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Stochastic process-based degradation modeling and RUL prediction: from Brownian motion to fractional Brownian motion

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Abstract Brownian motion (BM) has been widely used for degradation modeling and remaining useful life (RUL) prediction, but it is essentially Markovian. This implies that the future state in a BM-based degradation process relies only on its current state, independent of the past states. However, some practical industrial devices such as Li-ion batteries, ball bearings, turbofans, and blast furnace walls show degradations with long-range dependence (LRD), where the future degradation states depend on both the current and past degradation states. This type of degradation naturally brings two interesting problems, that is, how to model the degradations and how to predict their RULs. Recently, in contrast to the work that uses only BM, fractional Brownian motion (FBM) is introduced to model practical degradations. The most important feature of the FBM-based degradation models is the ability to characterize the non-Markovian degradations with LRD. Although FBM is an extension of BM, it is neither a Markovian process nor a semimartingale. Therefore, how to obtain the first passage time of an FBM-based degradation process has become a challenging task. In this paper, a review of the transition of RUL prediction from BM to FBM is provided. The peculiarities of FBM when addressing the LRD inherent in some practical degradations are discussed. We first review the BM-based degradation models of the past few decades and then give details regarding the evolution of FBM-based research. Interestingly, the existing BM-based models scarcely consider the effect of LRD on the prediction of RULs. Two practical cases illustrate that the newly developed FBM-based models are more generalized and suitable for predicting RULs than the BM-based models, especially for degradations with LRD. Along with the direction of FBM-based RUL prediction, we also introduce some important and interesting problems that require further study.

Keywords remaining useful life, degradation model, Brownian motion, fractional Brownian motion, long-range dependence

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1 Introduction

Over the past few decades, reliable and accurate remaining useful life (RUL) prediction for key engineering assets has attracted increasing attention in the fields of reliability and operational research [1–10]. Its key idea can be defined as calculating the probability density function (PDF) of the first passage time (FPT), namely, the time to reach a preset failure threshold for the first time. Accurate RUL prediction can provide effective information for scheduling maintenance strategies to avoid failures, thus ensuring the safety of a device. To achieve the exact prediction of the RUL distribution, an appropriate degradation model that fully characterizes the actual degradation process is usually required and can be used for

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subsequent forecasting and decision-making [5]. As a consequence, a number of degradation models have been developed to capture the degradation dynamics of a system [3–10].

Among these models, Brownian motion (BM) with a drift (namely, the Wiener process) has been extensively adopted to describe the degradation processes, since it possesses favorable mathematical characteristics and reasonable physical interpretations. It is appropriate for degradation processes that vary bidirectionally over time with a Gaussian noise [3]. The model usually includes a drift term and a standard BM with a diffusion coefficient and is also a Markov process with independent increments [11–13]. By virtue of the favorable properties of BM, the analytical distribution of the FPT can be obtained when the drift term is linear. Even when the degradation process has a nonlinear drift term, an approximate analytical FPT can be obtained [12]. In past decades, BM has been applied to many practical cases, such as light emitting diode (LED) lamps [11, 14, 15], rolling element bearings [16–21], Li-ion batteries [22–25], and fatigue crack dynamics [26–28].

However, one common assumption existing in the BM-based degradation models is that the degradation is an independent increment process, i.e., the increments of the degradation process are independent when the corresponding time intervals do not overlap [29, 30]. This independence leads to an analytical form of the RUL in the sense of the FPT. Unfortunately, it is in sharp contrast to some practical systems such as Li-ion batteries, bearings, turbofan engines, and blast furnace walls, in which the degradation processes have the property of long-range dependence (LRD) [29–32]. Owing to the external environments, operating conditions, and even the degradation mechanism itself, distant degradation states in these practical systems may be correlated. This kind of dependence can typically be characterized by a slow decaying autocorrelation function [33, 34] with the Hurst exponent as a key indicator [35]. If the Hurst exponent belongs to $(0.5, 1)$, then LRD exists.

It should be noted that degradation processes with LRD do not satisfy the prerequisite of independent increments. This implies that the existing BM-based models may not be suitable for characterizing this kind of dependence. Therefore, some interesting problems naturally arise in the field of RUL prediction. (1) Are there new stochastic process-based models suitable for degradations with LRD? (2) What is the main difference between the new and existing models? (3) Is it possible to obtain the exact analytical PDF of the RUL in the presence of LRD? Recently, a type of fractional Brownian motion (FBM)-based degradation model for characterizing the LRD inherent in some practical degradation processes was proposed [29, 30]. Although FBM is an extension of BM, it is essentially neither a Markovian process nor a semimartingale, with a Hurst exponent belonging to $(0, 0.5) \cup (0.5, 1)$. Hence, the assumption of independent increments in the existing BM-based models can be eliminated.

Although the BM-based degradation models have been reviewed in the prediction of RULs [3, 4, 6, 9, 10, 36], the burn-in testing for highly reliable systems [5, 9], the condition-based maintenance optimization [8, 9], and the health management of batteries [7], the LRD, which exists in many actual degradations, is totally ignored. Therefore, our main purpose is to introduce a new research direction for RUL prediction that considers the effect of LRD on the degradations. We first give a survey on the existing BM-based prediction of RULs, especially in recent decades. Then, a detailed evolution of the FBM-based research and two studies comparing BM- and FBM-based models are provided. Based on two practical cases, the newly developed FBM is shown to be more generalized and suitable than BM for predicting the RUL. We also discuss some important and interesting problems to be further studied, along with future directions for FBM-based prediction. Although some preliminary results on the prediction of RULs for FBM-based degradation models have already been obtained [29–32, 37], there still exist many interesting problems in the field of RUL prediction that require further consideration.

The rest of this paper is organized as follows. In Section 2, a survey on the BM-based degradation models used for the RUL prediction in past decades is given. In Section 3, the FBM-based degradation models, which can characterize the effect of the LRD on RULs, are introduced. Based on two practical cases of a blast furnace wall and a turbofan, studies comparing the BM- and FBM-based models are presented in Section 4. Some important and interesting problems inherent in the FBM-based degradation models are introduced in Section 5. The conclusion is given in Section 6.

2 BM-based degradation models

In the literature, one hopes to obtain suitable degradation models in accordance with practical cases. In fact, many interesting models based on BM have been proposed. In general, a degradation model is composed of a drift term and a diffusion term. The former describes the inherent degradation characteristic, which is usually deterministic and decided by the age, degradation state, or some covariates such as the stress, the environment, and the load. The latter describes the random effects associated with the degradation, which is usually implemented by a random term such as BM. In the past few decades, various degradation models and corresponding prediction methods have been proposed. In this section, we review the BM-based degradation models, which combine two different kinds of research lines. One is the degradation rate associated with the drift term, which can be constant or dependent on the age, degradation state, and covariates. The other is the consideration of the temporal variability (TV), individual variability (IV) (or unit-to-unit variability), measurement error (ME), and LRD for practical degradations. In this survey, the BM-based degradation models can be classified into four categories according to the first line, namely, models with a constant degradation rate (linear drift models), age-driven models, state-driven models, and covariate-driven models. For each category, we discuss whether these models consider the TV, IV, ME, and LRD.

2.1 Linear drift models

The basic linear model has a constant drift coefficient and a BM with a constant diffusion coefficient. In general, there are two kinds of expressions describing the linear model. Let $\{X(t), t \geqslant 0\}$ denote the degradation process; these two expressions are given by

$$X(t) = X(0) + \theta t + \sigma B(t), \tag{1}$$

$$\mathrm{d}X(t) = \theta \mathrm{d}t + \sigma \mathrm{d}B(t), \tag{2}$$

where θ denotes the drift coefficient, $\sigma > 0$ represents the diffusion coefficient, $B(t)$ is a standard BM, and $X(0)$ denotes the initial degradation state. Eq. (1) describes the degradation over time, and Eq. (2) denotes the degradation rate over time. It is clear that Eqs. (1) and (2) can equivalently describe the same degradation.

It is known that the above models can characterize a nonmonotonic degradation process and provide a good description of the system behavior owing to a constant intensity of use. When the failure threshold is constant, the distribution of the FPT for the above degradations can be formulated analytically, known as an inverse Gaussian distribution [11, 38, 39]. In the literature, model (1) has been widely investigated. Here, we focus on the recent studies regarding this model.

For model (1), Lee and Tang [11] studied a time-censored degradation. A modified EM algorithm was proposed to predict the failure times of the censored units [11]. Wang et al. [40] developed an adaptive prediction method that updates θ using a Kalman filter over time as new measurements become available. Furthermore, Huang et al. [41] extended this work using the closed skew normal distribution to weaken the normal hypothesis. Mishra and Vanli [42] applied principal component regression to extract damage-sensitive features and then used model (1) to describe the degradation and predict the RUL. Considering the influence of inspections, Zhang et al. [43] presented an adaptive estimation of the RUL when new observations arrive. Feng et al. [24] proposed an RUL prediction method based on a damage-marker bivariate degradation model, in which damage (MRC decay) and a composite marker constructed from the time-to-voltage-saturation and time-to-current-saturation are described by a two-dimensional Wiener diffusion process $W(t) = \{X(t), Y(t)\}$, with $X(t)$ and $Y(t)$ being interdependent; both of them can be formulated by (1), and the RUL was assumed to be the FPT of $X(t)$. Based on the same model, Shemehsavar and Amini [44] considered the case where $X(t)$ is a latent (unobservable) degradation state. For systems equipped with multi-sensors, Xi et al. [45] developed a degradation model with multi-dimensional observations and hidden degradation states. To predict the storage life for long-storage products, Feng et al. [46] proposed a multiphase Wiener degradation model, in which the drift coefficient of each phase is constant. Furthermore, Gao et al. [47] considered two situations where the change point in a two-phase degradation is constant or random.

