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State-feedback set stabilization of logical control networks with state-dependent delay

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Dear editor,

As an important mathematical model for the elucidation, analysis, and control of gene regulatory networks (GRNs), logical networks have attracted considerable attention from scientists in numerous fields of study. From a mathematical point of view, a logical network is parameter-free; thus, it can be utilized to reconstruct relatively large-scale GRNs. In studying logical networks, stabilization is highly critical as it has been widely used in maintaining human health and treating diseases [1, 2].

It is worth noting that the finite speed inherent in some biochemical reactions accounts for the prevalent time delay in the modeling of GRNs [3], which may cause undesirable performance or even annihilate the controllability of logical control networks. However, as an important class of time-delay systems, there exist few successful results by using logical control networks with state-dependent delay (SDD), and this motivates its consideration in this study.

A logical network in which the states and outputs have k different values is known as k-valued logical network. In this study, we convert the dynamical equations of a k-valued logical network into a bilinear discrete-time system by the semi-tensor product (STP) method, and then utilize classical control theory to analyze and control the system [4–7].

Before presenting the main results of this study, we introduce some necessary preparations, that will be used in what follows. For more details, please refer to [5].

Notations. $\mathbf{1}_r^{\mathrm{T}} = [\underbrace{1\ 1\ \cdots\ 1}_{r}]$. δ_n^i denotes the i-th column of the identity matrix I_n and $\Delta_n := \{\delta_n^i : 1 \leqslant i \leqslant n\}$. $W_{[m,n]}$ represents a swap matrix. A matrix L is called a logical matrix if each column of L has only one non-zero element, which is 1. Further, $\mathcal{L}_{s\times r}$ denotes the set of $s \times r$ logical matrices. $A > 0$ implies that every element of A is strictly positive.

Definition 1 ([5])**.** Assume $M \in \mathbb{M}_{s\times t}$, $N \in \mathbb{M}_{p\times r}$ and l is the least common multiple of t and p. We define the STP of M and N as

$$M \ltimes N = \left(M \otimes I_{\frac{l}{t}}\right)\left(N \otimes I_{\frac{l}{p}}\right).$$

Throughout this study, we will omit the symbol "$\ltimes$" if no confusion is raised.

Definition 2. Define the operator $D[p,q,s] = \mathbf{1}_p^{\mathrm{T}} \otimes I_q \otimes \mathbf{1}_s^{\mathrm{T}}$, and the power-reducing matrix $M_{r,k} = \delta_{k^2}[1, k+2, 2k+3, \ldots, k^2]$.

Problem description. The k-valued logical control network with SDD is described as follows:

$$\begin{cases} A_1(t+1) = f^1(X(t-\mu(X(t))), U(t)), \\ A_2(t+1) = f^2(X(t-\mu(X(t))), U(t)), \\ \vdots \\ A_n(t+1) = f^n(X(t-\mu(X(t))), U(t)), \end{cases} \tag{1}$$

where $X(t) = (A_1(t), A_2(t), \ldots, A_n(t))$ and $U(t) = (U_1(t), \ldots, U_m(t))$ represent the state and control input at time t, respectively; $\mu(X(t)) = g(X(t)) \in \{0, \ldots, \mu^*\}$ is an SDD, g is a pseudo-logical function, and f^i are logical functions for $i = 1, \ldots, n$.

Algebraic formulation. Following the conversion process presented in [5, 8], herein, we consider logical variables as canonical vectors and set $a(t) = \ltimes_{i=1}^n A_i(t) \in \Delta_{k^n}$, $b(t) = \ltimes_{i=t-\mu^*}^t A(i) \in \Delta_{k^{(\mu^*+1)n}}$, and $u(t) = \ltimes_{i=1}^m U_i(t) \in \Delta_{k^m}$. The SDD can be converted into the algebraic form as follows: $\mu(a(t)) = \hat{G}a(t)$. Then, the following equations can be derived:

$$\begin{aligned} A_i(t+1) &= V_i(\mu(a(t)))u(t)b(t) \\ &= [V_i(0)\cdots V_i(\mu^*)]\hat{G}a(t)u(t)b(t) \\ &= [V_i(0)\cdots V_i(\mu^*)]\hat{G}W_{[k^{m+(\mu^*+1)n},k^n]}(I_{k^{m+\mu^*n}} \\ &\quad \otimes M_{r,k^n})u(t)b(t) \\ &:= L_i u(t)b(t), \end{aligned} \tag{2}$$

where $i = 1, \ldots, n$.

