SCIENCE CHINA
Information Sciences

• **Supplementary File** •

Convolution theorem involving n-dimensional windowed fractional Fourier transform

Wen-Biao GAO[1,2] & Bing-Zhao LI[1,2*]

[1] *School of Mathematics and Statistics, Beijing Institute of Technology, Beijing* 100081, *China;*
[2] *Beijing Key Laboratory on MCAACI, Beijing Institute of Technology, Beijing* 100081, *China*

Appendix A Preliminaries

The n-dimensional FRFT with rotational angle α of $f(t)$ on $t \in \mathbb{R}^n$ is defined as

$$(\mathcal{F}_\alpha f)(\omega) = \widehat{f_\alpha}(\omega) = \int_{\mathbb{R}^n} f(t) K_\alpha(t,\omega) \mathrm{d}t, \quad \omega \in \mathbb{R}^n, \tag{A1}$$

where

$$K_\alpha(t,\omega) = \begin{cases} C_\alpha e^{\frac{i(|t|^2+|\omega|^2)\cot\alpha}{2} - i\langle t,\omega\rangle \csc\alpha}, & \alpha \neq n\pi, \\ \frac{e^{-i\langle t,\omega\rangle}}{(2\pi)^{\frac{n}{2}}}, & \alpha = \frac{\pi}{2}, \end{cases} \tag{A2}$$

and

$$C_\alpha = (2\pi i \sin\alpha)^{\frac{-n}{2}} e^{\frac{in\alpha}{2}} = [\pi(1-e^{-2i\alpha})]^{\frac{-n}{2}}, \quad \forall n \in \mathbb{Z}. \tag{A3}$$

The corresponding inversion formula is given by

$$f(t) = \mathcal{F}_\alpha^{-1}[\widehat{f_\alpha}](t) = \int_{\mathbb{R}^n} \widehat{f_\alpha}(\omega) \overline{K_\alpha(t,\omega)} \mathrm{d}\omega, \quad t \in \mathbb{R}^n, \tag{A4}$$

where

$$\overline{K_\alpha(t,\omega)} = \begin{cases} C'_\alpha e^{\frac{-i(|t|^2+|\omega|^2)\cot\alpha}{2} + i\langle t,\omega\rangle \csc\alpha}, & \alpha \neq n\pi, \\ \frac{e^{i\langle t,\omega\rangle}}{(2\pi)^{\frac{n}{2}}}, & \alpha = \frac{\pi}{2}, \end{cases} \tag{A5}$$

and

$$C'_\alpha = (2\pi i \sin\alpha)^{\frac{n}{2}} e^{\frac{-in\alpha}{2}} = [\pi(1-e^{-2i\alpha})]^{\frac{n}{2}}. \tag{A6}$$

Let $f, g \in L^2(\mathbb{R}^n)$, the convolution of f and g is defined by [1]

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y) g(x-y) \mathrm{d}y,$$

and the convolution operator of FRFT is defined by [2]

$$\begin{aligned} (f \star g)(a) &= C_\alpha \int_{\mathbb{R}^n} e^{-i\langle t, a-t\rangle \cot\alpha} f(t) g(a-t) \mathrm{d}t \\ &= C_\alpha e^{-ia^2\cot\alpha} (f_1 * g_1)(a), \quad a \in \mathbb{R}^n, \end{aligned} \tag{A7}$$

where $f_1(t) = e^{it\cot\alpha} f(t)$, $g_1(t) = e^{i\langle t,a\rangle \cot\alpha} g(t)$ and C_α is given by (A3). According to (A1) and (A7), we have

$$\mathcal{F}_\alpha[(f \star g)](\omega) = e^{\frac{-i|\omega|^2\cot\alpha}{2}} \widehat{f_\alpha}(\omega) \widehat{g_\alpha}(\omega). \tag{A8}$$

* Corresponding author (email: li_bingzhao@bit.edu.cn)