To describe the IV, θ and σ are usually assumed to be random variables. Note that the IV of this model is only for the population. Therefore, for a particular individual, the drift coefficient is still constant. Thus, these models are still considered linear. With nonsymmetric priors, Chakraborty et al. [48] proposed an RUL estimation method for the case where the stochastic parameters follow general distributions. Bian and Gebraeel [49] proposed a framework for estimating and continuously updating the RUL distributions of partially degraded components in both cases with or without an informative prior distribution. Using a recursive filter and the EM algorithm, Si et al. [50] proposed an online parameter estimation method and obtained an exact RUL distribution considering the parameter uncertainty. To integrate the population degradation information and individual degradation data, Hao and Su [51] proposed a Bayesian framework for RUL prediction on model (1). Utilizing historical degradations with failure and suspension events, Zhang et al. [52] developed the framework of a similarity-based model, in which the RUL was obtained by comparing the similarity between the operating and reference devices. An empirical two-stage estimation method was developed when the parameters in the multiphase degradation are assumed to be random [53]. Zhang et al. [54] considered the impact of changing-point variability on the RUL prediction.

For degradation processes affected by MEs, the observed degradation can be described by

$$Y(t) = A + BX(t) + \varepsilon(t), \tag{3}$$

where A denotes the bias error, B represents the linear relationship between the degradation $X(t)$ and the observation $Y(t)$, and $\varepsilon(t)$ denotes the stochastic error, which is usually assumed to be a white Gaussian noise. Si et al. [55] addressed the problem of the specification of the ME range to achieve a desirable lifetime estimation performance and developed guidelines regarding the formulation of specification limits to the distribution-related ME characteristics when $B = 1$. To denoise and merge multi-sensor vibration signals of the wind turbine gearbox, Pan et al. [56] developed a performance degradation assessment method based on a deep belief network and self-organizing map. To identify a degradation state that cannot be directly observed, Wei et al. [13] proposed a method for fusing multisensor observation data. To address shocks, a degradation-shock system model was developed by adding a jump term with an amplitude S in (1), and a prediction method was given for the model with MEs [57].

There are some studies that considered the three-source variability (i.e., TV, IV, and ME) simultaneously. Motivated by laser data, Peng and Tseng [58] addressed the effects of model mis-specification on the predicted mean time-to-failure when θ in model (1) follows a normal distribution. Then, they extended their work by assuming that θ follows a skew-normal distribution [59]. Based on [58], Peng and Hsu [60] further gave the closed-form determinant and inverse covariance matrix of the degradation model. Jin et al. [61] presented a Bayesian framework consisting of offline population degradation modeling, online degradation assessment, and RUL prediction for secondary batteries. In [23], an RUL prediction method for lithium-ion batteries based on the truncated normal distribution was proposed for model (1) with MEs. Tang et al. [62] analyzed the effects of model mis-specification on the parameter estimation, the RUL estimation, and the predictive maintenance decision-making. Si et al. [63] introduced three sources of variability contributing to the uncertainty of the estimated RUL. Zhang et al. [64] considered the problem of planning repeated degradation testing when the three-source variability is exhibited.

However, model (1) is suitable only for devices with a substantially constant degradation rate. For most real devices, this assumption is difficult to satisfy. Therefore, some variants based on model (1) have been studied. For devices with exponential degradation trends, an exponential model has been proposed, which is formulated by

$$S(t) = \phi + \beta \mathrm{e}^{\theta' t + \sigma B(t) - \frac{\sigma^2 t}{2}}. \tag{4}$$

Let $X(t) = \ln[S(t) - \phi]$; model (4) can then be transformed into model (1) with $X(0) = \ln\beta$ and $\theta = \theta' - \frac{\sigma^2}{2}$. Assuming that $\ln\beta$ and $\theta' - \frac{\sigma^2}{2}$ are independently normally distributed, Gebraeel et al. [16] developed Bayesian updating methods to update the stochastic parameters and the RUL distribution in the sense of non-FPT. Then, Gebraeel [65] extended this work to the case where $\ln\beta$ and $\theta' - \frac{\sigma^2}{2}$ follow a jointly bivariate normal distribution. Along this direction, Elwany and Gebraeel [66] developed a sensor-driven decision model for component replacement and spare parts inventory. Furthermore, a real-time updating method for the mean RUL of model (4) in the sense of FPT was proposed in [67]. Si et al. [68] provided an exact closed-form RUL distribution for model (4), as well as a method for updating the model parameters and RUL distribution based on Bayesian updating and the EM algorithm. Yu et al. [69] proposed an online RUL estimation method with a Bayesian-updated expectation-conditional-maximization algorithm and a modified Bayesian-model-averaging method. To describe the exponential degradation process after identifying the first change point of the bearing degradation, Wang et al. [17] used the Mahalanobis distance from a known healthy state to detect the beginning of the degradation stage and then studied the RUL prediction with θ as an adaptive parameter. However, the degradation model before the change point was ignored. To address this problem, a mixed-effects model that can simultaneously describe the two phases before and after the change point was proposed [70]. Li et al. [20] found that there were two shortcomings to the above exponential degradation model, i.e., the first prediction time was selected subjectively, and random errors of the stochastic process decreased the prediction accuracy; accordingly, they proposed an improved method, in which the first prediction time was selected adaptively and a particle filter was used to reduce random errors of the stochastic process. Considering multiple uncertainties of oscillator stochastic degradation processes, Liu et al. [71] developed an extreme-learning-based parameter-updating algorithm that combines the local and global similarity methods to eliminate the effects of multiple uncertainty sources. Park and Padgett [72] also provided an accelerated degradation model based on a transformation on the degradation $X(t)$, which is formulated as follows:

$$c[X(t)] - c[X(0)] = \theta t + \sigma B(t), \tag{5}$$

where $c(\cdot)$ is the damage accumulation function. Setting $c(x) = \log(x)$, the authors derived the PDF of the failure time. Chiang et al. [73] studied the accelerated degradation test strategies for model (5).

Another common variant of model (1) is based on a time-scale transformation $t' = \Lambda(t)$; with this transformation, the degradation model can be formulated by

$$X(t) = X(0) + \theta\Lambda(t) + \sigma B[\Lambda(t)], \tag{6}$$

where $\Lambda(t)$ is a monotonically increasing function representing a general time scale, t denotes the clock or calendar time, and $\Lambda(t)$ is the transformed time [74]. Doksum and Hóyland [75] applied the time-scale transformation to denote the variable-stress accelerated degradation and obtained the RUL using time-transformed inverse Gaussian distribution functions. Tang and Su [15] proposed an RUL estimator using the FPTs of the test units over certain predetermined nonfailure thresholds during the early stage of a degradation test. Based on model (6), a physics-statistics-based model was developed and an adaptive Kalman filter was used for RUL prediction [76]. Model (6) with IV was also considered. Tang et al. [77] described the unit-to-unit variability by the uncertainty on θ. Wang [78] considered the uncertainties on both θ and σ. Furthermore, considering that degradation signals are often compounded and contaminated by measurement errors, model (6) with both IV and MEs was studied in [79]. Considering the complex systems with multiple performance characteristics, a bivariate degradation model is proposed where the common factor of two performance characteristics and unit-to-unit variation are jointly taken into account [80]. Ye et al. [81] found that a unit with a higher degradation rate usually has a more volatile degradation path; motivated by this phenomenon, they introduced the correlation between θ and σ in degradation model (6) by setting $\sigma = \varsigma\theta$. Huang et al. [82] considered θ as an adaptive drift and proposed a state-space-based method using a Kalman filter and an EM algorithm to identify model parameters. Along this direction, Wang et al. [83] developed a more general degradation model, in which the time-scale transformations in the drift and diffusion terms are different, i.e.,

$$X(t) = X(0) + \theta\Lambda(t) + \sigma B[\tau(t)], \tag{7}$$

where $\Lambda(t)$ and $\tau(t)$ are both nondecreasing functions; however, for the case where $\Lambda(t) \neq \tau(t)$, only a Monte-Carlo-based method was proposed to predict the RUL. Then, Li et al. [84, 85] considered the case where an ME was included in the observation. Motivated by fatigue crack growth data, Wang et al. [86] proposed a Wiener process degradation model, in which a general quadratic variance function was used as a time-scale transformation of the diffusion, i.e., $\Lambda(t) = t$ and $\tau(t) = d_1 + d_2 t + d_3 t^2$ in model (7). Then, the ME was taken into account [87]. Inspired by the degradations of LED lamps, Tseng et al. [14] proposed a general model by taking transformations of both t and $X(t)$; the model is given by $\Omega[X(t)] = \theta\Lambda(t) + \sigma B[\Lambda(t)]$, where $\Omega(\cdot)$ denotes a strictly decreasing function.

