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Recursive filtering for nonlinear systems subject to measurement outliers

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Abstract In this paper, an innovative recursive filtering algorithm (RFA) is proposed for a class of nonlinear systems (NSs) subject to multiplicative noises (MNs) and measurement outliers (MOs). Initially, the MNs are employed to formulate the random influence on the NSs with the stochastic noises. Next, the outlier phenomenon could occur unpredictably during measurement transmission. Then, a self-adaptive saturation function is introduced to the constructed filter to mitigate the influence of MOs on the filter performance. In this paper, we design a resistant-outlier filter for NSs with MNs and MOs, and the filter gain ensures that the trace of the filtering error covariance matrix is minimized by solving the constructed Riccati-like difference equations. Moreover, the exponential boundedness of the filtering error in the sense of mean square is analyzed. Finally, the feasibility of the proposed RFA is illustrated by a simulation example when the MOs occur.

Keywords adaptive saturation, measurement outliers, multiplicative noises, nonlinear systems, recursive filtering

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1 Introduction

Over the past few decades, solving filtering problems has received significant attention with the rapid development of industrialization and digitization. A host of excellent filtering algorithms have been proposed by many researchers, such as the Kalman filtering [1], the extended Kalman filtering (EKF) [2,3], the H_∞ filtering [4,5], and the fusion filtering [6,7]. The classical Kalman filter has been widely employed in practice, particularly in engineering fields, because of its high filtering accuracy. Moreover, it has good real-time performance owing to the recursive calculation carried on, and it is also more suitable for online applications. Nevertheless, due to the inevitable emergence of nonlinearity in the practical systems, it is often impossible to obtain the optimal filter with the minimum variance of filtering error (FE). So far, many studies have been carried out to solve nonlinear characteristics, see, e.g., [8–11]. In [12], a recursive filtering algorithm (RFA) was proposed for stochastic nonlinear systems (NSs), where the obtained filter gain can ensure that the filtering error covariance matrix (FECM) has an upper bound (UB) and is optimized in the sense of matrix trace.

Most of the existing filtering algorithms deal with additive noise. However, as another type of practical noise, the attached multiplicative noise (MN) to the system state was firstly proposed in [13]. Using MN models seems more appropriate compared to the linearized models to approximate the NSs [14]. As a result, the MN problem has received great importance, see, e.g., [15–17]. An H_∞ filtering problem was studied by [16] for neural network systems, where the MNs were used to describe the random disturbances on the system states and also to analyze the influence of MNs on filtering performance. Besides, the MN is generally assumed to be described by a single random parameter. In [18], a mean-square-error estimator

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was designed for the systems with MNs and Markov jump, where the characterization of the scalar MN model was extended to the form of diagonal matrix MNs.

As for the filtering problems, the initial results assumed that the measured data are collected accurately. Such assumption, unfortunately, is not always realistic due to the inevitable incomplete data, the quantization error, and the sensor delay. These phenomena, if not adequately handled, could severely degrade the system performance and therefore this issue has triggered a growing research interest in solving some networked systems with these phenomena [19–27]. Recently, several studies have investigated the MOs that may come from the deception attacks, the abnormal pulse signals, and the environmental changes. The occurrence of the MOs would lead to abnormal measurement outputs and subsequently deteriorate the performance of the filter/estimator, which also affects the EKF performance. As a challenging research topic, the description of outliers has attracted tremendous attention, and several methods have been proposed, e.g., the T-distributed noise models [28, 29] and the Gamma-distributed noise models [30]. Recently, an innovative self-adaptive saturation method for mitigating MOs has been developed by [31, 32].

Considering the above mentioned discussion, we aim to develop an RFA for a class of NSs with MNs and MOs. An adaptive saturation method is introduced to reduce the influence of MOs on the system. Unlike the traditional saturation function, the dynamic saturation function is self-adaptive by adjusting the saturation level instantly. The highlights of this paper are as follows: (1) a self-adaptive saturation function is introduced into the developed EKF to deal with the MOs; (2) the UB of FECM is optimized by solving two Riccati-like difference equations (RLDEs); and (3) the mean-square exponential boundedness (MSEB) of the FE is guaranteed when the MOs occur.

The rest of the paper is shown as follows. In Section 2, a problem formulation is described in a concise way. In Section 3, a new RFA is proposed for the NSs with MNs and MOs, where an optimized UB of the FECM is obtained and the MSEB of the FE is analyzed. In Section 4, the feasibility of the presented RFA is verified by a simulation example. In Section 5, the conclusion is summed up succinctly.

Notation. In this paper, N^{T} represents the transpose of N. $\mathbb{R}^a$ means the a dimensional Euclidean space, $\mathbb{R}^{a\times b}$ is the $a \times b$ dimensional real matrix. $\beta \in \mathbb{R}$ represents that β is a scalar. I stands for the identity matrix. $M > 0$ means that M is a positive definite matrix. $\mathrm{diag}\{\cdot\}$ represents a diagonal matrix. $\mathrm{E}\{\alpha\}$ means the expectation of α and $\mathrm{Tr}\{\cdot\}$ denotes the trace of matrix. $\min\{x,y\}$ and $\max\{x,y\}$ represent the minimum and maximum value in variables x and y, respectively.

2 Problem formulation and preliminaries

Consider a class of discrete time-varying NSs with MNs described by

$$x_{s+1} = f(x_s) + \sum_{i=1}^{j} \alpha_{i,s} A_{i,s} x_s + B_s w_s, \tag{1}$$

$$y_s = C_s x_s + v_s, \tag{2}$$

where $x_s \in \mathbb{R}^n$ is the state vector, $\bar{x}_0$ and $P_{0|0}$ are the initial mean value and covariance, respectively. $y_s \in \mathbb{R}^m$ denotes the original measurement, $\alpha_{i,s} \in \mathbb{R}$ is a zero-mean Gaussian random variable with unit variance, the process noise $w_s \in \mathbb{R}^n$ and the measurement noise $v_s \in \mathbb{R}^m$ are zero-mean white Gaussian random vectors with covariance matrices $Q_s > 0$ and $Z_s > 0$, respectively. In this paper, it is assumed that $\alpha_{i,s}$, w_s, and v_s are mutually independent of each other at each time instant. The known nonlinear function $f(x_s)$ is continuously differentiable, and $A_{i,s}$, B_s, and C_s are known matrices.

As is well known, the measurement y_s plays a vital role in obtaining the system state. By utilizing the information from y_s, a recursive filter is modified as follows:

$$x_{s+1|s} = f(x_{s|s}), \tag{3}$$

$$x_{s+1|s+1} = x_{s+1|s} + L_{s+1}(y_{s+1} - C_{s+1} x_{s+1|s}), \tag{4}$$

where $x_{s|s}$ and $x_{s+1|s}$ are the estimated and one-step predicted values of x_s with an initial value $x_{0|0} = \bar{x}_0$ at time instant s, respectively. L_{s+1} represents the filter gain.

In practical engineering, due to various reasons, y_s could be occasionally an outlier which brings out some adverse impact on the performance of the filter if not handled. Taking this point into consideration, a novel filter is presented to resist the effect of the possible outlier by introducing a saturation function with an adaptive threshold. Therefore, the above recursive filter is updated as follows:

$$x_{s+1|s} = f(x_{s|s}), \tag{5}$$
$$x_{s+1|s+1} = x_{s+1|s} + L_{s+1}\mathrm{sat}_{\sigma_{s+1}}(y_{s+1} - C_{s+1}x_{s+1|s}), \tag{6}$$

where $\mathrm{sat}_{\sigma_{s+1}}(\cdot)$ is a vector of m adaptive saturation functions and σ_{s+1} is a vector of m non-negative dynamic saturation thresholds.

