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Finite-Time Adaptive Robust Simultaneous Stabilization of Nonlinear Delay Systems by the Hamiltonian Function Method

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Appendix A Proof of Theorem 1

Proof. Without loss of generality, we let $a = 1$ and $b = 2$ in the proof.

Substituting u into the system, one can obtain that

$$\begin{aligned}\dot{X}(t) \quad = \quad & A(X,\varepsilon)\nabla_X H(X) + B(X,\tilde{X},\varepsilon)\nabla_{\tilde{X}} H(\tilde{X}) + Q(X,\varepsilon) - k_1 sign(\nabla_X H(X))|\nabla_X H(X)|^{\alpha} \\ & - \nabla^T_{\tilde{X}} H(\tilde{X})\nabla_{\tilde{X}} H(\tilde{X})\frac{\nabla_X H(X)}{2\|\nabla_X H(X)\|^2} + G(X)v_2 + G(X)\Phi\tilde{\theta} + q(X)w,\end{aligned} \tag{A1}$$

where $H(X) = \sum_{i=1}^{2} H_i(X_i)$, $\nabla_X H(X) = [\nabla^T_{X_1} H_1(X_1), \nabla^T_{X_2} H_2(X_2)]^T$, $B(X,\tilde{X},\varepsilon) = Diag\{B_1(X_1,\tilde{X}_1,\varepsilon), B_2(X_2,\tilde{X}_2,\varepsilon)\}$, $q(X) = [q_1^T(X_1), q_2^T(X_2)]^T$, $G(X) = [g_1^T(X_1), g_2^T(X_2)]^T$, $Q(X,\varepsilon) = [Q_1^T(X_1,\varepsilon), Q_2^T(X_2,\varepsilon)]^T$, $\tilde{\theta} = \theta - \hat{\theta}$,

$$A(X,\varepsilon) = \begin{bmatrix} A_{11}(X_1,\varepsilon) & A_{12}(X_1,X_2) \\ A_{21}(X_1,X_2) & A_{22}(X_2,\varepsilon) \end{bmatrix} := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{A2}$$

$A_{11} = D_1(X_1,\varepsilon) - g_1(X_1)Kg_1^T(X_1)$, $A_{12} = g_1(X_1)Kg_2^T(X_2)$, $A_{21} = -g_2(X_2)Kg_1^T(X_1)$, $A_{22} = D_2(X_2,\varepsilon) + g_2(X_2)Kg_2^T(X_2)$.

Set $J(X) = 2H(X) + (\theta-\hat{\theta})^T\Upsilon^{-1}(\theta-\hat{\theta}) + \int_0^t(\|z(s)\|^2 - \gamma^2\|w(s)\|^2)ds$. Computing $\dot{H}(X)$, using v_2, Condition (i) of the theorem and $\nabla^T_X H(X)Q(X,\varepsilon) = 0$, we obtain $2\dot{H}(X)$

$$\begin{aligned} \leqslant \quad & \nabla^T_X H(X)\Big[A(X,\varepsilon) + A^T(X,\varepsilon) + B(X,\tilde{X},\varepsilon)B^T(X,\tilde{X},\varepsilon) + \zeta q(X)q^T(X) - \frac{1}{\gamma^2}G(X)G^T(X)\Big]\nabla_X H(X) \\ & + 2\nabla^T_X H(X)G(X)\Phi\tilde{\theta} - 2k_1\nabla^T_X H(X) sign(\nabla_X H(X))|\nabla_X H(X)|^{\alpha} - \|z\|^2 + \zeta^{-1}w^T w \\ \leqslant \quad & -2k_1\nabla^T_X H(X) sign(\nabla_X H(X))|\nabla_X H(X)|^{\alpha} - \|z\|^2 + \zeta^{-1}w^T w + 2\nabla^T_X H(X)G(X)\Phi\tilde{\theta}, \end{aligned} \tag{A3}$$

from which, $\gamma^2 \geqslant \zeta^{-1}$ and $\dot{\hat{\theta}} = \Upsilon\Phi^T G^T(X)\nabla_X H(X)$, we obtain $\dot{J}(X) \leqslant -2k_1\nabla^T_X H(X) sign(\nabla_X H(X))|\nabla_X H(X)|^{\alpha} \leqslant 0$.