Let $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $\frac{1}{p}+\frac{1}{q}=1$, $1 \leqslant p,q < \infty$, then

$$\|f \star g\|_\infty \leqslant \|f\|_p \|g\|_q. \tag{A9}$$

By multiplying the function $f \in L^2(\mathbb{R}^n)$ with a window function $\psi \in L^2(\mathbb{R}^n)$ before taking the FRFT, the WFRFT is obtained

$$(S_\psi f)(u,\xi) = \int_{\mathbb{R}^n} f(t)\overline{\psi}(t-u)K_\alpha(t,\xi)\mathrm{d}t, \quad u,\xi \in \mathbb{R}^n. \tag{A10}$$

Let $h(t,u) = f(t)\overline{\psi}(t-u)$, then

$$(S_\psi f)(u,\xi) = \int_{\mathbb{R}^n} h(t,u)K_\alpha(t,\xi)\mathrm{d}t = (\mathcal{F}_\alpha h)(\omega). \tag{A11}$$

According to (A4), we have

$$f(t)\overline{\psi}(t-u) = \int_{\mathbb{R}^n} (S_\psi f)(u,\xi)\overline{K_\alpha(t,\omega)}\mathrm{d}\omega. \tag{A12}$$

Multiplying both sides of (A12) from the right by $\psi(t-u)$ and integrating with respect to $\mathrm{d}u$, we get

$$\int_{\mathbb{R}^n} f(t)\overline{\psi}(t-u)\psi(t-u)\mathrm{d}u = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (S_\psi f)(u,\xi)\overline{K_\alpha(t,\omega)}\psi(t-u)\mathrm{d}u\mathrm{d}\omega.$$

Hence, we have

$$\|\psi\|_2^2 f(t) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (S_\psi f)(u,\xi)\overline{K_\alpha(t,\omega)}\psi(t-u)\mathrm{d}u\mathrm{d}\omega.$$

Let $\|\psi\|_2 \neq 0$, then we obtain the inversion formula of n-dimensional WFRFT

$$f(t) = \frac{1}{\|\psi\|_2^2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (S_\psi f)(u,\xi)\overline{K_\alpha(t,\omega)}\psi(t-u)\mathrm{d}u\mathrm{d}\omega.$$

According to (A7) and (A10), the WFRFT can be expressed in term of convolution

$$(S_\psi f)(u,\xi) = (\widetilde{f}_\xi \star \widetilde{\psi_u})(u), \tag{A13}$$

where

$$\widetilde{f}_\xi(t) = f(t)e^{\frac{i(|\xi|^2+|t|^2)\cot\alpha}{2} - i\langle t,\xi\rangle\csc\alpha}; \quad \widetilde{\psi_u}(t) = \overline{\psi}(-t)e^{i\langle u,t\rangle\cot\alpha - i|t|^2\cot\alpha}. \tag{A14}$$

The window convolution product associated with the n-dimensional FRFT is defined by

$$S_\psi(f \otimes g)(u,\xi) = (S_\psi f)(u,\xi)(S_\psi g)(u,\xi). \tag{A15}$$

It is shown that the application of the WFRFT to the convolution of two functions yields the product of their transforms.

Appendix B Proof of Theorem 1

Proof. Using (A1) and (A13), we have

$$\mathcal{F}_\alpha[(S_\psi f)(u,\xi)](\omega) = \mathcal{F}_\alpha[(\widetilde{f}_\xi \star \widetilde{\psi_u})(u)](\omega). \tag{B1}$$

Applying (A8) to (B1), we get

$$\mathcal{F}_\alpha[(S_\psi f)(u,\xi)](\omega) = e^{\frac{-i|\omega|^2\cot\alpha}{2}}\widehat{\widetilde{f}_{\xi,\alpha}}(\omega)\widehat{\widetilde{\psi_{u,\alpha}}}(\omega). \tag{B2}$$

Using (A15), we have

$$\mathcal{F}_\alpha[S_\psi(f \otimes g)(u,\xi)](\omega) = \mathcal{F}_\alpha[(S_\psi f)(u,\xi)(S_\psi g)(u,\xi)](\omega). \tag{B3}$$