More specifically, define the expression of σ_{s+1} as follows:

$$\sigma_{s+1} \triangleq [\sigma_{1,s+1}, \sigma_{2,s+1}, \ldots, \sigma_{m,s+1}]^{\mathrm{T}} \in \mathbb{R}^m.$$

For any vector $\iota \triangleq [\iota_1, \iota_2, \ldots, \iota_m]^{\mathrm{T}} \in \mathbb{R}^m$, $\mathrm{sat}_{\sigma_{\iota+1}}(\iota)$ is defined as follows:

$$\mathrm{sat}_{\sigma_{s+1}}(\iota) \triangleq [\mathrm{sat}_{\sigma_{1,s+1}}(\iota_1), \mathrm{sat}_{\sigma_{2,s+1}}(\iota_2), \ldots, \mathrm{sat}_{\sigma_{m,s+1}}(\iota_m)]^{\mathrm{T}}, \tag{7}$$

where $\mathrm{sat}_{\sigma_{i,s+1}}(\iota_i) \triangleq \max\{-\sigma_{i,s+1}, \min\{\sigma_{i,s+1}, \iota_i\}\}$. The evolution rules of the adaptive saturation function threshold are shown as follows:

$$\bar{\sigma}_{s+1} = \lambda\bar{\sigma}_s + (y_{s+1} - \hat{y}_{s+1})^{\mathrm{T}}R(y_{s+1} - \hat{y}_{s+1}), \tag{8}$$
$$\sigma_{i,s+1} = \sqrt{\bar{\sigma}_{s+1}/\pi_i}, \quad i = 1, 2, \ldots, m, \tag{9}$$

where $\bar{\sigma}_s \in \mathbb{R}$ with initial value $\overline{\sigma}_0$, $\lambda \in (0, 1)$, $R = R^{\mathrm{T}} > 0$, $\pi_i \in \mathbb{R}$ $(i = 1, 2, \ldots, m)$ are positive parameters, and $\hat{y}_{s+1} = C_{s+1}x_{s+1|s}$. Here, $\hat{y}_{s+1}$ could be seen as a prediction of the measurement y_{s+1} in some sense.

Remark 1. The evolution rules of σ_{s+1} in (6) can be formulated by (8) and (9). More specifically, the dynamic equation (8) determines the common variable $\bar{\sigma}_s$, which can be employed to determine all the thresholds $\sigma_{i,s+1}$ according to (9). The term involving λ in (8) is used to make the saturation level gradually approach 0. Also, the other term of (8) indicates that the saturation level depends on the changes of the error between the measurement output y_s and its prediction $\hat{y}_s$. In addition, π_i in (9) is used to represent the proportional value of the ith saturated component. According to the mechanism established by (8) and (9), if an outlier occurs, then $\bar{\sigma}_s$ as well as $\sigma_{i,s}$ will grow larger; otherwise, $\bar{\sigma}_s$ as well as $\sigma_{i,s}$ will become smaller such that the second term in (8) plays a tiny role.

Remark 2. The filtering algorithm based on Student-t distribution assumes that the noise obeys Student-t distribution to describe the possible outlier model, which will increase the complexity of the system model. Also, in practical application, the scale matrices and degrees of freedom parameters are difficult to obtain by simple calculation. The filtering algorithm proposed in this paper only uses the information of measurement values to adjust the saturation level to suppress the influence of outliers, which can greatly lighten the calculation burden and decrease the conservatism of the algorithm.

Define the one-step prediction error (OSPE) and FE as $e_{s+1|s} \triangleq x_{s+1} - x_{s+1|s}$ and $e_{s+1|s+1} \triangleq x_{s+1} - x_{s+1|s+1}$, respectively. The objective of this paper is arranged as follows:

(1) Design a filter with recursive form for mitigating the effects from MOs and obtain a UB of FECM, that is, there exists a matrix $\Phi_{s+1|s+1} > 0$ $(0 \leqslant s \leqslant N)$ which satisfies

$$\mathrm{E}\{e_{s+1|s+1}e_{s+1|s+1}^{\mathrm{T}}\} \leqslant \Phi_{s+1|s+1}, \quad \forall s \in \mathbb{N}; \tag{10}$$

(2) Calculate the filter gain that ensures the trace of the UB of FECM is minimum; and

(3) Discuss the MSEB of FE when the MOs occur.

3 Main results

Lemma 1 ([33])**.** For $\forall a, b \in \mathbb{R}$, there is a positive number $\epsilon \in [0, 1]$ satisfying

$$\mathrm{sat}_\sigma(a) - \mathrm{sat}_\sigma(b) = \epsilon(a - b), \tag{11}$$

where the function $\mathrm{sat}_\sigma(\cdot)$ is defined by (7).

Lemma 2 ([34]). For $\forall u, v \in \mathbb{R}^n$, the following inequality holds:

$$uv^{\mathrm{T}} + vu^{\mathrm{T}} \leqslant \delta uu^{\mathrm{T}} + \delta^{-1} vv^{\mathrm{T}}, \tag{12}$$

where δ is a positive scalar.

Lemma 3 ([35]). For known matrices $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$, where $\mathcal{E}$ satisfies $\mathcal{E}\mathcal{E}^{\mathrm{T}} \leqslant I$, if there exist a symmetric matrix $\mathcal{F} > 0$ and an auxiliary variable $\gamma > 0$ satisfying $\gamma^{-1}I - \mathcal{D}\mathcal{F}\mathcal{D}^{\mathrm{T}} > 0$, then the inequality

$$(\mathcal{B} + \mathcal{C}\mathcal{E}\mathcal{D})\mathcal{F}(\mathcal{B} + \mathcal{C}\mathcal{E}\mathcal{D})^{\mathrm{T}} \leqslant \mathcal{B}(\mathcal{F}^{-1} - \gamma\mathcal{D}^{\mathrm{T}}\mathcal{D})^{-1}\mathcal{B}^{\mathrm{T}} + \gamma^{-1}\mathcal{C}\mathcal{C}^{\mathrm{T}} \tag{13}$$

holds.

Lemma 4 ([36]). Given a matrix $\mathcal{M} = \mathcal{M}^{\mathrm{T}} > 0$ and two functions $\phi_s(\mathcal{M}) = \phi_s^{\mathrm{T}}(\mathcal{M}) \in \mathbb{R}^{n\times n}$ and $\psi_s(\mathcal{M}) = \psi_s^{\mathrm{T}}(\mathcal{M}) \in \mathbb{R}^{n\times n}$ for all $0 \leqslant s \leqslant N$, if

$$\phi_s(\mathcal{N}) \geqslant \phi_s(\mathcal{M}),\ \psi_s(\mathcal{N}) \geqslant \phi_s(\mathcal{N}), \quad \forall \mathcal{M} \leqslant \mathcal{N} = \mathcal{N}^{\mathrm{T}},$$

then one has

$$\mathcal{G}_{s+1} \leqslant \mathcal{H}_{s+1}, \tag{14}$$

where $\mathcal{G}_s$ and $\mathcal{H}_s$ are the solutions of the following recursive equations:

$$\mathcal{G}_{s+1} = \phi_s(\mathcal{G}_s),\ \mathcal{H}_{s+1} = \psi_s(\mathcal{H}_s),\ \mathcal{G}_0 = \mathcal{H}_0 > 0. \tag{15}$$

Lemma 5. For a given deterministic matrix $R = R^{\mathrm{T}} \in \mathbb{R}^{m\times m}$, one has

$$\mathrm{E}\left\{H^{\mathrm{T}}RH\right\} = \mathrm{Tr}\left\{\mathrm{E}\{RHH^{\mathrm{T}}\}\right\}, \tag{16}$$

where $H \triangleq [h_1, h_2, \ldots, h_m]^{\mathrm{T}} \in \mathbb{R}^m$ and $h_i \in \mathbb{R}$ $(i = 1, 2, \ldots, m)$.

Proof.

$$\begin{aligned}
\mathrm{E}\left\{H^{\mathrm{T}}RH\right\} &= \mathrm{E}\left\{\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ r_{21} & r_{22} & \cdots & r_{2m} \\ \vdots & \vdots & & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mm} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{bmatrix}\right\} \\
&= \mathrm{E}\{r_{11}h_1h_1 + r_{12}h_2h_1 + \cdots + r_{mm}h_mh_m\} \\
&= \mathrm{E}\left\{\sum_{i=1}^{m}\sum_{l=1}^{m} r_{il}h_lh_i\right\} \\
&= \mathrm{Tr}\left\{\mathrm{E}\{RHH^{\mathrm{T}}\}\right\}.
\end{aligned}$$

The proof is now complete.