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Multiplying equation (2) for all i yields the algebraic description of system (1) as follows:

$$a(t+1) = Lu(t)b(t), \quad (3)$$

where $L = L_1 * L_2 * \cdots * L_n \in \mathcal{L}_{k^n \times k^{m+(\mu^*+1)n}}$ and $*$ is Khatri-Rao product [5].

Via a straightforward calculation, we construct the augmented system as follows:

$$\begin{aligned} b(t+1) &= \ltimes_{i=t-\mu^*+1}^{t+1} a(i) \\ &= D[k^n, k^{\mu^* n}, 1] W_{[k^n, k^{(\mu^*+1)n}]} L(I_{k^m} \\ &\quad \otimes M_{r,k^{(\mu^*+1)n}}) u(t)b(t) \\ &:= Pu(t)b(t). \end{aligned} \quad (4)$$

Following Algorithm 1 presented in [8], one can reconstruct the original system (1) from the augmented system (4). Therefore, we have the following result.

Proposition 1. The original system (1) is equivalent to the augmented system (4).

Control invariance for logical control networks with SDD.

Definition 3. A nonempty set $\boldsymbol{\Lambda} \subseteq \Delta_{k^{(\mu^*+1)n}}$ is called a control invariant set for (4) if there exists a state feedback controller $u(t)$ given as

$$u_i(t) = h^i(a(t - \mu(a(t)))), \quad i = 1, \ldots, m, \quad (5)$$

under which $b(t) \in \boldsymbol{\Lambda}$ implies $b(t+1) \in \boldsymbol{\Lambda}$, where $h^i : \Delta_{k^n} \to \Delta_k$.

By utilizing the STP method, we obtain the algebraic description of a control $u(t)$ as follows:

$$u(t) = Hb(t). \quad (6)$$

Further, considering the augmented system (4), we construct the following matrix:

$$E = P\mathbf{1}_{k^m}, \quad (7)$$

which indicates whether any two states are reachable [9].

By setting $\boldsymbol{\Lambda} = \{\delta_{k^{(\mu^*+1)n}}^{\theta_1}, \ldots, \delta_{k^{(\mu^*+1)n}}^{\theta_\nu}\}$ with $\theta_1 < \cdots < \theta_\nu$, we simplify the matrix E as

$$E_{\boldsymbol{\Lambda}} := \begin{bmatrix} E_{\theta_1,\theta_1} & \cdots & E_{\theta_1,\theta_\nu} \\ \vdots & \vdots & \vdots \\ E_{\theta_\nu,\theta_1} & \cdots & E_{\theta_\nu,\theta_\nu} \end{bmatrix}. \quad (8)$$

Based on $E_{\boldsymbol{\Lambda}}$, we obtain the following conclusion.

Theorem 1. $\boldsymbol{\Lambda}$ is a control invariant set for the augmented system (4) if and only if $\sum_{i=1}^{\nu} \mathrm{Row}_{\theta_i}(E_{\boldsymbol{\Lambda}}) > 0$.

Set stabilization. Given a nonempty set $\boldsymbol{M} = \{\delta_{k^n}^{\alpha_1}, \ldots, \delta_{k^n}^{\alpha_r}\}$, by multiplying any $\mu^* + 1$ elements of $\boldsymbol{M}$, we construct the augmented state set $\boldsymbol{N} = \{\delta_{k^{(\mu^*+1)n}}^{\gamma_1}, \ldots, \delta_{k^{(\mu^*+1)n}}^{\gamma_{r^{\mu^*+1}}}\}$.

Based on the definition of set stabilization presented in [1] and Proposition 1, we have the following conclusion.

Proposition 2. Under the state feedback control, system (1) is stabilized to $\boldsymbol{M}$ if and only if system (4) is stabilized to $\boldsymbol{N}$.