From (B1), we obtain

$$\mathcal{F}_\alpha[S_\psi(f \otimes g)(u,\xi)](\omega) = \mathcal{F}_\alpha[(\widetilde{f}_\xi \star \widetilde{\psi_u}) \times (\widetilde{g}_\xi \star \widetilde{\psi_u})](\omega). \tag{B4}$$

Applying (B2) to the above expression, we get

$$e^{\frac{-i|\omega|^2\cot\alpha}{2}}\mathcal{F}_\alpha[U(t,\xi)(f \otimes g)](\omega)\widehat{\widetilde{\psi_{u,\alpha}}}(\omega) = \mathcal{F}_\alpha[(\widetilde{f}_\xi \star \widetilde{\psi_u}) \times (\widetilde{g}_\xi \star \widetilde{\psi_u})](\omega), \tag{B5}$$

where $U(t,\xi)=e^{\frac{i(|\xi|^2+|t|^2)\cot\alpha}{2}-i\langle t,\xi\rangle\csc\alpha}$. Therefore, we obtain

$$\mathcal{F}_\alpha[U(t,\xi)(f\otimes g)](\omega)=\frac{e^{\frac{i|\omega|^2\cot\alpha}{2}}}{\widehat{\widetilde{\psi}_{u,\alpha}}(\omega)}\mathcal{F}_\alpha[(\widetilde{f}_\xi\star\widetilde{\psi_u})\times(\widetilde{g}_\xi\star\widetilde{\psi_u})](\omega), \tag{B6}$$

provided, $\widehat{\widetilde{\psi}_{u,\alpha}}(\omega)\neq 0$. Hence

$$U(t,\xi)(f\otimes g)(t)=\mathcal{F}_\alpha^{-1}\left\{\frac{e^{\frac{i|\omega|^2\cot\alpha}{2}}}{\widehat{\widetilde{\psi}_{u,\alpha}}(\omega)}\mathcal{F}_\alpha[(\widetilde{f}_\xi\star\widetilde{\psi_u})\times(\widetilde{g}_\xi\star\widetilde{\psi_u})]\right\}(\omega). \tag{B7}$$

Applying (A4) to the above expression, we obtain

$$\begin{aligned}U(t,\xi)(f\otimes g)(t)&=\int_{\mathbb{R}^n}\overline{K_\alpha(t,\omega)}\frac{e^{\frac{i|\omega|^2\cot\alpha}{2}}}{\widehat{\widetilde{\psi}_{u,\alpha}}(\omega)}\mathcal{F}_\alpha[(\widetilde{f}_\xi\star\widetilde{\psi_u})\times(\widetilde{g}_\xi\star\widetilde{\psi_u})]\mathrm{d}\omega\\&=\int_{\mathbb{R}^n}\overline{K_\alpha(t,\omega)}\frac{e^{\frac{i|\omega|^2\cot\alpha}{2}}}{\widehat{\widetilde{\psi}_{u,\alpha}}(\omega)}\int_{\mathbb{R}^n}K_\alpha(x,\omega)(\widetilde{f}_\xi\star\widetilde{\psi_u})(x)\times(\widetilde{g}_\xi\star\widetilde{\psi_u})(x)\mathrm{d}x\mathrm{d}\omega.\end{aligned} \tag{B8}$$