Theorem 1. The OSPE $e_{s+1|s}$ and the FE $e_{s+1|s+1}$ are respectively given as

$$e_{s+1|s} = (F_s + D_s\aleph_sG_s)e_{s|s} + \sum_{i=1}^{j} \alpha_{i,s}A_{i,s}x_s + B_sw_s, \tag{17}$$

and

$$e_{s+1|s+1} = (I - L_{s+1}\Xi_{s+1}C_{s+1})\, e_{s+1|s} + L_{s+1}(\Xi_{s+1} - I)\sigma_{s+1} - L_{s+1}\Xi_{s+1}v_{s+1}, \tag{18}$$

where $F_s \triangleq \frac{\partial f(x_s)}{\partial x_s}|_{x_s=x_{s|s}}$, $\Xi_{s+1} \triangleq \mathrm{diag}\{\epsilon_{1,s+1}, \epsilon_{2,s+1}, \ldots, \epsilon_{m,s+1}\}$, $\aleph_s$ is an unknown time-varying matrix satisfying $\aleph_s\aleph_s^{\mathrm{T}} \leqslant I$, and D_s and G_s are two known time-varying matrices. The proof is in Appendix A.

Remark 3. In the existing results, Ξ_s is generally regarded as an uncertain term, which is usually enlarged to a unit matrix. Nevertheless, this processing method increases the conservatism of the obtained results. In this paper, it is obvious to see that the output estimation error and the saturation level in (A4) are available, and the matrix Ξ_s can be solved in terms of (11). Therefore, Ξ_s can be treated as a known matrix in the subsequent derivation, which will greatly reduce the conservatism and improve the accuracy of estimation.

Theorem 2. The OSPE covariance matrix $P_{s+1|s}$ and the FECM $P_{s+1|s+1}$ are given as follows:

$$P_{s+1|s} = (F_s + D_s\aleph_s G_s)P_{s|s}(F_s + D_s\aleph_s G_s)^{\mathrm{T}} + \sum_{i=1}^{j} A_{i,s}\mathrm{E}\{x_s x_s^{\mathrm{T}}\}A_{i,s}^{\mathrm{T}} + B_s Q_s B_s^{\mathrm{T}}, \tag{19}$$

and

$$\begin{aligned} P_{s+1|s+1} =& (I - L_{s+1}\Xi_{s+1}C_{s+1})\, P_{s+1|s}\, (I - L_{s+1}\Xi_{s+1}C_{s+1})^{\mathrm{T}} + \mathrm{E}\{\Upsilon_{s+1} + \Upsilon_{s+1}^{\mathrm{T}}\} \\ &+ L_{s+1}(\Xi_{s+1} - I)\mathrm{E}\left\{\sigma_{s+1}\sigma_{s+1}^{\mathrm{T}}\right\}(\Xi_{s+1} - I)^{\mathrm{T}} L_{s+1}^{\mathrm{T}} + L_{s+1}\Xi_{s+1}Z_{s+1}\Xi_{s+1}^{\mathrm{T}} L_{s+1}^{\mathrm{T}}, \end{aligned} \tag{20}$$

where

$$\Upsilon_{s+1} \triangleq (I - L_{s+1}\Xi_{s+1}C_{s+1})e_{s+1|s}\sigma_{s+1}^{\mathrm{T}}(\Xi_{s+1} - I)^{\mathrm{T}} L_{s+1}^{\mathrm{T}}.$$

Proof. According to (17), one has

$$\begin{aligned} P_{s+1|s} =& \mathrm{E}\{e_{s+1|s}e_{s+1|s}^{\mathrm{T}}\} \\ =& \mathrm{E}\Bigg\{(F_s + D_s\aleph_s G_s)e_{s|s}e_{s|s}^{\mathrm{T}}(F_s + D_s\aleph_s G_s)^{\mathrm{T}} + (F_s + D_s\aleph_s G_s)e_{s|s}\sum_{i=1}^{j} x_s^{\mathrm{T}} A_{i,s}^{\mathrm{T}}\alpha_{i,s}^{\mathrm{T}} + B_s w_s w_s^{\mathrm{T}} B_s^{\mathrm{T}} \\ &+ (F_s + D_s\aleph_s G_s)e_{s|s}w_s^{\mathrm{T}} B_s^{\mathrm{T}} + \sum_{i=1}^{j}\alpha_{i,s}A_{i,s}x_s e_{s|s}^{\mathrm{T}}(F_s + D_s\aleph_s G_s)^{\mathrm{T}} + \sum_{i=1}^{j}\alpha_{i,s}A_{i,s}x_s x_s^{\mathrm{T}} A_{i,s}^{\mathrm{T}}\alpha_{i,s}^{\mathrm{T}} \\ &+ \sum_{i=1}^{j}\alpha_{i,s}A_{i,s}x_s w_s^{\mathrm{T}} B_s^{\mathrm{T}} + B_s w_s e_{s|s}^{\mathrm{T}}(F_s + D_s\aleph_s G_s)^{\mathrm{T}} + B_s w_s\sum_{i=1}^{j} x_s^{\mathrm{T}} A_{i,s}^{\mathrm{T}}\alpha_{i,s}^{\mathrm{T}}\Bigg\} \\ =& (F_s + D_s\aleph_s G_s)P_{s|s}(F_s + D_s\aleph_s G_s)^{\mathrm{T}} + \sum_{i=1}^{j} A_{i,s}\mathrm{E}\{x_s x_s^{\mathrm{T}}\}A_{i,s}^{\mathrm{T}} + B_s Q_s B_s^{\mathrm{T}}. \end{aligned} \tag{21}$$

Similarly, the FECM $P_{s+1|s+1}$ in (20) is directly derived. The proof is now complete.

Note that Eq. (19) involves an uncertain term $\aleph_s$ such that $P_{s+1|s}$ and $P_{s+1|s+1}$ cannot be implemented in the algorithm. Next, we are going to calculate the optimal filter gain and the UB of FECM.

Define the following two RLDEs:

$$\begin{aligned} \Phi_{s+1|s} =& F_s(\Phi_{s|s}^{-1} - \gamma G_s^{\mathrm{T}} G_s)^{-1}F_s^{\mathrm{T}} + \gamma^{-1}D_s D_s^{\mathrm{T}} + B_s Q_s B_s^{\mathrm{T}} \\ &+ \sum_{i=1}^{j} A_{i,s}[(1+\delta_1)\Phi_{s|s} + (1+\delta_1^{-1})x_{s|s}x_{s|s}^{\mathrm{T}}]A_{i,s}^{\mathrm{T}}, \end{aligned} \tag{22}$$

and

$$\begin{aligned} \Phi_{s+1|s+1} =& (1+\delta_2)\,(I - L_{s+1}\Xi_{s+1}C_{s+1})\,\Phi_{s+1|s}\,(I - L_{s+1}\Xi_{s+1}C_{s+1})^{\mathrm{T}} \\ &+ (1+\delta_2^{-1})\Lambda_{s+1}L_{s+1}(\Xi_{s+1} - I)(\Xi_{s+1} - I)^{\mathrm{T}} L_{s+1}^{\mathrm{T}} + L_{s+1}\Xi_{s+1}Z_{s+1}\Xi_{s+1}^{\mathrm{T}} L_{s+1}^{\mathrm{T}} \end{aligned} \tag{23}$$

with positive define matrix solutions $\Phi_{s+1|s}$ and $\Phi_{s+1|s+1}$ with $\Phi_{0|0} = P_{0|0} > 0$ such that the following constraint

$$\gamma^{-1}I - G_s\Phi_{s|s}G_s^{\mathrm{T}} > 0 \tag{24}$$

is satisfied, where γ, δ_1, and δ_2 are three positive scalars, and

$$L_{s+1} \triangleq (1+\delta_2)\Phi_{s+1|s}C_{s+1}^{\mathrm{T}}\Xi_{s+1}^{\mathrm{T}}\Theta_{s+1}^{-1}, \tag{25}$$

$$\Theta_{s+1} \triangleq (1+\delta_2)\Xi_{s+1}C_{s+1}\Phi_{s+1|s}C_{s+1}^{\mathrm{T}}\Xi_{s+1}^{\mathrm{T}} + (1+\delta_2^{-1})\Lambda_{s+1}(\Xi_{s+1} - I)(\Xi_{s+1} - I)^{\mathrm{T}} + \Xi_{s+1}Z_{s+1}\Xi_{s+1}^{\mathrm{T}}, \tag{26}$$

$$\Lambda_{s+1} \triangleq \left[\lambda^{s+1}\overline{\sigma}_0 + \sum_{i=0}^{s}\lambda^{s-i}\mathrm{Tr}\{RC_{i+1}\Phi_{i+1|i}C_{i+1}^{\mathrm{T}} + Z_{i+1}\}\right]\sum_{i=1}^{m}\frac{1}{\pi_i}I. \tag{27}$$

