vector which includes demand or consumption. System (F1) may be rewritten as

$$A\frac{dx(t)}{dt} = Lx(t) + Ku(t), \tag{F2}$$

where $L = I_n - F, K = I_n, u(t) = -s(t)$. In multisector economic systems, production augmention in one sector often does not need the investment from all other sectors, and moreover, in practical cases only a few sectors can offer investment in capital to other sectors. Thus, most of the elements in A are zero except for a few. A is often singular. In this sense the system (F2) is a typical singular system. However, it might happen that L and K are subject to some random environmental effects ([6],[9]) such as $L = B + D$"noise" and $K = C + G$"noise". In this case, system (F2) can be expressed as

$$A\frac{dx(t)}{dt} = Bx(t) + Cu(t) + Dx(t)\text{"noise"} + Gu(t)\text{"noise"}, \tag{F3}$$

It turns out that a reasonable mathematical interpretation for the "noise" terms is the so-called white noise $\frac{dw(t)}{dt}$. By (F3), we have

$$Adx(t) = (Bx(t) + Cu(t))dt + (Dx(t) + Gu(t))dw(t), \tag{F4}$$

this is the form of system (1). In what follows, we will verify the effectiveness of Theorem 4. If for some concrete engineering practice, the following data are taken in (1):

$$A = \begin{bmatrix} 1 & 0 \\ 0 & N \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & I_2 \end{bmatrix}, C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, G = \begin{bmatrix} G_1 \\ 0 \end{bmatrix}, \tag{F5}$$

where $N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, C_1 = [1 \quad 1 \quad 0], C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, G_1 = [1 \quad 0 \quad 0]$. System (F4) can be expressed as

$$dx_1(t) = (x_1(t) + C_1u(t))dt + (2x_1(t) + G_1u(t))dw(t), \tag{F6}$$

$$0 = x_2(t) + C_2u(t), t > 0. \tag{F7}$$

Let $u(t) = MG = \begin{bmatrix} 0 \\ v(t) \end{bmatrix}, z(t) = 2x_1(t)$, where $v(t)$ is 2-dimension vector. Then $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, G_1M = [1 \quad 0 \quad 0]$. System (F6)-(F7) is changed into an equivalent form

$$-dx_1(t) = (x_1(t) - z(t) - [1 \quad 0]v(t))dt - z(t)dw(t), \tag{F8}$$

$$x_2(t) = -v(t), t > 0. \tag{F9}$$

Let $\Psi(t)$ be the solution of system $d\Psi(t) = \Psi(t)(dt - dw(t)), \Psi(0) = 1$. Then $\Psi(t) = e^{\frac{1}{2}t - w(t)}$. Since

$$E(\int_0^T (t-T)^2\Psi(t)[1 \quad 0](\Psi(t)[1 \quad 0])^{\mathrm{T}}dt) = \int_0^T (t-T)^2 e^{3t}dt > 0$$

for every $T > 0$, thus $E(\int_0^T (t-T)^2\Psi(t)[1 \quad 0](\Psi(t)[1 \quad 0])^{\mathrm{T}}dt)$ is invertible for every $T > 0$. By Theorem 4, system (F8)-(F9) is exactly null controllable.

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