Using (A7), we have

$$\begin{aligned}U(t,\xi)(f\otimes g)(t)&=\int_{\mathbb{R}^n}\overline{K_\alpha(t,\omega)}\frac{e^{\frac{i|\omega|^2\cot\alpha}{2}}}{\widehat{\widetilde{\psi}_{u,\alpha}}(\omega)}\int_{\mathbb{R}^n}K_\alpha(x,\omega)C_\alpha\int_{\mathbb{R}^n}e^{-i\langle y,x-y\rangle\cot\alpha}\widetilde{f}_\xi(y)\widetilde{\psi_u}(x-y)\mathrm{d}y\\&\quad\times C_\alpha\int_{\mathbb{R}^n}e^{-i\langle\eta,x-\eta\rangle\cot\alpha}\widetilde{g}_\xi(\eta)\widetilde{\psi_u}(x-\eta)\mathrm{d}\eta\mathrm{d}x\mathrm{d}\omega\\&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\overline{K_\alpha(t,\omega)}K_\alpha(x,\omega)C_\alpha C_\alpha\frac{\widetilde{\psi_u}(x-y)\widetilde{\psi_u}(x-\eta)}{\widehat{\widetilde{\psi}_{u,\alpha}}(\omega)}e^{\frac{i|\omega|^2\cot\alpha}{2}}\\&\quad\times e^{-i\langle y,x-y\rangle\cot\alpha}e^{-i\langle\eta,x-\eta\rangle\cot\alpha}U(y,\xi)f(y)U(\eta,\xi)g(\eta)\mathrm{d}x\mathrm{d}y\mathrm{d}\omega\mathrm{d}\eta.\end{aligned} \tag{B9}$$

Hence

$$\begin{aligned}(f\otimes g)(t)&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\widetilde{\psi_u}(x-y)\widetilde{\psi_u}(x-\eta)}{\widehat{\widetilde{\psi}_{u,\alpha}}(\omega)}\frac{U(y,\xi)U(\eta,\xi)}{U(t,\xi)}\overline{K_\alpha(t,\omega)}K_\alpha(x,\omega)\\&\quad\times C_\alpha^2e^{\frac{i|\omega|^2\cot\alpha}{2}}e^{-i\langle y,x-y\rangle\cot\alpha}e^{-i\langle\eta,x-\eta\rangle\cot\alpha}\mathrm{d}x\mathrm{d}\omega f(\mathrm{y})\mathrm{g}(\eta)\mathrm{d}y\mathrm{d}\eta.\end{aligned} \tag{B10}$$

By (A2) and (A5), we obtain

$$\begin{aligned}(f\otimes g)(t)&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}C_\alpha^2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\widetilde{\psi_u}(x-y)\widetilde{\psi_u}(x-\eta)}{\widehat{\widetilde{\psi}_{u,\alpha}}(\omega)}e^{i\frac{|x|^2+|y|^2+|\omega|^2+|\eta|^2+|\xi|^2}{2}\cot\alpha}e^{-i\langle\eta+y,x\rangle\cot\alpha}\\&\quad\times e^{i\langle t-x,\omega\rangle\csc\alpha-i\langle y+\eta-t,\xi\rangle\csc\alpha}\mathrm{d}x\mathrm{d}\omega f(\mathrm{y})\mathrm{g}(\eta)\mathrm{d}y\mathrm{d}\eta.\end{aligned} \tag{B11}$$

Appendix C Proof of Theorem 2

Proof. From Lemma 1, we have

$$\|\psi\|_2\|f\otimes g\|_2=\|S_\psi(f\otimes g)(u,\xi)\|_2. \tag{C1}$$

Applying (A15) to the above expression, we obtain

$$\|\psi\|_2\|f\otimes g\|_2=\|(S_\psi f)(S_\psi g)\|_2=\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|(S_\psi f)(u,\xi)(S_\psi g)(u,\xi)|^2\mathrm{d}u\mathrm{d}\xi\right)^{\frac{1}{2}}. \tag{C2}$$

Using (A13), we obtain

$$|(S_\psi g)(u,\xi)|=|(\widetilde{g}_\xi\star\widetilde{\psi_u})(u)|. \tag{C3}$$

Hence, by (A9) we have

$$|(S_\psi g)(u,\xi)|\leqslant\|g\|_2\|\psi\|_2. \tag{C4}$$

From (C2) and (C4), we obtain

$$\begin{aligned}\|\psi\|_2\|f\otimes g\|_2 &\leqslant \|g\|_2\|\psi\|_2\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|(S_\psi f)(u,\xi)|^2\mathrm{d}u\mathrm{d}\xi\right)^{\frac{1}{2}}\\ &= \|g\|_2\|\psi\|_2\|S_\psi f\|_2.\end{aligned}\tag{C5}$$

