SCIENCE CHINA
Information Sciences

• LETTER •

June 2021, Vol. 64 169302:1–169302:3
https://doi.org/10.1007/s11432-020-2909-5

Convolution theorem involving n-dimensional windowed fractional Fourier transform

Wenbiao GAO[1,2] & Bingzhao LI[1,2*]

[1]*School of Mathematics and Statistics, Beijing Institute of Technology, Beijing* 100081*, China;*
[2]*Beijing Key Laboratory on MCAACI, Beijing Institute of Technology, Beijing* 100081*, China*

Received 6 January 2020/Accepted 29 April 2020/Published online 26 April 2021

Citation Gao W B, Li B Z. Convolution theorem involving n-dimensional windowed fractional Fourier transform. Sci China Inf Sci, 2021, 64(6): 169302, https://doi.org/10.1007/s11432-020-2909-5

Dear editor,

The fractional Fourier transform (FRFT) was proposed by Wiener in 1929. As an important and powerful analyzing tool for time-frequency analysis, the FRFT has been applied in a lot of fields [1, 2] such as signal processing, optics, radar, and quantum mechanics. According to a similar definition of the FRFT, several other important fractional transforms [3, 4] were studied such as the fractional wavelet transform, the fractional cosine, the fractional sine, and the fractional Hankel transform.

The windowed fractional Fourier transform (WFRFT), also called short-time fractional Fourier transform, has been proposed, which is useful for guiding the time-frequency analysis in the FRFT domain. Yu et al. [5] proposed a novel image encryption algorithm that is based on the phase-truncated WFRFT and the hyper-chaotic system. Zhang [6] studied the phase retrieval problem of signals in the case when the measurements are coming from a discrete WFRFT.

The convolution product has a central place among the various modes of function compositions. It plays an important role in digital signal and image processing, Fourier series and solutions of convolution integral equation problems. Recently, some researchers have explored convolution and product theorems in different transform domains. In our previous work [7], we proposed the quaternion windowed linear canonical transform and investigated some properties of the quaternion windowed linear canonical transform. Kamalakkannan and Roopkumar [8] introduced a generalized fractional convolution on functions of several variables and derived their properties including convolution theorem and product theorem for the multidimensional FRFT. Upadhyay and Dubey [9] studied the wavelet convolution product and existence theorems by using the theory of n-dimensional FRFT. The demonstrated effectiveness of convolution and product theorems for signal processing problems in different transform domains motivate the study of window function associated with FRFT. The convolution and product theorems of the WFRFT have not been discussed, as far as known to the author.

In this letter, we propose an n-dimensional WFRFT. According to the n-dimensional WFRFT and the corresponding convolution product, we obtain the convolution theorem of the WFRFT. Then the existence theorems for WFRFT of convolution are investigated.

Preliminaries. Let us review some basic facts on the n-dimensional FRFT and n-dimensional WFRFT, which will be needed throughout the letter.

Definition 1. The n-dimensional FRFT with rotational angle α of $f(t)$ on $t \in \mathbb{R}^n$ is defined as [8]

$$(\mathcal{F}_\alpha f)(\omega) = \widehat{f_\alpha}(\omega) = \int_{\mathbb{R}^n} f(t) K_\alpha(t, \omega) \mathrm{d}t, \tag{1}$$

where $\omega \in \mathbb{R}^n$,

$$K_\alpha(t, \omega) = C_\alpha \mathrm{e}^{\frac{\mathrm{i}(|t|^2 + |\omega|^2)\cot\alpha}{2} - \mathrm{i}\langle t, \omega\rangle \csc\alpha},$$

and $C_\alpha = [\pi(1 - \mathrm{e}^{-2\mathrm{i}\alpha})]^{\frac{-n}{2}}, \forall n \in \mathbb{Z}, \alpha \neq n\pi$.