The linear degradation model is used in cases where the rate of degradation is not significantly affected by the cumulative damage, i.e., the degradation rate is approximately a constant. This model has a nice mathematical property in which the explicit distribution of the RUL can be obtained analytically. However, its scope of application is limited by the linearity assumption. To extend its scope of application, some variants have been proposed, such as models (4) and (6). However, there are still no approaches to guide how to choose a transformation function. In addition, this kind of transformation is applicable only to a degradation process that satisfies certain conditions [12]. The TV, IV, ME, and LRD in the above literature are summarized in Table 1.

Table 1 Linear drift models

Model type	Model description	Ref.	TV	IV	ME	LRD
Basic linear model	$X(t)=\varphi+\theta t+\sigma B(t)$	[11, 40–42]	Y	N	N	N
		[48–52, 66–68]	Y	Y	N	N
		[13, 55, 56]	Y	N	Y	N
		[23, 58–64]	Y	Y	Y	N
	$X(t)=\theta t+\sigma B(t)+\sum_{i=0}^{N_0^t}\gamma_i$	[43]	Y	N	N	N
	$X_k=X_{k-1}+\eta\tau_k+\sigma B(\tau_k)+I[C(t_k)-C(t_{k-1})]S$	[57]	Y	N	Y	N
	$\{X(t),Y(t)\}$	[24, 44]	Y	N	N	N
	$\boldsymbol{X}(t)=\boldsymbol{\eta}t+(\sigma\boldsymbol{I}+\boldsymbol{\Lambda})\boldsymbol{B}(t)$	[45]	Y	N	Y	N
	$X(t)=\begin{cases}X(0)+\eta_1 t+\sigma_1 B(t), 0<t\leqslant\tau_1\\ x_{\tau_1}+\eta_2(t-\tau_1)+\sigma_2 B(t-\tau_1), \tau_1<t\leqslant\tau_2\\ \cdots\\ x_{\tau_{n-1}}+\eta_2(t-\tau_{n-1})+\sigma_n B(t-\tau_{n-1}), \tau_{n-1}<t\end{cases}$	[46]	Y	N	N	N
		[53, 54]	Y	Y	N	N
	$\mathrm{d}X(t)=\theta_i\Delta t+\sigma_i\mathrm{d}B(t), i=1,2$	[47]	Y	N	N	N
Transformation on $X(t)$	$S(t)=\varphi+\theta\mathrm{e}^{\beta t+\sigma B(t)-\frac{\sigma^2}{2}t}$	[17]	Y	N	N	N
		[16, 20, 48, 65–68]	Y	Y	N	N
		[71]	Y	Y	Y	N
	$\ln X(t)=\ln\theta I(t>\gamma)+\alpha I(t\leqslant\gamma)+\beta(t-\gamma)I(t>\gamma)+[\sigma_1 I(t\leqslant\gamma)+\sigma_2 I(t>\gamma)]B(t)$	[70]	Y	Y	N	N
	$c[X(t)]-c[X(0)]=\theta t+\sigma B(t)$	[72, 73]	Y	N	N	N
Transformation on t	$X(t)=\beta\Lambda(t)+\sigma B[\Lambda(t)]$	[15, 74, 75]	Y	N	N	N
		[76–78]	Y	Y	N	N
		[79]	Y	Y	Y	N
	$X_i(t)=\alpha\beta_i\Lambda_i(t,\gamma_s)+\sigma_i B_i[\Lambda_i(t,\gamma_i)], i=1,2$	[80]	Y	Y	N	N
	$X(t)=\beta\Lambda(t)+\varsigma\beta B[\Lambda(t)]$	[81]	Y	Y	N	N
	$X(t)=X(0)+\eta\Lambda(t)+\sigma B[\Lambda(t)]$	[82]	Y	Y	N	N
	$X(t)=X(0)+\beta\Lambda(t)+\sigma B[\tau(t)]$	[83]	Y	Y	N	N
	$X(t)=\beta\Lambda(t)+\sigma B[\tau(t)]$	[84, 85]	Y	Y	Y	N
	$X(t)=a+bt+B(d_1+d_2t+d_3t^2)$	[86]	Y	N	N	N
		[87]	Y	N	Y	N
	$\Omega[X(t)]=\beta\Lambda(t)+\sigma B[\Lambda(t)]$	[14]	Y	N	N	N

2.2 Models with an age-dependent drift coefficient

In general, most devices have heterogeneous degradation rates. In one case, the degradation rate varies with the age of the device. In this subsection, we mainly review the models with age-dependent degradation rates, i.e., the drift coefficient of the model can be formulated as a function of the age. Some of the nonlinear degradation models given above are governed by a transformation of either t or $X(t)$. As stated in [12], not necessarily all nonlinear processes can be transformed in these ways. To address this problem, Tseng and Peng [88] proposed a nonlinear degradation directly without any transformation. The degradation model is given by

$$X(t)=M(t)+\int_0^t s(\tau)\mathrm{d}B(\tau), \tag{8}$$

where $M(t)$ denotes the mean degradation path and $B(t)$ is a standard BM. A recursive formula for the lifetime distribution can be obtained. However, the analytical expression of the FPT can be obtained only when $s^2(t)/M'(t)$ is a constant. In other words, the RUL prediction problem of the general nonlinear degradation process is still unresolved. To tackle this problem, Si et al. [12] derived approximate analytical expressions for a kind of Wiener process with nonlinear drift by introducing a time-space transformation; the proposed degradation model can be formulated by

$$X(t)=X(0)+\lambda\int_0^t \mu(\tau;\boldsymbol{\theta})\mathrm{d}\tau+\sigma B(t), \tag{9}$$

where $\mu(t;\boldsymbol{\theta})$ is a nonlinear drift. Along this direction, many studies have been conducted. Utilizing the Bayesian paradigm, Wang et al. [89] proposed an online RUL prediction method with λ as a stochastic parameter and $\boldsymbol{\theta}$ and σ as deterministic parameters. To describe a kind of hybrid deteriorating system, the drift $\lambda\int_0^t \mu(\tau;\boldsymbol{\theta})\mathrm{d}\tau$ was split into a linear term and a nonlinear term [90]. Based on model (9), Si [91] proposed an adaptive prognostic approach by updating a key parameter λ using Kalman filtering; the state space model is

$$\begin{aligned} \lambda_{t_i} &= \lambda_{t_{i-1}} + \eta_{t_i}, \\ x_{t_i} &= x_{t_{i-1}} + \lambda_{t_{i-1}} \int_{t_{i-1}}^{t_i} \mu(\tau;\boldsymbol{\theta})\mathrm{d}\tau + \sigma\epsilon_{t_i}, \end{aligned} \tag{10}$$

where $x_{t_i} = X(t_i)$, $\epsilon_{t_i} \sim \mathcal{N}(0, t_i - t_{i-1})$, and $\eta_{t_i} \sim \mathcal{N}(0,Q)$. Considering quantitative relationship between degradation rate and degradation variation, an improved Wiener process model where $\sigma = \sqrt{\zeta\lambda}$ was studied in [92]. Wang and Tsui [93] considered that model (10) has an underlying assumption: the predicted drift coefficient at the current point in time is equal to the a posteriori drift coefficient estimated at the previous time in the update phase of Kalman filtering; to relax this assumption, a model improved by replacing $\lambda_{t_{i-1}}\int_{t_{i-1}}^{t_i}\mu(\tau;\boldsymbol{\theta})\mathrm{d}\tau$ with $\lambda_{t_i}\int_0^{t_i}\mu(\tau;\boldsymbol{\theta})\mathrm{d}\tau - \lambda_{t_{i-1}}\int_0^{t_{i-1}}\mu(\tau;\boldsymbol{\theta})\mathrm{d}\tau$ was developed. Then, Wang et al. [94] extended the work in [91] by updating multiple parameters $\boldsymbol{\Phi} = [\lambda, \boldsymbol{\theta}]$. However, both Refs. [91, 94] used an autoregressive model of order 1 for the adaptive drift, which led to difficulties in both parameter estimation and RUL prediction. To improve the adaptive model, Zhai and Ye [95] assumed that λ is also a Wiener process, where $\lambda(t) = \lambda_0 + \kappa B(t)$, λ_0 and κ are constant. Using principal component analysis, Le et al. [96] constructed a degradation indicator and predicted the RUL when $\mu(t;\boldsymbol{\theta}) = \beta t^{\alpha}$. Lei et al. [97] and Dong et al. [98] studied model (9) when $\mu(t;\boldsymbol{\theta}) = \beta t^{\beta-1}$ with TV, IV, and MEs occurring simultaneously. Zheng et al. [99] considered the case in which the drift in the model is a weighted sum of several nonlinear functions. To eliminate the normality assumption, Zhang et al. [100] proposed a prediction method for a model with flexible random effects, in which the IV is described by the mixture of normal distributions according to the available information. Considering the step stress accelerated degradation test, Huang [101] extended model (9) to multiple phases. Zhang et al. [102] added a random jump term in model (9) to predict the RUL for devices that suffer from random impact effects. The randomly occurring jumps were modeled by a nonhomogeneous compound Poisson process. The same model was also studied in [103], where the authors regarded the random jumps as imperfect maintenances. For systems under prespecified periodical calibrations, Cui et al. [104] studied models in which the drift terms are influenced by calibrations.