Integrating $\dot{J}(X)$, using the zero state condition and $H(X) + (\theta-\hat{\theta})^T\Upsilon^{-1}(\theta-\hat{\theta}) \geqslant 0$, one can obtain

$$J(X) = 2H(X) + (\theta-\hat{\theta})^T\Upsilon^{-1}(\theta-\hat{\theta}) + \int_0^t(\|z(s)\|^2 - \gamma^2\|w(s)\|^2)ds \leqslant 0, \tag{A4}$$

which implies that $\int_0^T \|z(t)\|^2 dt \leqslant \gamma^2 \int_0^T \|w(t)\|^2 dt$.

Now, we prove $X \in \Omega := \cup_{i=1}^{2}\Omega_i$ for $\forall t > 0$, $\phi = 0$, ε and $w \in \Theta$. From (A4), we obtain

$$2H(X) \leqslant -(\theta-\hat{\theta})^T\Upsilon^{-1}(\theta-\hat{\theta}) - \int_0^t(\|z(s)\|^2 - \gamma^2\|w(s)\|^2)ds \leqslant \gamma^2\int_0^t \|w(s)\|^2 ds, \tag{A5}$$

with which and Assumption 2, one can obtain that

$$H(X) \leqslant 0.5\gamma^2\int_0^t \|w(s)\|^2 ds \leqslant \frac{\gamma^2}{2\mu^2}. \tag{A6}$$

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In addition, noting that $H(X) = H_1(X_1) + H_2(X_2)$ and Assumption 3, we have $H(X) \geqslant \alpha_1\|X_1\|^2 + \alpha_2\|X_2\|^2 \geqslant \min\{\alpha_1,\alpha_2\}[\|X_1\|^2 + \|X_2\|^2] = \min\{\alpha_1,\alpha_2\}\|X\|^2$, from which one can obtain that

$$\|X\|^2 \leqslant \frac{\gamma^2}{2\mu^2\min\{\alpha_1,\alpha_2\}}. \tag{A7}$$

Using the S-procedure [1], we have

$$Z^T\begin{bmatrix} 2s-\frac{\gamma^2}{2\mu^2\min\{\alpha_1,\alpha_2\}} & -s(\sigma_\kappa)^T \\ -s\sigma_\kappa & I_{n_1+n_2}\end{bmatrix}Z \geqslant 0,\quad \kappa = 1,2,\cdots,n_1+n_2,$$

where $Z = [1, X^T]^T$ and $s > 0$ is a scalar. From Condition (ii), one can obtain that X remains in Ω for $w \in \Theta$.

Third, under $w = 0$, we show the system (A1) is finite-time stability. Obviously, the system is asymptotically stable if $w = 0$, which implies that $\hat\theta \to \theta$. Therefore, there exists some constant number $\nu > 0$ such that $\|\hat\theta(t)\| \leqslant \nu,\ \ t > 0$.

Now, we show that the derivative condition $\dot V(t,\phi) \leqslant -\kappa(V(t,\phi))^{\frac{1}{\beta}}$ holds for $w = 0$.

From (A4) and $\dot J(X) \leqslant -2k_1\nabla_X^T H(X)sign(\nabla_X H(X))|\nabla_X H(X)|^\alpha \leqslant 0$, we have

$$2\dot H(X) \leqslant -2k_1\nabla_X^T H(X)sign(\nabla_X H(X))|\nabla_X H(X)|^\alpha + 2(\theta-\hat\theta)^T\Upsilon^{-1}\dot{\hat\theta}. \tag{A8}$$