Based on the nonempty set $\boldsymbol{\Lambda}$, we construct the following row vector:

$$\Xi_{\boldsymbol{\Lambda}}^{\phi} = \Sigma_{i=1}^{\nu} \mathrm{Row}_{\theta_i}(E^{\phi}), \quad (9)$$

where E is defined in (7) and ϕ is a positive integer.

Theorem 2. System (1) is stabilized to $\boldsymbol{M}$ via the state feedback controller (4) if and only if there exist a positive integer $\phi \leqslant k^{(\mu^*+1)n}$ and a control invariant set $\boldsymbol{\Lambda} \subseteq \boldsymbol{N}$ satisfying

$$\Xi_{\boldsymbol{\Lambda}}^{\phi} > 0. \quad (10)$$

Proof. Let $Y_\rho(\boldsymbol{\Lambda})$ consist of all the initial state sequences that reach the set $\boldsymbol{\Lambda}$ at the ρ-th step.

Suppose (10) holds; then, based on Definition 3, we have that

$$\begin{cases} \boldsymbol{\Lambda} \subseteq Y_1(\boldsymbol{\Lambda}), \\ Y_\phi(\boldsymbol{\Lambda}) = \Delta_{k^{(\mu^*+1)n}}. \end{cases} \quad (11)$$

Now, define

$$Y_s^{\circ}(\boldsymbol{\Lambda}) = Y_s(\boldsymbol{\Lambda}) \backslash Y_{s-1}(\boldsymbol{\Lambda}), \quad s = 1, \ldots, \phi. \quad (12)$$

In particularly, $Y_0(\boldsymbol{\Lambda}) := \emptyset$. Then, for any state $\delta_{k^{(\mu^*+1)n}}^{\sigma}$, there exists a unique positive integer $\phi_\sigma \leqslant \phi$ such that $\delta_{k^{(\mu^*+1)n}}^{\sigma} \in Y_{\phi_\sigma}^{\circ}(\boldsymbol{\Lambda})$. Moreover, if $\phi_\sigma = 1$, there exists a positive integer $l_\sigma \leqslant k^m$ such that $\delta_{k^{(\mu^*+1)n}}^{\theta(1;\delta_{k^{(\mu^*+1)n}}^{\sigma}, \delta_{k^m}^{l_\sigma})} \in \boldsymbol{\Lambda}$. Furthermore, if $2 \leqslant \phi_\sigma \leqslant k^{(\mu^*+1)n}$, there exists a positive integer $l_\sigma \leqslant k^m$ such that $\delta_{k^{(\mu^*+1)n}}^{\theta(1;\delta_{k^{(\mu^*+1)n}}^{\sigma}, \delta_{k^m}^{l_\sigma})} \in Y_{\phi_\sigma - 1}(\boldsymbol{\Lambda})$.

Set $H = \delta_{k^m}[l_1 \ l_2 \ \cdots \ l_{k^{(\mu^*+1)n}}] \in \mathcal{L}_{k^m \times k^{(\mu^*+1)n}}$. For every $b(0) = \delta_{k^{(\mu^*+1)n}}^{\sigma}$, under the control $u(t) = Hb(t)$, we have that

$$\begin{aligned} b(1) \ &= \ PHb(0)b(0) = \delta_{k^{(\mu^*+1)n}}^{\theta(1;\delta_{k^{(\mu^*+1)n}}^{\sigma}, \delta_{k^m}^{l_\sigma})} \\ &\in \begin{cases} \boldsymbol{\Lambda}, \ \text{if } \phi_\sigma = 1; \\ Y_{\phi_\sigma - 1}(\boldsymbol{\Lambda}), \ \text{if } 2 \leqslant \phi_\sigma \leqslant \phi. \end{cases} \end{aligned} \quad (13)$$

Define $\kappa = \max\{\phi_1, \ldots, \phi_{k^{(\mu^*+1)n}}\}$. When $t \geqslant \kappa$, one can easily obtain that $b(t) \in \boldsymbol{\Lambda} \subseteq \boldsymbol{N}$; thus, system (4) is stabilized to $\boldsymbol{N}$ via $u(t) = \delta_{k^m}[l_1 \ l_2 \cdots l_{k^{(\mu^*+1)n}}]b(t)$. By Proposition 2, system (1) is stabilized to $\boldsymbol{M}$ via $u(t) = Hb(t)$.