The corresponding inversion formula is given by

$$f(t) = \mathcal{F}_\alpha^{-1}[\widehat{f_\alpha}](t) = \int_{\mathbb{R}^n} \widehat{f_\alpha}(\omega) \overline{K_\alpha(t, \omega)} \mathrm{d}\omega, \tag{2}$$

where $t \in \mathbb{R}^n$ and $\overline{K_\alpha(t, \omega)}$ is the conjugate of $K_\alpha(t, \omega)$.

If $\alpha = \frac{\pi}{2}$, then the FRFT becomes the Fourier transform.

Definition 2. Let $\psi \in L^2(\mathbb{R}^n)$ be a window function, $f \in L^2(\mathbb{R}^n)$. Then the n-dimensional WFRFT is defined as

$$(S_\psi f)(u, \xi) = \int_{\mathbb{R}^n} f(t) \overline{\psi}(t - u) K_\alpha(t, \xi) \mathrm{d}t, \tag{3}$$

where $u, \xi \in \mathbb{R}^n$.

From (1) and (3), we can obtain the relationship between the FRFT and WFRFT:

$$(S_\psi f)(u, \xi) = \int_{\mathbb{R}^n} h(t, u) K_\alpha(t, \xi) \mathrm{d}t = (\mathcal{F}_\alpha h)(\omega),$$

where $h(t, u) = f(t)\overline{\psi}(t - u)$.

* Corresponding author (email: li_bingzhao@bit.edu.cn)

According to (2), we have

$$f(t)\overline{\psi}(t-u)=\int_{\mathbb{R}^n}(S_\psi f)(u,\xi)\overline{K_\alpha(t,\omega)}\mathrm{d}\omega. \tag{4}$$

Let $\|\psi\|_2\neq 0$, and then we obtain the inversion of n-dimensional WFRFT:

$$f(t)=\frac{1}{\|\psi\|_2^2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(S_\psi f)(u,\xi)\times\overline{K_\alpha(t,\omega)}\psi(t-u)\mathrm{d}u\mathrm{d}\omega.$$

Based on the n-dimensional WFRFT, we study the convolution for the n-dimensional WFRFT, which satisfies the following form:

$$S_\psi(f\otimes g)(u,\xi)=(S_\psi f)(u,\xi)(S_\psi g)(u,\xi). \tag{5}$$

It shows that the application of the WFRFT to the convolution of two functions yields the product of their transforms.

Related theorem for the WFRFT. In this part, we obtain the window convolution theorem and existence theorems associated with the n-dimensional WFRFT by the above definitions.

Lemma 1. Let $\psi\in L^2(\mathbb{R}^n)$ be a window function, $f\in L^2(\mathbb{R}^n)$ and $g\in L^2(\mathbb{R}^n)$. Then the Parseval theorem of WFRFT can be obtained:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(S_\psi f)(u,\xi)\overline{(S_\psi g)(u,\xi)}\mathrm{d}u\mathrm{d}\xi=\|\psi\|_2^2\int_{\mathbb{R}^n}f(t)\overline{g(t)}\mathrm{d}t. \tag{6}$$

If $f=g$, then

$$\|(S_\psi f)(u,\xi)\|_2^2=\|\psi\|_2^2\|f\|_2^2. \tag{7}$$

The main results of this letter are summarized as the following theorems.

Theorem 1 (Convolution theorem for the WFRFT)**.** Let $\psi\in L^2(\mathbb{R}^n)$ be a window function, $f\in L^2(\mathbb{R}^n)$ and $g\in L^2(\mathbb{R}^n)$. Then for $t\in\mathbb{R}^n$, the window convolution is obtained by

$$(f\otimes g)(t)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}A_\xi(y,\eta,t)f(y)g(\eta)\mathrm{d}y\mathrm{d}\eta, \tag{8}$$

where

$$A_\xi(y,\eta,t)=C_\alpha^2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\widetilde{\psi_u}(x-y)\widetilde{\psi_u}(x-\eta)}{\widehat{\widetilde{\psi_{u,\alpha}}}(\omega)}\times\mathrm{e}^{\mathrm{i}\frac{|x|^2+|y|^2+|\omega|^2+|\eta|^2+|\xi|^2}{2}\cot\alpha}\mathrm{e}^{-\mathrm{i}\langle\eta+y,x\rangle\cot\alpha}\times\mathrm{e}^{\mathrm{i}\langle t-x,\omega\rangle\csc\alpha-\mathrm{i}\langle y+\eta-t,\xi\rangle\csc\alpha}\mathrm{d}x\mathrm{d}\omega,$$