Theorem 3. Consider the OSPE covariance matrix $P_{s+1|s}$ and the FECM $P_{s+1|s+1}$ in (19) and (20). Then, the matrix $\Phi_{s+1|s+1}$ is a UB of FECM, i.e.,

$$P_{s+1|s+1} \leqslant \Phi_{s+1|s+1}. \tag{28}$$

Moreover, the filter gain L_{s+1} in (25) ensures that the trace of the $\Phi_{s+1|s+1}$ is minimized. For the proof, please see Appendix B.

Up to now, the filter gain has been obtained. Next, we aim to analyze the MSEB of the FE.

Assumption 1. There exist positive numbers $\underline{a}$, $\overline{a}$, $\underline{b}$, $\overline{b}$, $\underline{c}$, $\overline{c}$, $\underline{d}$, $\overline{d}$, $\underline{f}$, $\overline{f}$, $\underline{q}$, $\overline{q}$, $\underline{z}$, and $\overline{z}$ satisfying

$$\begin{aligned}
&\underline{a} \leqslant \|A_{i,s}\| \leqslant \overline{a}, \quad \underline{b} \leqslant \|B_s\| \leqslant \overline{b}, \quad \underline{c} \leqslant \|C_s\| \leqslant \overline{c},\\
&\underline{d} \leqslant \|D_s\| \leqslant \overline{d}, \quad \underline{f} \leqslant \|F_s\| \leqslant \overline{f}, \quad \underline{q} \leqslant \|Q_s\| \leqslant \overline{q},\\
&\underline{z} \leqslant \|Z_s\| \leqslant \overline{z}
\end{aligned}$$

for any i and s.

Lemma 6 ([37]). Assume that $\mathcal{V}_s(\eta_s)$ represents a stochastic process and positive numbers $\underline{\tau}$, $\overline{\tau}$, ρ, and ς satisfy

$$\underline{\tau}\,\|\eta_s\|^2 \leqslant \mathcal{V}_s(\eta_s) \leqslant \overline{\tau}\,\|\eta_s\|^2, \tag{29}$$

and

$$\mathrm{E}\left\{\mathcal{V}_s(\eta_s)|\eta_{s-1}\right\} \leqslant (1-\varsigma)\,\mathcal{V}_{s-1}(\eta_{s-1}) + \rho, \tag{30}$$

where $\varsigma \in (0,1)$, and then the MSEB of η_s is proved, i.e.,

$$\mathrm{E}\{\|\eta_s\|^2\} \leqslant \frac{\overline{\tau}}{\underline{\tau}}\mathrm{E}\{\|\eta_0\|^2\}\,(1-\varsigma)^s + \frac{\rho}{\underline{\tau}}\sum_{i=1}^{s}(1-\varsigma)^i. \tag{31}$$

Lemma 7 ([38]). Given constant matrices Γ_1, Γ_2, Γ_3 where $0 < \Gamma_1 = \Gamma_1^{\mathrm{T}}$ and $0 < \Gamma_2 = \Gamma_2^{\mathrm{T}}$, then $\Gamma_1 - \Gamma_3\Gamma_2\Gamma_3^{\mathrm{T}} \geqslant 0$ if and only if

$$\begin{bmatrix}\Gamma_1 & \Gamma_3^{\mathrm{T}}\\ \Gamma_3 & \Gamma_2^{-1}\end{bmatrix} \geqslant 0 \quad \text{or} \quad \begin{bmatrix}\Gamma_2^{-1} & \Gamma_3\\ \Gamma_3^{\mathrm{T}} & \Gamma_1\end{bmatrix} \geqslant 0 \quad \text{or} \quad \Gamma_2^{-1} - \Gamma_3^{\mathrm{T}}\Gamma_1^{-1}\Gamma_3 \geqslant 0. \tag{32}$$

Lemma 8. The two RLDEs $\Phi_{s+1|s}$ and $\Phi_{s+1|s+1}$ are bounded, i.e.,

$$\Phi_{s+1|s} \leqslant \left(\ell + \gamma^{-1}\overline{d}^2 + \overline{b}^2\overline{q}\right)I, \tag{33}$$

$$\Phi_{s+1|s+1} \leqslant (1+\delta_2)\Phi_{s+1|s}, \tag{34}$$

where $\ell \triangleq \mathrm{Tr}\{F_s(\Phi_{s|s}^{-1} - \gamma G_s^{\mathrm{T}} G_s)^{-1} F_s^{\mathrm{T}} + \sum_{i=1}^{j} A_{i,s}[(1+\delta_1)\Phi_{s|s} + (1+\delta_1^{-1}) x_{s|s} x_{s|s}^{\mathrm{T}}] A_{i,s}^{\mathrm{T}}\}$.

Proof. Based on (22) and Assumption 1, one easily obtains (33). Substituting (25) into (23) yields

$$\Phi_{s+1|s+1} \leqslant (1+\delta_2)\Phi_{s+1|s} - (1+\delta_2)^2\Phi_{s+1|s}C_{s+1}^{\mathrm{T}}\Xi_{s+1}^{\mathrm{T}}\Theta_{s+1}^{-1}\Xi_{s+1}C_{s+1}\Phi_{s+1|s}^{\mathrm{T}} \leqslant (1+\delta_2)\Phi_{s+1|s}. \tag{35}$$

The proof is now complete.

Theorem 4. Consider the NSs with MNs described by (1) and (2). Based on the conditions of Theorem 3 and Assumption 1, if the following condition:

$$(1+\delta_1)(\delta_2 - 1) > 2j^2, \tag{36}$$

where δ_1 and δ_2 are positive scalars, holds, then the MSEB of FE is guaranteed in the presence of MOs. Please refer to Appendix C for the proof.

Remark 4. Until now, this paper has addressed the RFA in the consideration of nonlinearities, MNs, and MOs. The distinctive features are exhibited as follows: (1) the designed adaptive saturation function is critical for resolving the problems of the difficulty of outlier model description and the uncertainty of outlier occurrence time; (2) the analysis process is much complicated due to the introduction of the nonlinearities, the MNs and the MOs; and (3) the saturation matrix Ξ_s can be obtained accurately in the presented algorithm, which improves the accuracy of the estimation and decreases the conservatism of the approach.