Conversely, suppose system (1) is stabilized to $\boldsymbol{M}$ via $u(t) = Hb(t)$; then, the augmented system (4) is stabilized to $\boldsymbol{N}$. Via a straightforward calculation, we obtain a closed-loop system given by

$$b(t+1) = \hat{Q}b(t), \quad (14)$$

where $\hat{Q} = PHM_{r,k^{(\mu^*+1)n}} \in \mathcal{L}_{k^{(\mu^*+1)n} \times k^{(\mu^*+1)n}}$.

Let $\boldsymbol{\Lambda}$ comprise all the attractors of the system represented by (14). Clearly, $\boldsymbol{\Lambda} \subseteq \boldsymbol{N}$ and $\boldsymbol{\Lambda} \subseteq Y_1(\boldsymbol{\Lambda})$. Moreover, Definition 3 implies that $\boldsymbol{\Lambda}$ is a control invariant set of system (4). Let $\phi \leqslant k^{(\mu^*+1)n}$ denote the transient period of system (14). Then, we have that $Y_\phi(\boldsymbol{\Lambda}) = \Delta_{k^{(\mu^*+1)n}}$, which implies that Eq. (10) holds.

Next, we exploit a numerical example to illustrate the results obtained in this study.

Example 1. Consider the following logical control network with SDD:

$$\begin{cases} A_1(t+1) = u(t) \wedge \{\neg A_1(t - \mu(A(t))) \vee \neg A_2(t)\}, \\ A_2(t+1) = u(t) \wedge \{\neg A_1(t - \mu(A(t))) \wedge \neg A_2(t)\}, \end{cases}$$

where $\mu(0,0) = 0$ and $\mu(0,1) = \mu(1,0) = \mu(1,1) = 1$.

We aim to design a state-feedback controller (if it exists) such that system (15) is stabilized to the set $\boldsymbol{M} = \{(0,0), (1,1)\}$.

Set $a(t) = \ltimes_{i=1}^{2} A_i(t)$, $b(t) = \ltimes_{i=t-1}^{t} a(i)$. Then, we have $\mu(a(t)) = \delta_2[1\ 2\ 2\ 2]a(t)$. Based on the research in [5], every dynamical equation in (15) can be converted into the following algebraic description:

$$\begin{cases} A_1(t+1) = \delta_2[2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 1\ 1\ 2\ 1\ 1\ 1\ 2 \\ \ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2]u(t)b(t), \\ A_2(t+1) = \delta_2[2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2 \\ \ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2]u(t)b(t). \end{cases}$$

Multiplying these equations yields the algebraic form of system (15) as

$$a(t+1) = \delta_4[4\ 2\ 4\ 2\ 4\ 2\ 4\ 2\ 4\ 1\ 2\ 1\ 4\ 1\ 2\ 1\ 4\ 4\ 4 \\ 4\ 4\ \ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4]u(t)b(t).$$

From (4), we derive the augmented system as

$$b(t+1) = Pu(t)b(t), \tag{15}$$

where $P = \delta_{16}[4\ 6\ 12\ 14\ 4\ 6\ 12\ 14\ 4\ 5\ 10\ 13\ 4\ 5$ $10\ 13\ 4\ 8\ 12\ 16\ 4\ 8\ 12\ 16\ 4\ 8\ 12\ 16\ 4\ 8\ 12\ 16]$.

Moreover, the canonical vector form of $\boldsymbol{M}$ is $\{\delta_4^1, \delta_4^4\}$; thus, $\boldsymbol{N} = \{\delta_{16}^1, \delta_{16}^4, \delta_{16}^{13}, \delta_{16}^{16}\}$.

From a straightforward calculation, one can find a control invariant set $\boldsymbol{N'} = \{\delta_{16}^{16}\} \subseteq \boldsymbol{N}$, such that $\Xi_{B'}^2 > 0$.

Hence, by Theorem 2, system (15) is stabilized to the set $\boldsymbol{M}$ by

$$u(t) = \delta_{16}[1\ 2\ 2\ 2\ 1\ 2\ 1\ 2\ 2\ 2\ 2\ 2\ 1\ 2\ 2\ 2]b(t).$$

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