$\widetilde{\psi_u}(t)=\overline{\psi}(-t)\mathrm{e}^{\mathrm{i}\langle u,t\rangle\cot\alpha-\mathrm{i}|t|^2\cot\alpha}$, and $\widehat{\widetilde{\psi_{u,\alpha}}}(\omega)\neq 0$, obviously, we have $f\otimes g=g\otimes f$.

Proof. In order to obtain Theorem 1, we first introduce the convolution operator of FRFT defined by

$$(f\star g)(a)=C_\alpha\int_{\mathbb{R}^n}\mathrm{e}^{-\mathrm{i}\langle t,a-t\rangle\cot\alpha}f(t)g(a-t)\mathrm{d}t=C_\alpha\mathrm{e}^{-\mathrm{i}a^2\cot\alpha}(f_1*g_1)(a),\quad a\in\mathbb{R}^n,$$

where $*$ is the classical convolution operator. According to Definition 1, Definition 2 and (5), we obtain

$$\mathcal{F}_\alpha[S_\psi(f\otimes g)(u,\xi)](\omega)=\mathcal{F}_\alpha\left[\left(\widetilde{f_\xi}\star\widetilde{\psi_u}\right)\times\left(\widetilde{g_\xi}\star\widetilde{\psi_u}\right)\right](\omega).$$

Using (2), we have

$$U(t,\xi)(f\otimes g)(t)=\mathcal{F}_\alpha^{-1}\left\{\frac{\mathrm{e}^{\frac{\mathrm{i}|\omega|^2\cot\alpha}{2}}}{\widehat{\widetilde{\psi_{u,\alpha}}}(\omega)}\mathcal{F}_\alpha\Big[\left(\widetilde{f_\xi}\star\widetilde{\psi_u}\right)\times\left(\widetilde{g_\xi}\star\widetilde{\psi_u}\right)\Big]\right\}(\omega),$$

where $U(t,\xi)=\mathrm{e}^{\frac{\mathrm{i}(|\xi|^2+|t|^2)\cot\alpha}{2}-\mathrm{i}\langle t,\xi\rangle\csc\alpha}$. Applying (2) to the above expression, we obtain

$$U(t,\xi)(f\otimes g)(t)=\int_{\mathbb{R}^n}\overline{K_\alpha(t,\omega)}\frac{\mathrm{e}^{\frac{\mathrm{i}|\omega|^2\cot\alpha}{2}}}{\widehat{\widetilde{\psi_{u,\alpha}}}(\omega)}\times\int_{\mathbb{R}^n}K_\alpha(x,\omega)\left(\widetilde{f_\xi}\star\widetilde{\psi_u}\right)(x)\times\left(\widetilde{g_\xi}\star\widetilde{\psi_u}\right)(x)\mathrm{d}x\mathrm{d}\omega.$$

By the convolution operator of FRFT, we can have the result.

Please refer to Appendix B for the detailed proof.

Remark 1. Let $\alpha=\frac{\pi}{2}$, in the above theorem, we obtain a convolution theorem for the windowed Fourier transform.

Next, according to the above theorem and definitions, we present three existence theorems for WFRFT of convolution.

Theorem 2. Let $\psi\in L^2(\mathbb{R}^n)$ be a window function, $f\in L^2(\mathbb{R}^n)$ and $g\in L^2(\mathbb{R}^n)$. Then

$$\|f\otimes g\|_2\leqslant\|f\|_2\|g\|_2\|\psi\|_2. \tag{9}$$

Proof. Please refer to Appendix C for this proof.