For $\nabla_X^T H(X)sign(\nabla_X H(X))|\nabla_X H(X)|^\alpha$, using Assumption 3 and Lemma 2, it is easy to obtain that

$$\begin{aligned}\nabla_X^T H(X)sign(\nabla_X H(X))|\nabla_X H(X)|^\alpha &= |\frac{\partial H(X)}{\partial x_{11}}|^{1+\alpha} + \cdots + |\frac{\partial H(X)}{\partial x_{1n_1}}|^{1+\alpha} + |\frac{\partial H(X)}{\partial x_{21}}|^{1+\alpha} + \cdots + |\frac{\partial H(X)}{\partial x_{2n_2}}|^{1+\alpha}\\ \geqslant \Big((\frac{\partial H(X)}{\partial x_{11}})^2 + \cdots + (\frac{\partial H(X)}{\partial x_{2n_2}})^2\Big)^{\frac{1+\alpha}{2}} &\geqslant m^{\frac{1+\alpha}{2}}(H_1(X_1)+H_2(X_2))^{\frac{1+\alpha}{2}} = m^{\frac{1+\alpha}{2}}(H(X))^{\frac{1+\alpha}{2}},\end{aligned} \tag{A9}$$

where $m := \min\{\iota_1\varrho_1^{-1},\ \iota_2\varrho_2^{-1}\}$. Substituting (A9) into (A8), and noting that $\dot{\hat\theta} = \Upsilon\Phi^T\Big(g_1^T(X_1)\nabla_{X_1}H_1(X_1) + g_2^T(X_2)\nabla_{X_2}H_1(X_2)\Big)$, one can obtain

$$\dot H(X) \leqslant -k_1 m^{\frac{1+\alpha}{2}}(H(X))^{\frac{1+\alpha}{2}} + (\theta-\hat\theta)^T\Phi^T G^T(X)\nabla_X H(X). \tag{A10}$$

For $(\theta-\hat\theta)^T\Phi^T G^T(X)\nabla_X H(X)$, noting that $\|\hat\theta(t)\| \leqslant \nu$, θ is a bounded constant vector, we obtain

$$(\theta-\hat\theta)^T\Phi^T G^T(X)\nabla_X H(X) \leqslant \|(\theta-\hat\theta)^T\|\|\Phi^T G^T(X)\nabla_X H(X)\| \leqslant M\|\Phi^T G^T(X)\nabla_X H(X)\|, \tag{A11}$$

where $M := \max\{\|\theta\|\} + \nu > 0$ is a constant number.

In the following, we will prove that $\|\Phi^T G^T(X)\nabla_X H(X)\|$ is higher degree of $(H(X))^{\frac{1+\alpha}{2}}$. To do this, we first show that there exists a matrix $\chi > 0$ such that $G(X)\Phi\Phi^T G^T(X) \leqslant \chi H(X)$ is true. In fact, noting that $B_i(X_i,\tilde X_i,\varepsilon)\triangle_{H_i}(\tilde X_i,\varepsilon) = g_i(X_i)\Phi\theta$ and $B_i(X_i,\tilde X_i,\varepsilon) := \zeta_i(X_i)L_i(\tilde X_i,\varepsilon)$, one can obtain that $g_i(X_i)\Phi\theta = \zeta_i(X_i)L_i(\tilde X_i,\varepsilon)\triangle_{H_i}(\tilde X_i,\varepsilon)$, thus we have $g_i(X_i)\Phi\theta\theta^T\Phi^T g_i^T(X_i) \leqslant \lambda_{\max}\{L_i(\tilde X_i,\varepsilon)\triangle_{H_i}(\tilde X_i,\varepsilon)\triangle_{H_i}^T(\tilde X_i,\varepsilon)L_i^T(\tilde X_i,\varepsilon)\}\zeta_i(X_i)\zeta_i^T(X_i)$, from which and $g_i(X_i)\Phi\theta\theta^T\Phi^T g_i^T(X_i) \geqslant \lambda_{\min}\{\theta\theta^T\}g_i(X_i)\Phi\Phi^T g_i^T(X_i)$, we obtain that $g_i(X_i)\Phi\Phi^T g_i^T(X_i) \leqslant$

$$\lambda_{\max}\{L_i(\tilde X_i,\varepsilon)\triangle_{H_i}(\tilde X_i,\varepsilon)\triangle_{H_i}^T(\tilde X_i,\varepsilon)L_i^T(\tilde X_i,\varepsilon)\}\zeta_i(X_i)\zeta_i^T(X_i)(\lambda_{\min}\{\theta\theta^T\})^{-1} \leqslant m_{i1}\zeta_i(X_i)\zeta_i^T(X_i), \tag{A12}$$