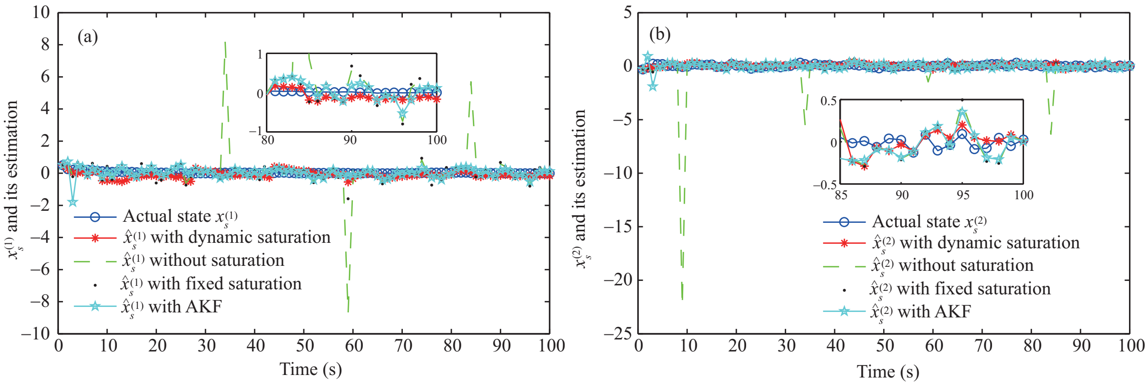

Figure 1 (Color online) Actual states x_s and their estimates. (a) Actual state $x_s^{(1)}$ and its estimate $\hat{x}_s^{(1)}$; (b) actual state $x_s^{(2)}$ and its estimate $\hat{x}_s^{(2)}$.

4 Simulation example

In this section, a simulation example is provided to illustrate that the proposed adaptive saturation recursive filter can effectively suppress possible measurement outliers. The parameters in system (1) are selected as

$$f(x_s) = \begin{bmatrix} 0.8x_s^{(1)} + \sin(x_s^{(1)}x_s^{(2)}) \\ 0.5x_s^{(2)} + \sin(x_s^{(1)}x_s^{(2)}) \end{bmatrix}, \quad A_{1,s} = \begin{bmatrix} 0.2+\sin(2s) & 0 \\ 0 & 0.15 \end{bmatrix},$$

$$B_s = \begin{bmatrix} 0.02 \\ 0.03 + 0.2\mathrm{e}^{-6s} \end{bmatrix}, \quad C_s = \begin{bmatrix} 0.9 & 0 \\ 0 & -1.2 + 0.01\sin(s+3) \end{bmatrix}.$$

In addition, the initial value of the state and the initial dynamic saturation level are $x_{0|0} = [0.5 \ -0.3]^{\mathrm{T}}$ and $\overline{\sigma}_0 = 0.5$, respectively. The fixed saturation level is 0.5. The other parameters are chosen as $\Phi_{0|0} = 35I \in \mathbb{R}^{2\times 2}$, $\lambda = 0.45$, $\pi_i = \mathrm{diag}\{160, 7.5\}$, $R = \mathrm{diag}\{0.01, 0.3\}$, $l = 1$, $\delta_1 = 1.5$, $\delta_2 = 2$, $\gamma = 0.02$, $D_s = \mathrm{diag}\{0.1, 0.1\}$, and $G_s = \mathrm{diag}\{0.1, 0.2\}$.

In order to describe the possible MOs, we assume that the period of occurrence is $T = 25$, and the time of the first outlier is a random number between 0 and 10. The amplitude of the outliers is zero-mean white Gaussian with covariance 400. Figure 1 depicts the actual states $x_s^{(i)} (i = 1, 2)$ and the estimated values of states with adaptive saturation and with traditional methods including the traditional RFA, the fixed saturation method, and the adaptive Kalman filter (AKF). As shown in Figure 1, we can see that when the outliers occur, the estimation of traditional methods has a great deviation, which makes a great impact on the filtering performance. Nevertheless, owing to the introduction of adaptive saturation, the filtering algorithm has reduced the influence from outliers and improved the accuracy of estimation to a great extent. Dynamic saturation levels are displayed in Figure 2, which shows that the changing trend of saturation levels is consistent with that of the output estimation error. The curves of lg(FECM), lg(MSEB) of FE and their upper bounds are drawn in Figures 3(a) and (b), respectively, which verify the effectiveness of the scheme proposed in this paper. To show the superiority of the proposed filtering algorithm, the filtering effects under dynamic saturation method and ordinary methods are compared by the root mean square error (RMSE) of each state variable, which are displayed in Figure 4. It can be seen from Figure 4 that, in the case of outliers occurring, the RMSE considering the adaptive saturation is smaller than that of other methods. Also, the adverse effect of the MOs receives a good suppression due to the introduction of the adaptive saturation function.

5 Conclusion

In this paper, we studied the RFA for a class of NSs with MNs and MOs. An innovative self-adaptive saturation function was introduced for the designed RFA to resist the effects on the filtering performance from MOs. Using some well-known inequality techniques, a UB of the FECM was derived by solving two RLDEs, and the minimization of the UB was guaranteed by the designed filter gain. Moreover, the

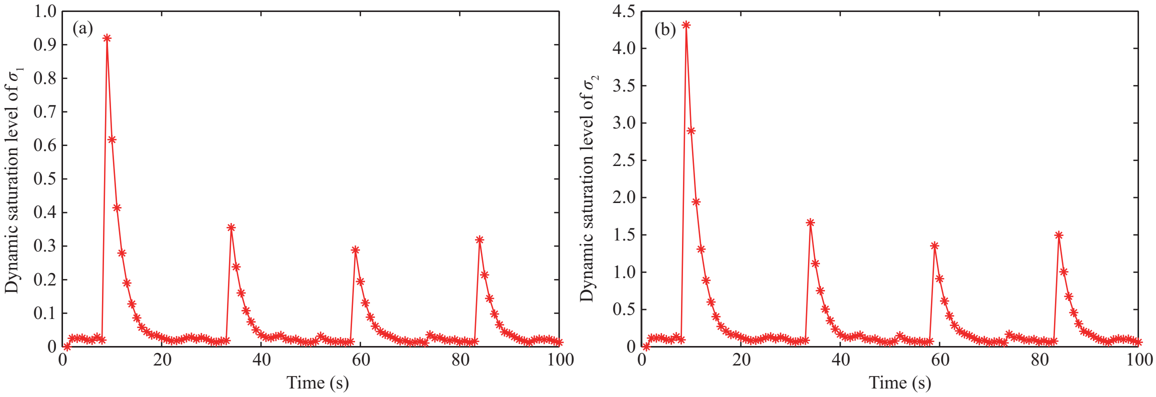

Figure 2 (Color online) Dynamic saturation levels of (a) σ_1 and (b) σ_2.

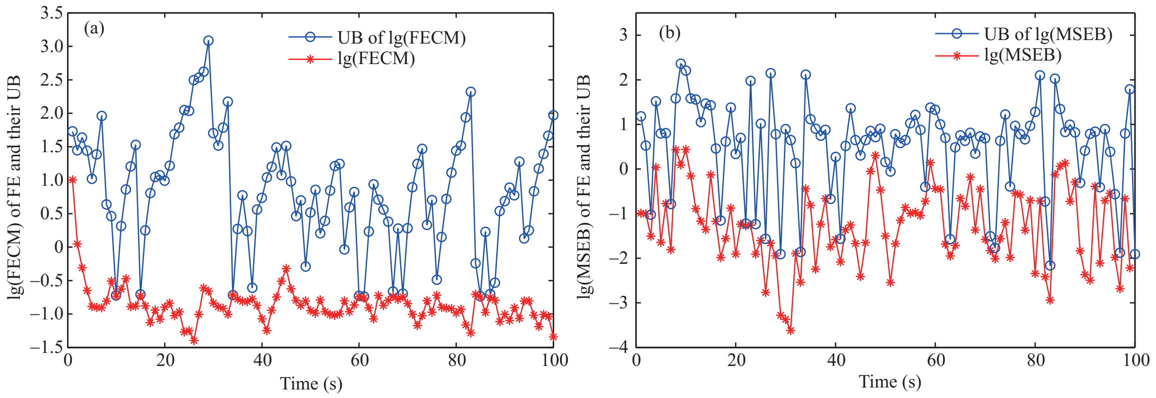

Figure 3 (Color online) (a) lg(FECM) and (b) lg(MSEB) of FE and their UB.