For degradation processes that cannot be directly observed, Feng et al. [105] assumed that the measurement equation is $Y(t) = g[X(t);\boldsymbol{\xi}] + \varepsilon(t)$ and presented a state-space-based prognostic method, where $g[X(t);\boldsymbol{\xi}]$ is a nonlinear function with an unknown parameter set $\boldsymbol{\xi}$ and $\varepsilon(t)$ is a white Gaussian noise.

Wang et al. [106] and Huang et al. [107] considered the prediction of the RUL for a product with two or more performance characteristics. Assuming that the concerned product has p-dimensional performance characteristics and the degradation of each performance characteristic is governed by model (9), the RUL is defined as the FPT when at least one performance characteristic reaches its threshold. Wang et al. [106] proposed a simulation-based method and an approximate method to predict the RUL when performance characteristics are dependent. Huang et al. [107] proposed an improved likelihood function for degradation model (9) and used the Frank copula function to obtain the RUL. Wang et al. [108] introduced a time-scale transformation in the diffusion term, which makes the model more general.

Compared with the model with a transformation on either $X(t)$ or t, the model with an age-dependent drift coefficient is more general and thus more widely applicable. However, these models still have many shortcomings. One prerequisite is that the degradation rate depends entirely on the age of the concerned system. In addition, the existing theory cannot obtain the exact analytical PDF of the FPT of these models. The TV, IV, ME, and LRD in the abovementioned models are summarized in Table 2.

2.3 Models with a state-dependent drift coefficient

The above models assume that the degradation rates depend only on the age. However, the degradation rates of some devices depend not only on their current age but also on the current state of degradation. For example, bearing wear can affect the vibration amplitude and frequency. Different vibration characteristics may lead to different wear speeds. Therefore, how to model the state-dependent degradation rate is a problem worth studying.

Table 2 Models with an age-dependent drift coefficient

Model type	Model description	Ref.	TV	IV	ME	LRD
Nonlinear	$X(t) = M(t) + \sigma \int_0^t s(x)\mathrm{d}B(x)$	[88]	Y	N	N	N
	$X(t) = X(0) + \lambda \int_0^t \mu(\tau;\boldsymbol{\theta})\mathrm{d}\tau + \sigma B(t)$	[101, 107]	Y	N	N	N
		[12, 89–94, 106]	Y	Y	N	N
		[105]	Y	Y	Y	N
	$X(t) = X(0) + \int_0^t \beta\gamma^{\alpha}\mathrm{d}\gamma + \sigma B(t)$	[96]	Y	N	N	N
		[97, 98]	Y	Y	Y	N
	$X(t) = X(0) + \boldsymbol{f}(t;\boldsymbol{\theta}_1)^{\mathrm{T}}\boldsymbol{\theta}_2 + \sigma B(t)$	[100]	Y	Y	N	N
		[99]	Y	Y	Y	N
	$X(t) = X(0) + \int_0^t \mu(\tau;\boldsymbol{\theta})\mathrm{d}\tau + \sigma B(t) + \sum_{j=0}^{N(i)} V_j$	[102, 103]	Y	N	N	N
	$X(t) = \mu_n(t) + \sigma B(t)$ $\mu_n(t) = \mu_{n-1}(t) - \theta_n\left[\mu_{n-1}(\sum_{i=1}^{n} d_i) - X(0)\right]$ $\mu_n(t) = \mu_{n-1}(t - \sum_{i=1}^{n} d_i + t_n^*)$	[104]	Y	N	N	N
	$X(t) = X(0) + \lambda \int_0^t \mu(\tau;\boldsymbol{\theta})\mathrm{d}\tau + \sigma B[s(t;\gamma)]$	[108]	Y	Y	N	N
	$X(t) = \int_0^t \upsilon(\tau)\mathrm{d}S(\tau;\alpha) + \sigma B(t)$ $\upsilon(t) = \upsilon_0 + \kappa W(t)$	[95]	Y	Y	N	N

To assess the reliability integrated circuit, Laurenciu and Cotofana [109] considered that the drift of a degradation model is composed of a deterministic term and a variable term, which can be governed as follows:

$$\mathrm{d}X(t) = g(x,t)\mathrm{d}t + \alpha f(x,t)\mathrm{d}t + \sqrt{\sigma}\mathrm{d}B(t), \tag{11}$$

where $g(x,t)$ denotes the degradation rate of a fixed, deterministic component and $\alpha f(x,t)$ denotes the degradation rate of a variable one; the fixed and variable drift terms depend on both the age and state. Based on model (11), an end-of-life and reliability estimation framework was also proposed in [109].

Zhang et al. [18] proposed a general age- and state-dependent degradation model, which can be formulated by

$$\mathrm{d}X(t) = \mu[X(t),t;\boldsymbol{\theta}]\mathrm{d}t + \sigma[X(t),t;\boldsymbol{\theta}]\mathrm{d}B(t). \tag{12}$$

With model (12), an approximated analytical RUL distribution was obtained using the Lamperti transformation and a time-space transformation. Li et al. [110] studied a similar degradation model, in which the diffusion coefficient $\sigma[X(t),t;\boldsymbol{\theta}]$ in model (12) was replaced by $\sigma(t;\boldsymbol{\theta})$. Furthermore, the unit-to-unit variability in an age- and state-dependent degradation model was considered in [111].

Starting from statistical properties of the degradation records and expert opinion, Deng et al. [112] proposed a general age- and state-dependent Ornstein-Uhlenbeck (OU) process to model the degradation:

$$Y(t) = Y_0 + \int_0^t [a(s)Y_s + b(s)]\mathrm{d}s + \int_0^t \sigma(s)\mathrm{d}B_s. \tag{13}$$

The PDF of the RUL for model (13) was obtained using the Volterra integral equation of the second kind with a nonsingular kernel.

The models mentioned in this subsection possess age- and state-dependent drift or diffusion coefficients. Therefore, they are more general than the linear models and age-dependent models. However, since the degradation state itself is included in these models, the closed-form state equation is unavailable. Only approximate likelihood functions can be obtained using some approximate algorithms, such as the discrete maximum likelihood method and Hermite polynomial expansion method. In addition, the age- and state-dependent drift or diffusion coefficients make the exact analytical PDF of the FPT more difficult to obtain. Therefore, the parameter estimation and RUL prediction of these models still require further research. The TV, IV, ME, and LRD in the above literature are summarized in Table 3.

2.4 Models with a covariate-dependent drift coefficient

In general, during the operation of a device, its environment, operating conditions, and loads will inevitably change with time, resulting in different statistical characteristics of its degradation rate. In this subsection, we focus on models whose degradation rates are dependent on covariates. The covariates can be environmental parameters, stresses, or operating condition labels.

Table 3 Models with a state-dependent drift coefficient

Model type	Model description	Ref.	TV	IV	ME	LRD
State-dependent	$\mathrm{d}X(t) = g(x,t)\mathrm{d}t + \alpha f(x,t)\mathrm{d}t + \sqrt{\sigma}\mathrm{d}B(t)$	[109]	Y	Y	Y	N
	$\mathrm{d}X(t) = \mu[X(t),t;\boldsymbol{\theta}]\mathrm{d}t + \sigma[X(t),t;\boldsymbol{\theta}]\mathrm{d}B(t)$	[18]	Y	N	N	N
	$s(t) = s_k + \int_{t_k}^{t} \mu[s(\tau),\tau]\mathrm{d}\tau + \int_{t_k}^{t} \sigma(\tau)\mathrm{d}B(\tau)$	[110]	Y	Y	N	N
	$s(t) = s_k + a\int_{t_k}^{t} \mu[s(\tau),\tau,b]\mathrm{d}\tau + \int_{t_k}^{t} \sigma\mathrm{d}B(\tau)$	[111]	Y	Y	N	N
	$Y(t) = Y_0 + \int_0^t [a(s)Y_s + b(s)]\mathrm{d}s + \int_0^t \sigma(s)\mathrm{d}B_s$	[112]	Y	N	N	N

To incorporate covariates into degradation modeling, Rishel [113] used two interdependent stochastic processes to describe the degradation processes and covariates; the model is given by

$$\begin{aligned} \mathrm{d}X(t) &= \rho[X(t),Y(t)]\mathrm{d}t, \\ \mathrm{d}Y(t) &= f[X(t),Y(t)]\mathrm{d}t + \sigma[X(t),Y(t)]\mathrm{d}B(t), \end{aligned} \tag{14}$$

where $X(t)$ denotes the degradation process and $Y(t)$ can be considered as an environmental variable influencing the degradation rate. However, this work considered only the optimal control problems of model (14). Based on model (14), Lefebvre et al. calculated the explicit mean FPT when $X(t)$ reaches 0 [114] and $Y(t)$ reaches a preset threshold [115].