Theorem 3. Let $f\in L^p(\mathbb{R}^n)$, $g\in L^q(\mathbb{R}^n)$, $1<p,q<\infty$ and $\psi\in L^{p'}(\mathbb{R}^n)\cap L^{q'}(\mathbb{R}^n)$ is a window function, $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$. Then

$$|S_\psi(f\otimes g)(u,\xi)|\leqslant\|f\|_p\|\psi\|_{p'}\|g\|_q\|\psi\|_{q'}. \tag{10}$$

Proof. Please refer to Appendix D for this proof.

Theorem 4. Let $\psi\in L^2(\mathbb{R}^n)$ be a window function, and $f\in L^2(\mathbb{R}^n)$, $g\in L^2(\mathbb{R}^n)$. Then

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|S_\psi(f\otimes g)(u,\xi)|\mathrm{d}u\mathrm{d}\xi\right|\leqslant\|\psi\|_2^2\|f\|_2\|g\|_2. \tag{11}$$

Proof. Please refer to Appendix E for this proof.

Theorem 5. Let $\psi\in L^q(\mathbb{R}^n)$ be a window function, $f\in L^p(\mathbb{R}^n)$, $g\in L^p(\mathbb{R}^n)$, $1\leqslant p,q<\infty$, $\frac{1}{p}+\frac{1}{q}-1\geqslant 0$ and $f*g\in L^r(\mathbb{R}^n)$, $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1$. Then

$$\|S_\psi(f\otimes g)\|_r\leqslant C_\alpha\|f\|_p\|g\|_p\|\psi\|_q^2. \tag{12}$$

Proof. Please refer to Appendix F for this proof.

Remark 2. Let $\alpha=\frac{\pi}{2}$, in the above theorems, we obtain the existence theorems for the windowed Fourier transform of convolution.

Conclusion. In this letter, according to the definition of the n-dimensional FRFT, we propose the n-dimensional WFRFT and define a window convolution product associated with the n-dimensional FRFT. Moreover, based on the convolution product, an elegant analytical expression of window convolution is obtained in the n-dimensional fractional Fourier transform domain. Finally, using the new convolution of the n-dimensional WFRFT, several kinds of existence theorems for n-dimensional WFRFT are introduced. The results in this letter are new in the literature.

Further investigations on this topic are now under investigation such as the uncertainty principle, sampling theorem of the n-dimensional WFRFT. They will be reported in a forthcoming paper.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 61671063) and Foundation for Innovative Research Groups of National Natural Science Foundation of China (Grant No. 61421001).

Supporting information Appendixes A–F. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

References

1 Zhang C W, Shi J, Zhang Z M, et al. FRFT-based interference suppression for OFDM systems in IoT environment. IEEE Commun Lett, 2019, 23: 2068–2072

2 Tao R, Wang Y. Fractional Fourier Transform and Its Applications. Beijing: Tsinghua University Press, 2009

3 Upadhyay S K, Khatterwani K. Fractional wavelet transform through heat equation. J Thermal Stresses, 2019, 42: 1386–1414

4 Shi J, Zhang N T, Liu X P. A novel fractional wavelet transform and its applications. Sci China Inf Sci, 2012, 55: 1270–1279

5 Yu S S, Zhou N R, Gong L H, et al. Optical image encryption algorithm based on phase-truncated short-time fractional Fourier transform and hyper-chaotic system. Optics Lasers Eng, 2020, 124: 105816

6 Zhang Q. Uniqueness guarantees for phase retrieval from discrete windowed fractional Fourier transform. Optik, 2018, 158: 1491–1498

7 Gao W B, Li B Z. Quaternion windowed linear canonical transform of two-dimensional signals. Adv Appl Clifford Algebr, 2020, 30: 16

8 Kamalakkannan R, Roopkumar R. Multidimensional fractional Fourier transform and generalized fractional convolution. Integral Transforms Special Functions, 2020, 31: 152–165

9 Upadhyay S K, Dubey J K. Wavelet convolution product involving fractional fourier transform. Fractional Calculus Appl Anal, 2017, 20: 173–189