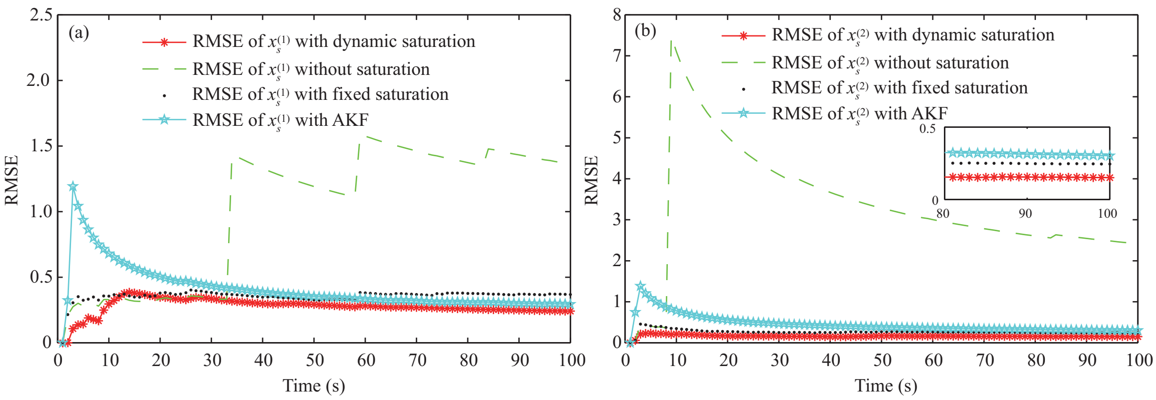

Figure 4 (Color online) RMSEs of (a) $x_s^{(1)}$ and (b) $x_s^{(2)}$.

MSEB of the FE was analyzed. Finally, the effectiveness of the RFA, which can significantly reduce the impacts from the outliers on system performance, was investigated by a simulation example. The future research direction includes extending the proposed algorithm to other more complex systems and other filtering methods, such as the unscented Kalman filtering and the cubature Kalman filtering, in which the outliers' characteristics would be captured more accurately and actively.

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Appendix A Proof of Theorem 1

By means of (1) and (5), the OSPE is expressed as

$$e_{s+1|s} = x_{s+1} - x_{s+1|s} = f(x_s) + \sum_{i=1}^{j} \alpha_{i,s} A_{i,s} x_s - f(x_{s|s}) + B_s w_s. \tag{A1}$$

Expanding $f(x_s)$ by Taylor formula around $x_{s|s}$, one has

$$f(x_s) = f(x_{s|s}) + F_s e_{s|s} + o(|e_{s|s}|), \tag{A2}$$

where F_s and $o(|e_{s|s}|)$ denote the Jacobian matrix and the high-order terms of Taylor series expression, respectively. Following the literature[1)], the $o(|e_{s|s}|)$ is equivalent to the following equation:

$$o(|e_{s|s}|) = D_s \aleph_s G_s e_{s|s}, \tag{A3}$$

where D_s is a scaling matrix, G_s is a degree of freedom function, and $\aleph_s$ represents the linearization errors. Consequently, one has (17).

Next, taking (2), (6) and (11) into consideration, the FE $e_{s+1|s+1}$ is calculated as follows:

$$\begin{aligned} e_{s+1|s+1} &= x_{s+1} - x_{s+1|s+1} \\ &= x_{s+1} - x_{s+1|s} - L_{s+1}\mathrm{sat}_{\sigma_{s+1}}\left(y_{s+1} - C_{s+1}x_{s+1|s}\right) \\ &= x_{s+1} - x_{s+1|s} - L_{s+1}\Xi_{s+1}(C_{s+1}x_{s+1} + v_{s+1} - C_{s+1}x_{s+1|s}) + L_{s+1}(\Xi_{s+1} - I)\sigma_{s+1} \\ &= (I - L_{s+1}\Xi_{s+1}C_{s+1})\, e_{s+1|s} + L_{s+1}(\Xi_{s+1} - I)\sigma_{s+1} - L_{s+1}\Xi_{s+1}v_{s+1}. \end{aligned} \tag{A4}$$

Finally, the proof is complete.

Appendix B Proof of Theorem 3

First of all, we need to solve the unknown terms of (19). According to Lemma 3, one obtains

$$(F_s + D_s\aleph_s G_s)P_{s|s}(F_s + D_s\aleph_s G_s)^{\mathrm{T}} \leqslant F_s(P_{s|s}^{-1} - \gamma G_s^{\mathrm{T}} G_s)^{-1} F_s^{\mathrm{T}} + \gamma^{-1} D_s D_s^{\mathrm{T}}. \tag{B1}$$

Applying (12) to the other uncertain term of the (19) yields

$$\begin{aligned} \sum_{i=1}^{j} A_{i,s}\mathrm{E}\{x_s x_s^{\mathrm{T}}\}A_{i,s}^{\mathrm{T}} &\leqslant \sum_{i=1}^{j} A_{i,s}\mathrm{E}\{(1+\delta_1)e_{s|s}e_{s|s}^{\mathrm{T}} + (1+\delta_1^{-1})x_{s|s}x_{s|s}^{\mathrm{T}}\}A_{i,s}^{\mathrm{T}} \\ &= \sum_{i=1}^{j} A_{i,s}[(1+\delta_1)P_{s|s} + (1+\delta_1^{-1})x_{s|s}x_{s|s}^{\mathrm{T}}]A_{i,s}^{\mathrm{T}}, \end{aligned} \tag{B2}$$

which leads to

$$\begin{aligned} P_{s+1|s} \leqslant\, & F_s(P_{s|s}^{-1} - \gamma G_s^{\mathrm{T}} G_s)^{-1}F_s^{\mathrm{T}} + \gamma^{-1}D_s D_s^{\mathrm{T}} + B_s Q_s B_s^{\mathrm{T}} \\ & + \sum_{i=1}^{j} A_{i,s}[(1+\delta_1)P_{s|s} + (1+\delta_1^{-1})x_{s|s}x_{s|s}^{\mathrm{T}}]A_{i,s}^{\mathrm{T}}. \end{aligned} \tag{B3}$$

Next, we shall handle the uncertainty on the right side of (20) in a similar way. Again, it follows from (12) that

$$\begin{aligned} \mathrm{E}\{\Upsilon_{s+1} + \Upsilon_{s+1}^{\mathrm{T}}\} \leqslant\, & \delta_2(I - L_{s+1}\Xi_{s+1}C_{s+1})P_{s+1|s}(I - L_{s+1}\Xi_{s+1}C_{s+1})^{\mathrm{T}} \\ & + \delta_2^{-1}L_{s+1}(\Xi_{s+1} - I)\mathrm{E}\{\sigma_{s+1}\sigma_{s+1}^{\mathrm{T}}\}(\Xi_{s+1} - I)^{\mathrm{T}}L_{s+1}^{\mathrm{T}}. \end{aligned} \tag{B4}$$

According to (8), (9), and (16), one has

$$\begin{aligned} \mathrm{E}\{\sigma_{s+1}\sigma_{s+1}^{\mathrm{T}}\} &\leqslant \mathrm{E}\left\{\sum_{i=1}^{m}\sigma_{i,s+1}^2 I\right\} \\ &= \mathrm{E}\left\{\bar{\sigma}_{s+1}\sum_{i=1}^{m}\frac{1}{\pi_i}I\right\} \\ &= \mathrm{E}\left\{[\lambda\bar{\sigma}_s + (C_{s+1}e_{s+1|s} + v_{s+1})^{\mathrm{T}}R(C_{s+1}e_{s+1|s} + v_{s+1})]\sum_{i=1}^{m}\frac{1}{\pi_i}I\right\} \\ &= \mathrm{E}\left\{[\lambda\bar{\sigma}_s + \mathrm{Tr}\{R(C_{s+1}P_{s+1|s}C_{s+1}^{\mathrm{T}} + Z_{s+1})\}]\sum_{i=1}^{m}\frac{1}{\pi_i}I\right\} \\ &\leqslant \left[\lambda^{s+1}\overline{\sigma}_0 + \sum_{i=0}^{s}\lambda^{s-i}\mathrm{Tr}\{R(C_{i+1}\Phi_{i+1|i}C_{i+1}^{\mathrm{T}} + Z_{i+1})\}\right]\sum_{i=1}^{m}\frac{1}{\pi_i}I. \end{aligned} \tag{B5}$$