To model the degradation under a time-varying stress level, Lim and Yum [116] proposed the following model based on a relative stress level:

$$X(t) = X(0) + \int_0^t \lambda[w(\tau)]\mathrm{d}\tau + \sigma B(t), \tag{15}$$

where $w(t) \in s_1, s_2, \ldots, s_n$ denotes the stress level at time t and s_i is the ith stress level. Bian and Gebraeel [117] considered the degradation with varying environmental profiles and extended $w(t)$ in model (15) to a continuous variable with a range of $[0,\infty)$. However, this work accounted only for the scenario in which the future environment is deterministic and known. This scenario may deviate from the real situation. To address this problem, Si et al. [118] assumed that $w(t)$ is a two-state continuous-time homogeneous Markov process. Furthermore, Bian et al. [119] and Li et al. [120] considered that the degradation signal exhibited jumps at environmental transition epochs. Bian et al. [119] studied the RUL prediction for model (15) with a random jump term added. In addition, Li et al. [120] proposed a two-factor state-space model, in which changes of the degradation rate are introduced into a state transition function, and jumps are introduced into a measurement function. Taking the bearing temperature as a dynamic covariate, Jin et al. [121] developed a physics-of-failure-based degradation model for the momentum wheel; the model is a special case of model (15) with $w(t) = \beta_0 \mathrm{e}^{-b/T(t)}$, where $w(t)$ characterizes the effect of temperature on the degradation, $T(t)$ denotes the bearing temperature at time t, and b is an unknown parameter. For the lumen degradation of LEDs, Tsai et al. [122] proposed a model in which the drift coefficient is decided by the ambient temperature $L_1(t)$ and the drive current $L_2(t)$; their relationship is formulated by the generalized Eyring model. Liao and Tian [123] provided a framework of RUL prediction for a single unit under time-variant operating conditions by assuming that the future stress level was known. Liu et al. [124] predicted the RUL using the Monte Carol simulation in the case where the drift is a stress-dependent linear accelerated function, i.e., $\lambda(S) = a + bS$ in model (15), where a and b are stochastic parameters with IV and S denotes the stress. To model the step-stress accelerated degradation test, Liao and Tseng [125] proposed a piecewise degradation model, where the drift coefficient of each segment depends on the temperature stress; they also discussed how to choose the optimal settings of variables such as the sample size, measurement frequency, and termination time. For products characterized by multiple performance parameters, Sun et al. [126] proposed a model with a stress-dependent drift and a transformed diffusion to describe each performance parameter and then used a copula function to integrate them. For the case where the diffusion term is also covariate-dependent, Liao and Elsayed [127] proposed an accelerated degradation testing model, in which the drift and diffusion terms are dependent on external stresses. For the age- and covariate-dependent degradation rates, Liu et al. [128,129] and Chen et al. [130] studied a covariate-dependent degradation model based on the time transformation. The model is given by

$$X(t) = X(0) + \theta_i \Lambda(t) + \sigma B[\Lambda(t)], \tag{16}$$

where θ_i denotes the drift coefficient under accelerated stress type i. Liu et al. [128] assumed $\log(\theta_i) = A + \sum_i B_i\varphi(s_i) + \eta$, where A and B_i are constant parameters, $\varphi(s_i)$ denotes a function of accelerated stress s_i, and η is a white Gaussian noise, which is used to denote the IV in the model. For a higher estimation accuracy and a wider range of use, Chen et al. [130] considered the effects of the stress level, IV, and ME simultaneously. The relationship between θ_i and the accelerated stress type can be formulated by $\theta_i = \alpha\exp[-\beta\varphi(s_i)]$, where $\alpha \sim \mathcal{N}(\mu_\alpha, \sigma_\alpha^2)$, and the measurement error is assumed to be normally distributed with zero mean. Furthermore, to describe more general degradations, Liu et al. [129] studied a degradation model with different time-scale transformations in the drift and diffusion.

Peng et al. [131] indicated that modern complex systems are generally characterized by multiple degradation indicators and suffer from dynamic operating conditions; to cope with this situation, they developed a model that characterizes various combinations of dependent degradation indicators and the influences of external factors, which can be presented by

$$Y_i(t) = \eta_i(t; \boldsymbol{X}^E, \boldsymbol{X}^o, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^R) + \sigma_i B(t), \tag{17}$$

where $Y_i(t)$ is the ith degradation process in a complex system and $Y_i(0) = 0$. In model (17), $\boldsymbol{X}^E$ and $\boldsymbol{X}^o$ denote covariates of the environmental conditions and operating profiles, respectively, $\boldsymbol{\theta}_i^F$ and $\boldsymbol{\theta}_i^R$ denote deterministic parameters and random parameters, respectively, and $\eta_i(\cdot)$ is the degradation mean function. The dependence among these degradation processes is formulated by a copula function. However, since the model is too complicated, only a numerical prediction method was proposed in [131]. Hao et al. [132] also studied the covariate-dependent multiple degradations and assumed that the degradation rates have linear relationships with the workloads and proposed methods to actively control the degradation as well as to predict the RUL of each unit by dynamically adjusting its workloads.

The models with covariate-dependent drift coefficients can describe degradation processes that are adaptable to different environmental changes. For these models, a key question is how to predict changes in future environmental variables and incorporate them into the RUL prediction. Judging from the existing research results, this problem has not been solved very well. Some studies developed models that depend on covariates but assumed that for a degenerate individual, the covariates are constant; the RUL prediction of these models is the same as that for models that do not depend on covariates, such as [116, 130]. Some other studies considered the time-variant nature of the covariates but assumed that future covariates are known, which are not in line with the actual engineering context, such as [117, 120, 122, 123, 125, 127, 129]. For the case where the future covariates are unknown, an analytical RUL distribution is still unavailable, and only numerical results can be obtained, such as [118,119,121,124,131]. In addition, the TV, IV, ME, and LRD in the above models are summarized in Table 4.

3 FBM-based degradation models

3.1 Motivations for introducing FBM

As stated in [3], the BM-based degradation models have Markov properties. Given a degradation state at a certain moment, a future degradation process is independent of the degradation before that moment. Subsequently, a question naturally arises: does the degradation process of any actual device match this kind of assumption? To answer this question, we provide actual data for four different devices, i.e., the degradations of a blast furnace wall, a turbofan [133], a ball bearing [134], and a Li-ion battery [135]. The four degradation paths are illustrated in Figure 1.

The Hurst exponent has been widely used to evaluate whether a stochastic process has either the Markov property or LRD [136]. If the Hurst exponent equals 0.5, the process is a Markov process with independent increments. If the Hurst exponent is greater than 0.5, the process has LRD. The corresponding Hurst exponents of the four actual degradation processes are shown in Figure 1. It can be found that the Hurst exponents of these four devices are greater than 0.5, which indicates that LRD does exist in the degradation processes of these real devices. In fact, the degradation processes of some actual devices face difficulty in satisfying the Markov hypothesis. However, as shown in Tables 1–4, the BM-based degradation models do not consider the LRD in the degradation processes. As a consequence, it is not suitable to use the models mentioned in Section 2 to describe the degradation processes of these devices. Hence, a new model that can describe the LRD of degradation processes is desired. Here, we reproduce the definition of LRD [137].

Table 4 Models with a covariate-dependent drift coefficient

Model type	Model description	Ref.	TV	IV	ME	LRD
Covariate-dependent	$\mathrm{d}X(t) = \rho[X(t), Y(t)]\mathrm{d}t$ $\mathrm{d}Y(t) = f[X(t), Y(t)]\mathrm{d}t + \sigma[X(t), Y(t)]\mathrm{d}B(t)$	[113–115]	Y	N	N	N
	$X(t) = x(0) + \int_0^t \lambda[\upsilon(u)]\mathrm{d}u + \sigma B(t)$	[116–118]	Y	N	N	N
	$S(t) = s(0) + \int_0^t r[\psi(v)]\mathrm{d}v + \sum_{j=1}^{N(t)} J[\psi(A_j)] + \gamma B(t)$	[119]	Y	N	N	N
	$x_k = x_{k-1} + r_{p_{k-1}}\eta\Delta t_k + \omega_{k-1}$ $y_k = a_{p_k}(b_{p_k} + x_k^c + v_k)$	[120]	Y	N	Y	N
	$X(t) = \beta_0 \int_0^t \mathrm{e}^{-b/T(\tau)}\mathrm{d}\tau + B(t)$	[121]	Y	N	N	N
	$X(t) = x(0) + \int_0^t \lambda[L_1(u), L_2(u)]\mathrm{d}u + \sigma B(t)$	[122]	Y	N	N	N
	$\mathrm{d}X(t, z) = g(z; \alpha)\mathrm{d}t + \sigma \mathrm{d}B(t)$	[123]	Y	N	N	N
	$X(t) = \varphi + (a + bS)t + \sigma B(t)$	[124]	Y	Y	N	N
	$X(t)=\begin{cases}\eta_1\tau + \sigma B(\tau), & 0\leqslant\tau<\tau_1\\ \eta_2(\tau-\tau_1)+\eta_1\tau_1+\sigma B(\tau), & \tau_1\leqslant\tau<\tau_2\\ \cdots & \\ \eta_m(\tau-\tau_{m-1})+\sum_{i=1}^{m-1}\eta_i(\tau_i-\tau_{i-1})+\sigma B(\tau), & \tau_{m-1}\leqslant\tau<\tau_m\end{cases}$ $\eta_i = \exp(a + \frac{b}{273+S_i}), \tau_0 = 0$	[125]	Y	N	N	N
	$X(t) = X(0) + \mathrm{e}^{a+b\varphi(S_i)} \int_0^t \lambda(t; \boldsymbol{\theta})\mathrm{d}t + \sigma B[\tau(t; \gamma)]$	[126]	Y	N	N	N
	$\mathrm{d}X(t; z) = h(z; \boldsymbol{\theta})\mathrm{d}t + f(z; \boldsymbol{\beta})\mathrm{d}B(t)$	[127]	Y	N	N	N
	$X(t) = X(0) + \mathrm{e}^{A+\sum_i B_i\phi(s_i)+\eta}\Lambda(t) + \sigma B[\Lambda(t)]$	[128]	Y	Y	N	N
	$X(t) = X(0) + \alpha\mathrm{e}^{-\beta\varphi(s_i)}\Lambda(t) + \sigma B[\Lambda(t)] + \sigma_\varepsilon\varepsilon$	[130]	Y	Y	Y	N
	$X(t) = \mu(\boldsymbol{S}; \boldsymbol{\eta})\Lambda(t, \theta) + \sigma B[\tau(t; \gamma)]$	[129]	Y	Y	N	N
	$Y_i(t) = \eta_i(t; \boldsymbol{X}^E, \boldsymbol{X}^o, \boldsymbol{\theta}_i^F, \boldsymbol{\theta}_i^R) + \sigma_i B(t)$	[131]	Y	Y	N	N
	$\mathrm{d}X_i(t) = \beta_i u_i(t)\mathrm{d}t + \sigma_i B(t)$	[132]	Y	N	N	N