Using Lemma 1, we obtain

$$\|f\otimes g\|_2 \leqslant \|f\|_2\|g\|_2\|\psi\|_2.\tag{C6}$$

Appendix D Proof of Theorem 3

Proof. Using (A15), we have

$$|S_\psi(f\otimes g)(u,\xi)| = |(S_\psi f)(u,\xi)(S_\psi g)(u,\xi)|.\tag{D1}$$

From (A13), we have

$$|S_\psi(f\otimes g)(u,\xi)| = |(\widetilde{f}_\xi\star\widetilde{\psi_u})(u)||(\widetilde{g}_\xi\star\widetilde{\psi_u})(u)|.\tag{D2}$$

By (A9), we get

$$|S_\psi(f\otimes g)(u,\xi)| \leqslant \|f\|_p\|\psi\|_{p'}\|g\|_q\|\psi\|_{q'}.\tag{D3}$$

Appendix E Proof of Theorem 4

Proof. Using (A15), we have

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|S_\psi(f\otimes g)(u,\xi)|\mathrm{d}u\mathrm{d}\xi\right| = \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|(S_\psi f)(u,\xi)(S_\psi g)(u,\xi)|\mathrm{d}u\mathrm{d}\xi\right|.\tag{E1}$$

By Lemma 1, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|S_\psi(f\otimes g)(u,\xi)|\mathrm{d}u\mathrm{d}\xi\right| &= \|\psi\|_2^2\left|\int_{\mathbb{R}^n}f(t)g(t)\mathrm{d}t\right|\\ &\leqslant \|\psi\|_2^2\|f\|_2\|g\|_2.\end{aligned}\tag{E2}$$

Appendix F Proof of Theorem 5

Proof. According to (A7) and the Young's inequality [1, 3], we obtain

$$\begin{aligned}\|f\star\psi\|_r &= \left(\int_{\mathbb{R}^n}|C_\alpha e^{-ia^2\cot\alpha}(f_1*\psi_1)(a)|^r\mathrm{d}a\right)^{\frac{1}{r}}\\ &= C_\alpha\|f_1*\psi_1\|_r\\ &\leqslant C_\alpha\|f_1\|_p\|\psi_1\|_q,\end{aligned}\tag{F1}$$

where $f_1(t)=e^{it\cot\alpha}f(t)$, $\psi_1(t)=e^{i\langle t,a\rangle\cot\alpha}\psi(t)$ and C_α is given by (A3).

Note that $\|f_1\|_p=\|f\|_p$, $\|\psi_1\|_q=\|\psi\|_q$. Hence, we get

$$\|f\star\psi\|_r \leqslant C_\alpha\|f\|_p\|\psi\|_q.\tag{F2}$$

Using (A13) and (F2), we have

$$\|S_\psi f\|_r = \|\widetilde{f}_\xi\star\widetilde{\psi}_u\|_r \leqslant C_\alpha\|f\|_p\|\psi\|_q,\tag{F3}$$

where $\widetilde{f}_\xi$ and $\widetilde{\psi}_u$ are given by (A14).

By (A15) and the proof of Theorem 2. we have

$$\begin{aligned}\|S_\psi(f\otimes g)\|_r &= \|(S_\psi f)\times(S_\psi g)\|_r\\ &= \|g\|_p\|\psi\|_q\|S_\psi f\|_r.\end{aligned}\tag{F4}$$

Substituting (F3) into (F4), we obtain

$$\|S_\psi(f\otimes g)\|_r \leqslant C_\alpha\|f\|_p\|g\|_p\|\psi\|_q^2.\tag{F5}$$

References

1 Chui C K. Atlantis Studies in Mathematics for Engineering and Science. World Scientific, 2009

2 Upadhyay S K, Dubey J K. Wavelet convolution product involving fractional fourier transform. Fract Calc Appl Anal., 2017, 20: 173-189

3 Kecks W. The Convolution Product and Some Applications, D Reidel Publishing Co., Dordrecht, Holland, 1982