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Substituting (B4) and (B5) into (20) yields

$$\begin{aligned} P_{s+1|s+1} \leqslant & (1+\delta_2)\left(I - L_{s+1}\Xi_{s+1}C_{s+1}\right) P_{s+1|s} \left(I - L_{s+1}\Xi_{s+1}C_{s+1}\right)^{\mathrm{T}} \\ & + (1+\delta_2^{-1})\Lambda_{s+1} L_{s+1}(\Xi_{s+1} - I)(\Xi_{s+1} - I)^{\mathrm{T}} L_{s+1}^{\mathrm{T}} + L_{s+1}\Xi_{s+1}Z_{s+1}\Xi_{s+1}^{\mathrm{T}} L_{s+1}^{\mathrm{T}}. \end{aligned} \tag{B6}$$

Then, by utilizing Lemma 4, it follows from (23) and (B6) that

$$P_{s+1|s+1} \leqslant \Phi_{s+1|s+1}. \tag{B7}$$

Finally, making the partial derivative of the trace of (23) about L_{s+1} equal to zero, one has

$$\begin{aligned} \frac{\partial \mathrm{Tr}\left\{\Phi_{s+1|s+1}\right\}}{\partial L_{s+1}} = & -2(1+\delta_2)\left(I - L_{s+1}\Xi_{s+1}C_{s+1}\right)\Phi_{s+1|s}C_{s+1}^{\mathrm{T}}\Xi_{s+1}^{\mathrm{T}} \\ & + 2(1+\delta_2^{-1})\Lambda_{s+1}L_{s+1}(\Xi_{s+1} - I)(\Xi_{s+1} - I)^{\mathrm{T}} + 2L_{s+1}\Xi_{s+1}Z_{s+1}\Xi_{s+1}^{\mathrm{T}} \\ = & \, 0. \end{aligned} \tag{B8}$$

In view of (B8), the optimal L_{s+1} is easily calculated in (25). The proof is now complete.

Appendix C Proof of Theorem 4

Define the quadratic function like the following:

$$\mathcal{V}_s(e_{s|s}) = e_{s|s}^{\mathrm{T}}\Phi_{s|s}^{-1}e_{s|s}. \tag{C1}$$

Combining (17), (A4) and $x_s = x_{s|s} + e_{s|s}$, one has

$$\begin{aligned} e_{s+1|s+1} = & \mathfrak{H}_{s+1}\mathfrak{M}_s e_{s|s} + \sum_{i=1}^{j}\alpha_{i,s}\mathfrak{H}_{s+1}A_{i,s}(x_{s|s} + e_{s|s}) + \mathfrak{H}_{s+1}B_s w_s \\ & + L_{s+1}(\Xi_{s+1} - I)\sigma_{s+1} - L_{s+1}\Xi_{s+1}v_{s+1}, \end{aligned} \tag{C2}$$

where $\mathfrak{H}_{s+1} \triangleq I - L_{s+1}\Xi_{s+1}C_{s+1}$ and $\mathfrak{M}_s \triangleq F_s + D_s\aleph_s G_s$.

Next, calculating $\mathcal{V}_{s+1}(e_{s+1|s+1})$ and taking the mathematical expectation, one has

$$\begin{aligned} & \mathrm{E}\left\{\mathcal{V}_{s+1}(e_{s+1|s+1})|e_{s|s}\right\} \\ & = \mathrm{E}\Bigg\{\sigma_{s+1}^{\mathrm{T}}\left(\Xi_{s+1} - I\right)^{\mathrm{T}} L_{s+1}^{\mathrm{T}}\Phi_{s+1|s+1}^{-1}\mathfrak{H}_{s+1}\mathfrak{M}_s e_{s|s} + \sigma_{s+1}^{\mathrm{T}}\left(\Xi_{s+1} - I\right)^{\mathrm{T}} L_{s+1}^{\mathrm{T}}\Phi_{s+1|s+1}^{-1}L_{s+1}\left(\Xi_{s+1} - I\right)\sigma_{s+1} \\ & \quad + \sum_{i=1}^{j}\sum_{l=1}^{j}\left(x_{s|s} + e_{s|s}\right)^{\mathrm{T}} A_{i,s}^{\mathrm{T}}\mathfrak{H}_{s+1}^{\mathrm{T}}\Phi_{s+1|s+1}^{-1}\mathfrak{H}_{s+1}A_{l,s}\left(x_{s|s} + e_{s|s}\right) + w_s^{\mathrm{T}}B_s^{\mathrm{T}}\mathfrak{H}_{s+1}^{\mathrm{T}}\Phi_{s+1|s+1}^{-1}\mathfrak{H}_{s+1}B_s w_s \\ & \quad + e_{s|s}^{\mathrm{T}}\mathfrak{M}_s^{\mathrm{T}}\mathfrak{H}_{s+1}^{\mathrm{T}}\Phi_{s+1|s+1}^{-1}\mathfrak{H}_{s+1}\mathfrak{M}_s e_{s|s} + e_{s|s}^{\mathrm{T}}\mathfrak{M}_s^{\mathrm{T}}\mathfrak{H}_{s+1}^{\mathrm{T}}\Phi_{s+1|s+1}^{-1}L_{s+1}\left(\Xi_{s+1} - I\right)\sigma_{s+1} \\ & \quad + v_{s+1}^{\mathrm{T}}\Xi_{s+1}^{\mathrm{T}}L_{s+1}^{\mathrm{T}}\Phi_{s+1|s+1}^{-1}L_{s+1}\Xi_{s+1}v_{s+1}\Bigg\}. \end{aligned} \tag{C3}$$

To deal with the cross terms in (C3), one has

$$\begin{aligned} & \mathrm{E}\big\{e_{s|s}^{\mathrm{T}}\mathfrak{M}_s^{\mathrm{T}}\mathfrak{H}_{s+1}^{\mathrm{T}}\Phi_{s+1|s+1}^{-1}L_{s+1}\left(\Xi_{s+1} - I\right)\sigma_{s+1} + \sigma_{s+1}^{\mathrm{T}}\left(\Xi_{s+1} - I\right)^{\mathrm{T}} L_{s+1}^{\mathrm{T}}\Phi_{s+1|s+1}^{-1}\mathfrak{H}_{s+1}\mathfrak{M}_s e_{s|s}\big\} \\ & \leqslant \mathrm{E}\big\{e_{s|s}^{\mathrm{T}}\mathfrak{M}_s^{\mathrm{T}}\mathfrak{H}_{s+1}^{\mathrm{T}}\Phi_{s+1|s+1}^{-1}\mathfrak{H}_{s+1}\mathfrak{M}_s e_{s|s} + \sigma_{s+1}^{\mathrm{T}}\left(\Xi_{s+1} - I\right)^{\mathrm{T}} L_{s+1}^{\mathrm{T}}\Phi_{s+1|s+1}^{-1}L_{s+1}\left(\Xi_{s+1} - I\right)\sigma_{s+1}\big\}. \end{aligned} \tag{C4}$$