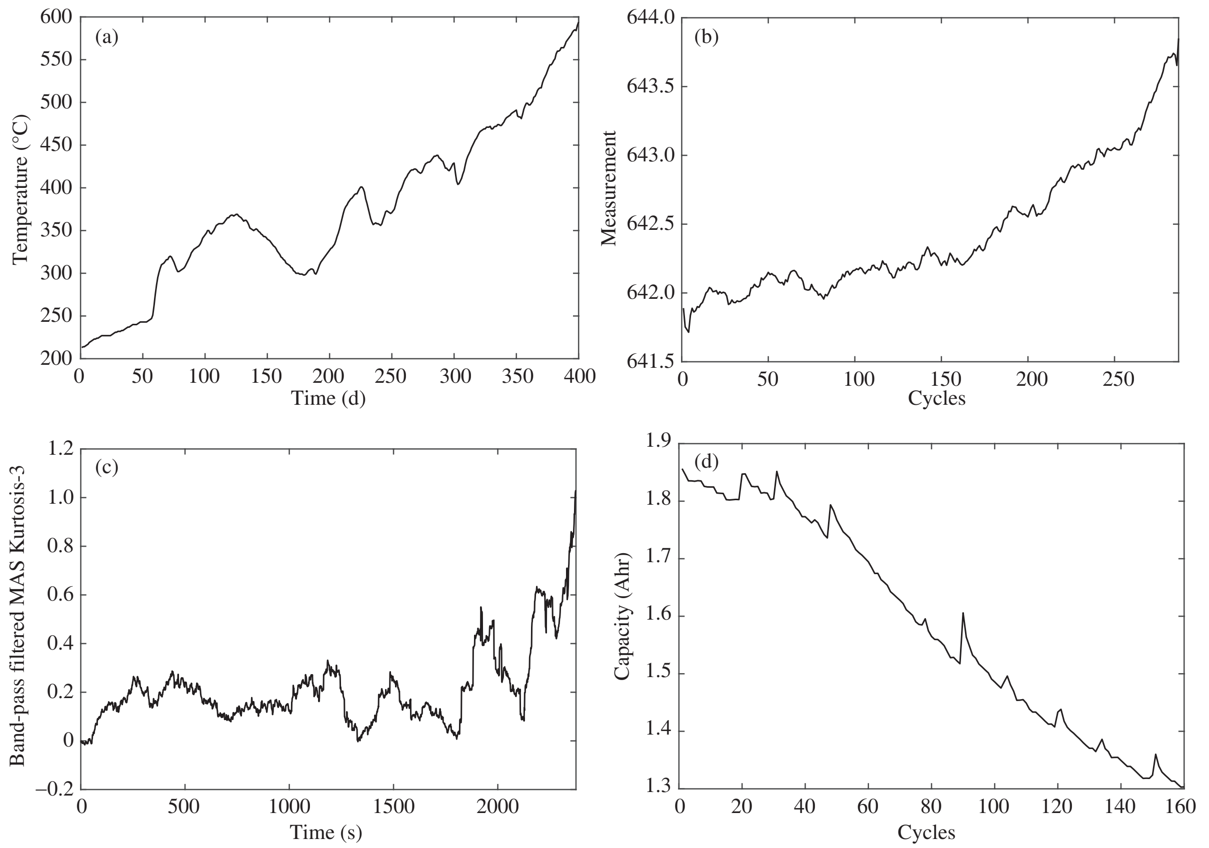

Figure 1 The degradation paths of four different devices. (a) A blast furnace wall ($H = 0.7979$); (b) a turbofan ($H = 0.7413$); (c) a ball bearing ($H = 0.6120$); (d) a Li-ion battery ($H = 0.7322$).

Definition 1. For a sequence $x_0, x_1, \ldots$, if there exist constants C and $\alpha \in (0, 1)$ that make the following formula hold, then the sequence is referred to as having LRD:

$$\lim_{i\to\infty}\frac{\rho(i)}{Ci^{-\alpha}}=1, \tag{18}$$

where $\rho(i)=\mathrm{cov}(x_k,x_{k+i})$ is the autocovariance function.

To describe the LRD in the degradation process, FBM was introduced in degradation modeling [29–32, 37]. For a standard FBM $\{B_H(t),t\geqslant 0\}$ with the Hurst exponent H, it holds that $E\left[B_H\left(t\right)B_H\left(s\right)\right]=0.5(t^{2H}+s^{2H}-|s-t|^{2H})$. According to Definition 1, when $0.5<H<1$, the FBM $\{B_H(t),t\geqslant 0\}$ has LRD. In addition, when $H=0.5$, the FBM is equivalent to a BM, i.e., an FBM-based degradation model reduces to a BM-based degradation model when the Markov assumption holds. In other words, for any BM-based degradation model, there always exists an FBM-based degradation model that can cover it because the BM-based model serves as a special case. Therefore, the FBM-based model can not only describe the degradation processes with LRD but also model the Markovian degradation processes. In addition, all the models mentioned in Section 2 can be extended to more general forms using the FBM. This also provides a new research direction for the study of degradation modeling and RUL prediction.

3.2 FBM-based degradation models and their RUL prediction

As stated in Subsection 3.1, the Markov assumption does not always hold in general, and LRD does exist in many real devices, which implies that the future degradation trend depends not only on the current degradation state but also on the historical degradation path. Because the FBM-based degradation models do not satisfy the Markov property, there exist some difficulties in identifying models and solving the FPT.

Inspired by the FBM, Xi et al. [29] developed a degradation model with LRD, which is formulated by the following fractional diffusion process $\{X(t),t\geqslant 0\}$:

$$\mathrm{d}X(t)=\mu(t;\boldsymbol{\theta})\mathrm{d}t+\sigma_H\mathrm{d}B_H(t), \tag{19}$$

where $\{B_H(t),t\geqslant 0\}$ is an FBM with a Hurst exponent $H\in(0,1)$, $\mu(t;\boldsymbol{\theta})$ is a continuous function of the age t, and $\boldsymbol{\theta}$ represents a set of parameters involved in $\mu(t;\boldsymbol{\theta})$. Note that the function $\mu(t;\boldsymbol{\theta})$ can be linear or nonlinear and its form does not affect the LRD of the degradation process. However, the modeling advantages brought by FBM also pose some challenges for solving the FPT. It is known that FBM is essentially neither a Markovian process nor a semimartingale when the Hurst exponent is not equal to 0.5 [30]. In addition, the semimartingale is the most generalized class of stochastic processes where Ito's integral can be defined. Therefore, many mature mathematical theorems cannot be directly applied. In [29], only a Monte-Carlo-based algorithm was provided to obtain a numerical RUL prediction. In [138], an expectation for the FPT solution of non-Markovian processes in confinement was given, including FBM. However, the expectation is not enough for RUL prediction. For actual devices, the operating environment has randomness, and the manufacturing process has heterogeneity. In addition to the uncertainty caused by load change and human operation, the degradation process and RUL naturally have uncertainty. However, the expectation of the RUL cannot characterize these uncertainties. Therefore, predicting the distribution of the RUL is more valuable. Unfortunately, there has been no theoretical result on the distribution of the FPT for FBM. To solve this problem, Zhang et al. [30] proposed an approximate explicit PDF of the FPT for model (19) with a weak convergence theory [139]; the general solution of FPT-based process (19) is formulated by

$$\begin{aligned}f_{L_k}(l_k)=&\frac{[\sigma(l_k)]^{\frac{1}{2}}}{\sqrt{2\pi\,[\sigma(0)]^{\frac{1}{2}}\int_0^{l_k}[\sigma(s)]^{\frac{1}{2}}\,\mathrm{d}s}}\exp\left\{-\frac{[\omega-x_k-\int_{t_k}^{l_k+t_k}\mu(s;\boldsymbol{\theta})\mathrm{d}s]^2}{2\,[\sigma(0)]^{\frac{1}{2}}\int_0^{l_k}[\sigma(s)]^{\frac{1}{2}}\,\mathrm{d}s}\right\}\\&\times\left\{\frac{\omega-x_k}{\int_0^{l_k}[\sigma(s)]^{\frac{1}{2}}\mathrm{d}s}+\frac{\mu\left(l_k+t_k;\boldsymbol{\theta}\right)}{[\sigma(l_k)]^{\frac{1}{2}}}-\frac{1}{\int_0^{l_k}[\sigma(s)]^{\frac{1}{2}}\mathrm{d}s}\int_{t_k}^{l_k+t_k}\mu\left(s;\boldsymbol{\theta}\right)\mathrm{d}s\right\}\end{aligned} \tag{20}$$

where

$$\sigma(t)=\lim_{\tau\to 0}\sigma_H^2\left\{\begin{aligned}&\sum_{i=1}^{\lfloor\frac{t}{\tau}\rfloor}\left[\int_{(i-1)\tau}^{i\tau}c_Hs^{\frac{1}{2}}\int_{\lfloor\frac{t}{\tau}\rfloor\tau}^{\lfloor\frac{t+\tau}{\tau}\rfloor\tau}(u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}\mathrm{d}u\mathrm{d}s\right]^2\\&+\left[\int_{\lfloor\frac{t}{\tau}\rfloor\tau}^{\lfloor\frac{t+\tau}{\tau}\rfloor\tau}c_Hs^{\frac{1}{2}}\int_s^{\lfloor\frac{t+\tau}{\tau}\rfloor\cdot\tau}(u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}\mathrm{d}u\mathrm{d}s\right]^2\end{aligned}\right\} \tag{21}$$

Table 5 FBM-based degradation models

Model type	Model description	Ref.	TV	IV	ME	LRD
Age-dependent	$\mathrm{d}X(t) = \mu(t;\boldsymbol{\theta})\mathrm{d}t + \sigma_H \mathrm{d}B_H(t)$	[29, 30]	Y	N	N	Y
		[31, 140]	Y	Y	N	Y
State-dependent	$\mathrm{d}X(t) = \mu[X(t), t; \boldsymbol{\theta}]\mathrm{d}t + \sigma_H \mathrm{d}B_H(t)$	[32]	Y	N	N	Y
Covariate-dependent	$X(t) = X(0) + \int_0^t \lambda[\Phi(s)]\mathrm{d}s + \sigma_H \mathrm{d}B_H(t)$	[37]	Y	N	N	Y

and $c_H=\sqrt{\frac{2H\Gamma(\frac{3}{2}-H}{\Gamma(\frac{1}{2}+H)\Gamma(2-2H)}}$ is a constant [30]. To obtain the PDF of the RUL, a sufficiently small τ needs to be determined to approximately calculate the numerical limitation and integrals involved in (20) and (21). With the accuracy set by τ, it needs $\mathcal{O}(l_k/\tau)$ floating-point operations to calculate $f_{L_k}(l_k)$. As $\sigma(t)$ for $t \in [0, l_k]$ should be stored during the computational procedure, it requires $\mathcal{O}(l_k/\tau)$ memory bits. It is easy to prove that when $H = 0.5$, Eq. (20) is equivalent to the PDF of the RUL obtained in [12]. Furthermore, Xi et al. [31] studied the RUL prediction of model (19) with IV, where the parameters in the drift $\mu(t;\boldsymbol{\theta})$ are supposed to follow normal distributions. Zhang et al. [140] considered the IV in model (19) by regarding the parameters in the drift term as hidden state variables; an unscented particle filter was used to posteriorly estimate the distribution of these multiple hidden state variables.

To describe the degradation process with an age- and state-dependent degradation rate, Zhang et al. [32] extended model (19) to a more general form, which can be formulated by

$$\mathrm{d}X(t) = \mu[X(t), t; \boldsymbol{\theta}]\mathrm{d}t + \sigma_H \mathrm{d}B_H(t). \tag{22}$$

Since an FBM-based process has nonindependent increments, its likelihood function contains the inverse of the covariance matrix. The size of the matrix increases over the runtime of the device, resulting in a high computational complexity for model identification. To solve this problem, the authors used a measure transformation to transform the original degradation process into an independent increment process. Using the Radon-Nikodym derivative of the transformation, the likelihood ratio function can be formulated, and the parameter estimates can be obtained. Similar to [30], an approximate explicit solution of the RUL was also proposed in [32].

Considering devices operating under varying conditions, Zhang et al. [37] studied the RUL prediction for degradation processes with multiple modes; the degradation model is given by

$$X(t) = X(0) + \int_0^t \lambda[\Phi(s)]\mathrm{d}s + \sigma_H \mathrm{d}B_H(t), \tag{23}$$

where $\Phi(t)$ denotes the mode at time t, which is described by a continuous-time Markov chain. Owing to the uncertainty of mode switching, the drift term and diffusion term have uncertainty simultaneously, which poses new challenges for RUL prediction. The authors computed the uncertainty of mode switching in a future interval, then considered this uncertainty together with that of the degradation process, and finally obtained the RUL prediction.

The TV, IV, ME, and LRD in the FBM-based models are listed in Table 5. At present, FBM-based degradation modeling and RUL prediction are still in the initial stage with few research results. There are still many follow-up studies that can be investigated in depth, such as degradation processes with an ME or a time-varying Hurst exponent. The detailed issues to be studied further are discussed in Section 5.

4 Comparative cases

4.1 Case 1: a blast furnace wall

To illustrate the superiority of FBM, a comparative study on the degradation process of a blast furnace wall is conducted. The degradation data are collected from an actual operating blast furnace, the capacity of which is 2650 m^3. Because the temperature of the blast furnace wall is a key measurement of its remaining thickness, it usually serves as an indicator of the degradation [30, 102]. The degradation path of the blast furnace wall is shown in Figure 1(a). Here, we intend to predict the FPT when the temperature of the blast furnace wall reaches 590°C, where the corresponding failure time from $t = 0$ is the 399th day.

Table 6 The estimated parameters and model performance in Case 1

	$\hat{a}$	$\hat{b}$	$\hat{\sigma}_H$	$\hat{H}$	p	AIC	$\overline{\text{MSE}}$
M_a	0.1712	1.2848	2.6226	–	3	1510.5	1159.7
M_b	1.0851×10^{-12}	5.4651	2.2608	0.7979	4	1447.2	65.1

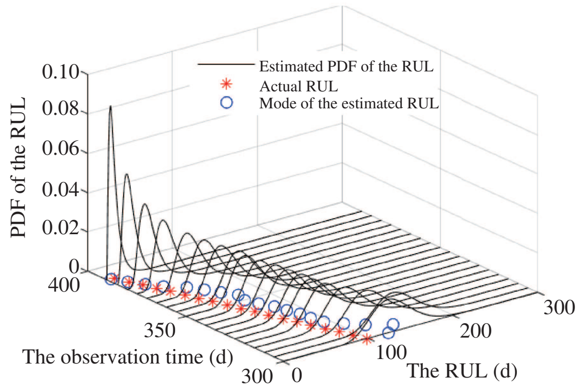

Figure 2 (Color online) The predicted RULs in Case 1 based on M_a.

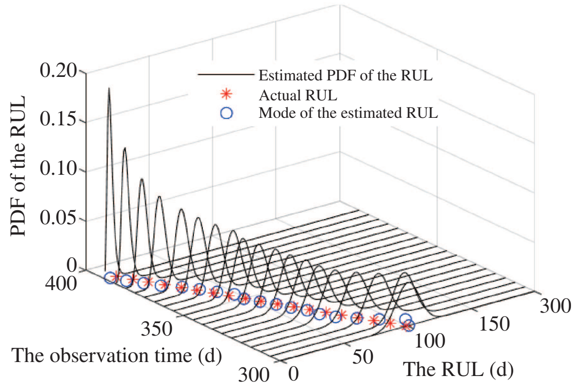

Figure 3 (Color online) The predicted RULs in Case 1 based on M_b.

Here, we respectively use a BM-based and an FBM-based model to describe the degradation shown in Figure 1(a). The degradation models used for RUL prediction are given as (24) and (25). Note that the drift terms of the two models are the same. However, the diffusion terms are driven by BM and FBM, respectively.

$$M_a : \mathrm{d}X\left(t\right) = abt^{(b-1)} + \sigma B\left(t\right), \tag{24}$$

$$M_b : \mathrm{d}X\left(t\right) = abt^{(b-1)} + \sigma B_H\left(t\right). \tag{25}$$

According to the properties of BM and FBM, the joint probability distribution of differential observations can be written. Then, the maximum likelihood estimation method can be used to obtain the estimates of unknown parameters in the above two models. The detailed estimation methods can be found in [12, 30]. The estimated results are listed in Table 6.