Similarly, the third item in (C3) satisfies

$$\begin{aligned} & \mathrm{E}\left\{\sum_{i=1}^{j}\sum_{l=1}^{j}\left(x_{s|s} + e_{s|s}\right)^{\mathrm{T}} A_{i,s}^{\mathrm{T}}\mathfrak{H}_{s+1}^{\mathrm{T}}\Phi_{s+1|s+1}^{-1}\mathfrak{H}_{s+1}A_{l,s}\left(x_{s|s} + e_{s|s}\right)\right\} \\ & \leqslant \mathrm{E}\left\{2\sum_{i=1}^{j}\sum_{l=1}^{j}e_{s|s}^{\mathrm{T}}A_{i,s}^{\mathrm{T}}\mathfrak{H}_{s+1}^{\mathrm{T}}\Phi_{s+1|s+1}^{-1}\mathfrak{H}_{s+1}A_{l,s}e_{s|s} + 2\sum_{i=1}^{j}\sum_{l=1}^{j}x_{s|s}^{\mathrm{T}}A_{i,s}^{\mathrm{T}}\mathfrak{H}_{s+1}^{\mathrm{T}}\Phi_{s+1|s+1}^{-1}\mathfrak{H}_{s+1}A_{l,s}x_{s|s}\right\} \\ & \leqslant \mathrm{E}\left\{2j\sum_{i=1}^{j}e_{s|s}^{\mathrm{T}}A_{i,s}^{\mathrm{T}}\mathfrak{H}_{s+1}^{\mathrm{T}}\Phi_{s+1|s+1}^{-1}\mathfrak{H}_{s+1}A_{i,s}e_{s|s} + 2j\sum_{i=1}^{j}x_{s|s}^{\mathrm{T}}A_{i,s}^{\mathrm{T}}\mathfrak{H}_{s+1}^{\mathrm{T}}\Phi_{s+1|s+1}^{-1}\mathfrak{H}_{s+1}A_{i,s}x_{s|s}\right\}. \end{aligned} \tag{C5}$$

Applying Lemma 3 to (22) and (23), one obtains

$$\Phi_{s+1|s} \geqslant \mathfrak{M}_s\Phi_{s|s}\mathfrak{M}_s^{\mathrm{T}}, \tag{C6}$$

$$\Phi_{s+1|s} \geqslant (1+\delta_1)\sum_{i=1}^{j}A_{i,s}\Phi_{s|s}A_{i,s}^{\mathrm{T}}, \tag{C7}$$

$$\Phi_{s+1|s+1} \geqslant (1+\delta_2)\mathfrak{H}_{s+1}\Phi_{s+1|s}\mathfrak{H}_{s+1}^{\mathrm{T}}, \tag{C8}$$

which follows from Lemma 7 that

$$\mathfrak{M}_s^{\mathrm{T}} \Phi_{s+1|s}^{-1} \mathfrak{M}_s \leqslant \Phi_{s|s}^{-1}, \tag{C9}$$

$$\sum_{i=1}^{j} A_{i,s}^{\mathrm{T}} \Phi_{s+1|s}^{-1} A_{i,s} \leqslant \frac{j}{1+\delta_1} \Phi_{s|s}^{-1}, \tag{C10}$$

$$\mathfrak{H}_{s+1}^{\mathrm{T}} \Phi_{s+1|s+1}^{-1} \mathfrak{H}_{s+1} \leqslant \frac{1}{1+\delta_2} \Phi_{s+1|s}^{-1}. \tag{C11}$$

Based on Lemma 5, one has

$$\begin{aligned} \mathrm{E}\{v_{s+1}^{\mathrm{T}} L_{s+1}^{\mathrm{T}} \Phi_{s+1|s+1}^{-1} L_{s+1} v_{s+1}\} &= \mathrm{Tr}\big\{\mathrm{E}\{L_{s+1}^{\mathrm{T}} \Phi_{s+1|s+1}^{-1} L_{s+1} v_{s+1} v_{s+1}^{\mathrm{T}}\}\big\} \\ &= \mathrm{Tr}\big\{\mathrm{E}\{L_{s+1}^{\mathrm{T}} \Phi_{s+1|s+1}^{-1} L_{s+1} Z_{s+1}\}\big\} \\ &= \mathrm{Tr}\big\{L_{s+1}^{\mathrm{T}} \Phi_{s+1|s+1}^{-1} L_{s+1} Z_{s+1}\big\}. \end{aligned} \tag{C12}$$

Similarly, one obtains

$$\mathrm{E}\{w_s^{\mathrm{T}} B_s^{\mathrm{T}} \Phi_{s+1|s}^{-1} B_s w_s\} = \mathrm{Tr}\big\{B_s^{\mathrm{T}} \Phi_{s+1|s}^{-1} B_s Q_s\big\}, \tag{C13}$$

and

$$\begin{aligned} \mathrm{E}\{\sigma_{s+1}^{\mathrm{T}} L_{s+1}^{\mathrm{T}} \Phi_{s+1|s+1}^{-1} L_{s+1} \sigma_{s+1}\} &= \mathrm{Tr}\{L_{s+1}^{\mathrm{T}} \Phi_{s+1|s+1}^{-1} L_{s+1} \mathrm{E}\{\sigma_{s+1} \sigma_{s+1}^{\mathrm{T}}\}\} \\ &\leqslant \mathrm{Tr}\{L_{s+1}^{\mathrm{T}} \Phi_{s+1|s+1}^{-1} L_{s+1} \Lambda_{s+1}\}. \end{aligned} \tag{C14}$$

Substituting (C4)–(C14) into (C3) yields

$$\begin{aligned} &\mathrm{E}\left\{\mathcal{V}_{s+1}(e_{s+1|s+1}) | e_{s|s}\right\} \\ &\leqslant \frac{2}{1+\delta_2} e_{s|s}^{\mathrm{T}} \mathfrak{M}_s^{\mathrm{T}} \Phi_{s+1|s}^{-1} \mathfrak{M}_s e_{s|s} + \mathrm{E}\Bigg\{ \frac{1}{1+\delta_2} w_s^{\mathrm{T}} B_s^{\mathrm{T}} \Phi_{s+1|s}^{-1} B_s w_s + v_{s+1}^{\mathrm{T}} L_{s+1}^{\mathrm{T}} \Phi_{s+1|s+1}^{-1} L_{s+1} v_{s+1} \\ &\quad + \frac{2j}{1+\delta_2} \sum_{i=1}^{j} e_{s|s}^{\mathrm{T}} A_{i,s}^{\mathrm{T}} \Phi_{s+1|s}^{-1} A_{i,s} e_{s|s} + \frac{2j}{1+\delta_2} \sum_{i=1}^{j} x_{s|s}^{\mathrm{T}} A_{i,s}^{\mathrm{T}} \Phi_{s+1|s}^{-1} A_{i,s} x_{s|s} + 2\sigma_{s+1}^{\mathrm{T}} L_{s+1}^{\mathrm{T}} \Phi_{s+1|s+1}^{-1} L_{s+1} \sigma_{s+1} \Bigg\} \\ &\leqslant \varrho e_{s|s}^{\mathrm{T}} \Phi_{s|s}^{-1} e_{s|s} + \kappa, \end{aligned} \tag{C15}$$

where

$$\varrho \triangleq \frac{2j^2 + 2(1+\delta_1)}{(1+\delta_2)(1+\delta_1)},$$

and

$$\begin{aligned} \kappa \triangleq &\frac{1}{1+\delta_2} \mathrm{Tr}\big\{B_s^{\mathrm{T}} \Phi_{s+1|s}^{-1} B_s Q_s\big\} + \mathrm{Tr}\big\{L_{s+1}^{\mathrm{T}} \Phi_{s+1|s+1}^{-1} L_{s+1} Z_{s+1}\big\} \\ &+ \frac{2j}{1+\delta_2} \sum_{i=1}^{j} x_{s|s}^{\mathrm{T}} A_{i,s}^{\mathrm{T}} \Phi_{s+1|s}^{-1} A_{i,s} x_{s|s} + 2\mathrm{Tr}\{L_{s+1}^{\mathrm{T}} \Phi_{s+1|s+1}^{-1} L_{s+1} \Lambda_{s+1}\}. \end{aligned}$$

Furthermore, by means of the properties of the conditional expectations, one has

$$\mathrm{E}\left\{\mathcal{V}_{s+1}(e_{s+1|s+1})\right\} \leqslant \varrho \mathcal{V}_s(e_{s|s}) + \kappa. \tag{C16}$$

As can be seen from (8), (9), (C9), and (C11) that $\Phi_{s+1|s}^{-1}$, $\Phi_{s+1|s+1}^{-1}$, L_{s+1}, and σ_{s+1} are all bounded. At this point, according to Lemma 6 and condition (36), the MSEB of FE is confirmed which completes the proof.