To quantify the prediction performance, the Akaike information criterion (AIC) is used to evaluate the fitting degree of the models, which can be calculated by

$$\mathrm{AIC} = -2\left(\max \ell\right) + 2p, \tag{26}$$

where ℓ denotes the likelihood function and p represents the number of unknown parameters in the model. Based on (26), the AIC of each model is calculated and listed in Table 6. Because a smaller AIC means a better fitting degree, the FBM-based model M_b is more suitable for describing the degradation process shown in Figure 1(a).

Moreover, the RUL prediction results obtained based on models M_a and M_b are compared. Using the prediction methods proposed in [12, 30], the RULs of the degradation from several points in time are obtained and plotted in Figures 2 and 3, respectively. To compare the RULs obtained by the two models, the peak value of each PDF is marked in Figures 2 and 3, namely, the mode of the RUL representing the most likely time of failure. Then, to quantify the accuracy of the prediction results, the mean squared error (MSE) of the predicted results obtained by each model is calculated by

$$\mathrm{MSE}_k = \int_0^{\infty} \left(l_k - \tilde{l}_k\right)^2 f_{L_k}\left(l_k\right) \mathrm{d}l_k, \tag{27}$$

where $f_{L_k}(\cdot)$ refers to the predicted PDF of the RUL and $\tilde{l}_k$ denotes the true RUL from t_k. The average MSE of each model is calculated and also listed in Table 6. From Figures 2 and 3, the RUL modes obtained by model M_b are closer to the true RULs than those obtained by model M_a. From Table 6, the predicted PDF of the RUL obtained by model M_b has a smaller average MSE than that obtained by M_a. To summarize, the FBM-based model M_b is better than the BM-based model M_a for the degradation of the blast furnace wall.

Table 7 The estimated parameters and model performance in Case 2

	$\hat{a}$	$\hat{b}$	$\hat{\sigma}_H$	$\hat{H}$	p	AIC	$\overline{\text{MSE}}$
M_a	1.4882×10^{-11}	4.4984	0.0252	–	3	−1390.2	32.68
M_b	1.5117×10^{-26}	10.5326	0.0281	0.7413	4	−1394.7	13.56

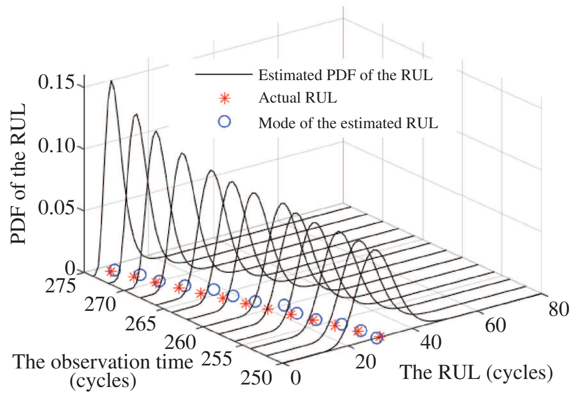

Figure 4 (Color online) The predicted RULs in Case 2 based on M_a.

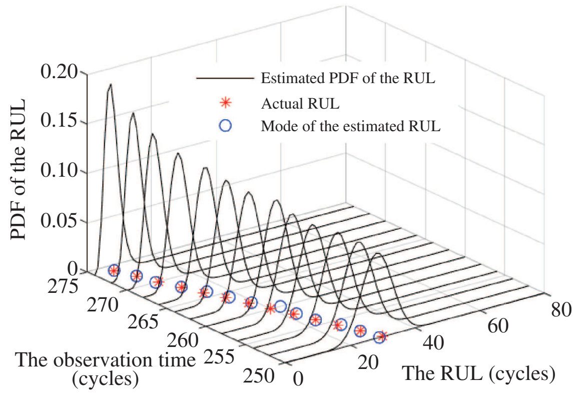

Figure 5 (Color online) The predicted RULs in Case 2 based on M_b.

4.2 Case 2: a turbofan

In this subsection, the degradation of a turbofan is further carried out to illustrate the advantage of the FBM-based model. The degradation data are obtained from the Prognostics CoE Center of Excellence of NASA Ames Research Center [133]. The degradation path is shown in Figure 1(b). The failure threshold is set as 643.67, and the corresponding failure time is the 280th cycle.

Here, we also use models M_a (24) and M_b (25) to predict the RUL of the degradation shown in Figure 1(b). Using the same method as in Subsection 4.1, the parameters are estimated and the RULs at several instants are predicted. The estimated parameters, the AIC of the two models, and the average MSEs of the predicted RULs are shown in Table 7. The PDFs of RULs obtained by M_a and M_b are plotted in Figures 4 and 5, respectively.

From Table 7 and Figures 4 and 5, the AIC of model M_b is smaller than that of model M_a, and the average MSE of the RULs predicted by M_b is smaller than that predicted by M_a. This indicates that M_b is more suitable for the degradation of the turbofan.

5 Further problems for FBM-based prediction

As stated in the above sections, the current BM-based models completely ignore the LRD in the degradation processes. However, LRD exists in some practical devices. Inspired by this observation, the FBM is introduced into the degradation models, and some preliminary results for predicting the RULs of the FBM-based degradation models are obtained. Note that a degradation process with LRD is neither a Markovian process nor a semimartingale. This leads to the obvious difference between the RUL predictions for BM- and FBM-based degradation models. Some important and interesting problems to be further studied are summarized in the following.

Problem 1: How to obtain the exact analytical PDF of an RUL?

As stated in [29, 30], because the degradation process with LRD is neither a Markovian process nor a semimartingale, the exact analytical FPT is difficult to derive. In the preliminary studies, a weak convergence theory is introduced to approximate the FBM to a BM with a time-varying coefficient, and an approximate explicit solution of the FPT is obtained [30]. However, the approximate degree of this solution is evaluated only by a comparison with the Monte Carlo simulations, and no quantitative analysis is obtained. It is necessary and important to obtain the exact analytical PDF of the RUL.

Problem 2: How to obtain the online estimation of the Hurst exponent?

In the preliminary studies, the Hurst exponent was estimated in an offline manner. If sufficient historical data are obtained, the various methods can be used to estimate the Hurst exponent, such as R/S analysis [141], Whittle analysis [142], and wavelet analysis [143]. However, these methods are offline and cannot update the model parameters online if new data are obtained. In fact, as stated in the current BM-based studies, the online update parameters can improve the accuracy of the model as the number of observations increases, thereby improving the predictive performance of the RUL. Therefore, how to update the Hurst exponent online is also a problem worth further study.

Problem 3: How to predict the RUL for devices with multiple and dependent LRD degradation processes?

In the preliminary studies, the proposed FBM-based degradation models focused on a single degradation process. However, with the increase in complexity of modern devices, it is difficult to describe the degradation of a device by a single stochastic process. In the case of multiple dependent degradation processes, these processes are usually coupled with each other. Thus, how to model dependent LRD degradations and predict the RUL is an important problem to consider.

Problem 4: How to predict the RUL of a degradation if the Hurst exponent belongs to (0, 0.5)?

The existing studies on BM-based models considered the case where the degradation process has no temporal dependence, i.e., the Hurst exponent equals 0.5. In the FBM-based degradation modeling and RUL prediction studies, only the case where the degradation process has LRD is considered, i.e., the Hurst exponent belongs to $(0.5, 1)$. According to the definition of FBM, the range of the Hurst exponent is $(0, 1)$. When the Hurst exponent belongs to $(0, 0.5)$, the degradation process has short-range dependence. In this case, the weak convergence theory usually no longer holds. Therefore, the RUL for the degradation with a Hurst exponent belonging to $(0, 0.5)$ is difficult to predict and thus deserves further study.

Problem 5: How to predict the RUL of an FBM-based degradation with indirect data?

For actual devices, the observations are often mixed with various noises. Therefore, it is difficult to obtain direct degradation data. In general, this problem can be addressed by filtering methods based on the state-space equations of the degradations. Since the FBM-based model has dependent increments, the system noise in its state-space equation is colored. Therefore, how to estimate those degradation states contaminated by noise is a problem that needs to be solved. Specifically, the estimation of the Hurst exponent in the presence of measurement noise becomes more difficult. In addition, for complex systems, how to integrate information collected from multiple sensors, identify the hidden degradation state, and predict the RUL are also challenging problems.

Problem 6: How to obtain the RUL for more general degradations?

Although the FBM-based degradation models are more general than the existing BM-based degradation models, there are still some assumptions that may be difficult to satisfy in practical cases. For example, FBM has stationary increments, making it difficult to describe a degradation process with nonstationary increments. In the literature, several variants of FBM have been proposed, such as subfractional Brownian motion and bifractional Brownian motion. However, these stochastic processes have not been applied to the modeling of degradation processes. To expand the scope of application of degradation models, it can be advantageous to introduce more general forms of FBM into the degradation processes.

6 Conclusion

This paper reviewed the BM-based degradation models from the following two aspects.

- The models can be classified according to the degradation rate relative to the age, state, or covariates.
- The representative models are listed based on whether TV, IV, ME, and LRD are considered or not.

It is illustrated that the existing BM-based models completely ignore the LRD of the degradation processes. However, LRD exists in the degradation processes of some actual devices. Therefore, this study also summarizes recent studies on FBM-based degradation models, which can address the LRD in degradations. Two case studies show that FBM is more suitable for degradation modeling than BM. Furthermore, some important and interesting research perspectives are highlighted for future work.

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