where $\lambda_{\max}\{L_i(\tilde X_i,\varepsilon)\triangle_{H_i}(\tilde X_i,\varepsilon)\triangle_{H_i}^T(\tilde X_i,\varepsilon)L_i^T(\tilde X_i,\varepsilon)\}\lambda_{\min}\{\theta_i\theta_i^T\}^{-1} := m_{i1}$. For $\zeta_i(X_i)\zeta_i^T(X_i)$, based on Lemma 5, we have $g_i(X_i)\Phi\Phi^T g_i^T(X_i) \leqslant m_{i1}M_i(X_i)X_iX_i^T I_{n_i\times n_i}M_i^T(X_i) \leqslant m_{i1}M_i(X_i)M_i^T(X_i)\lambda_{\max}\{X_iX_i^T\}$

$$:\leqslant m_{i1}m_{i2}\lambda_{\max}\{X_iX_i^T\} \leqslant m_{i1}m_{i2}(x_{i1}^2 + x_{i2}^2 + \cdots + x_{in_i}^2) = m_{i1}m_{i2}\|X_i\|^2, \tag{A13}$$

with which, Lemma 5 and Assumption 3, $G(X)\Phi\Phi^T G^T(X) \leqslant 2\sum_{i=1}^2 g_i(X_i)\Phi\Phi^T g_i^T(X_i)I_{n_1+n_2} \leqslant (2m_{11}m_{12}\alpha_1^{-1}\alpha_1\|X_1\|^2 + 2m_{21}m_{22}\alpha_2^{-1}\alpha_2\|X_2\|^2)I_{n_1+n_2} \leqslant \max\{2m_{11}m_{12}\alpha_1^{-1}, 2m_{21}m_{22}\alpha_2^{-1}\}[H_1(X_1)+H_2(X_2)]I_{n_1+n_2} = \chi H(X)I_{n_1+n_2}$. (A14)

Using (A14) and Assumption 3, it is easy to obtain $\|\Phi^T G^T(X)\nabla_X H(X)\|$

$$\begin{aligned}
&= (\nabla_X^T H(X)G(X)\Phi\Phi^T G^T(X)\nabla_X H(X))^{\frac12} = (H(X))^{\frac12}(\nabla_X^T H(X)\chi\nabla_X H(X))^{\frac12}\\
&\leqslant (H(X))^{\frac12}(\lambda_{\max}\{\chi\})^{\frac12}(\nabla_X^T H(X)\nabla_X H(X))^{\frac12} = (H(X))^{\frac12}(\lambda_{\max}\{\chi\})^{\frac12}(\nabla_{X_1}^T H_1(X_1)\nabla_{X_1}H_1(X_1)\\
&\quad + \nabla_{X_2}^T H_2(X_2)\nabla_{X_2}H_2(X_2))^{\frac12}\\
&\leqslant (H(X))^{\frac12}(\lambda_{\max}\{\chi\})^{\frac12}(\eta_1\|X_1\|^2 + \eta_2\|X_2\|^2)^{\frac12} \leqslant (H(X))^{\frac12}(\lambda_{\max}\{\chi\})^{\frac12}(\eta_1\alpha_1^{-1}H(X_1) + \eta_2\alpha_2^{-1}H(X_2))^{\frac12}\\
&\leqslant (H(X))^{\frac12}(\lambda_{\max}\{\chi\})^{\frac12}(\max\{\eta_1\alpha_1^{-1},\eta_2\alpha_2^{-1}\})^{\frac12}(H(X_1)+H(X_2))^{\frac12}\\
&= (\max\{\eta_1\alpha_1^{-1},\eta_2\alpha_2^{-1}\})^{\frac12}(H(X))^{\frac12}(\lambda_{\max}\{\chi\})^{\frac12}(H(X))^{\frac12} = (\max\{\eta_1\alpha_1^{-1},\eta_2\alpha_2^{-1}\})^{\frac12}(\lambda_{\max}\{\chi\})^{\frac12}H(x).
\end{aligned} \tag{A15}$$

Thus, $\|\Phi^T G^T(X)\nabla_X H(X)\|$ is higher degree than $(H(X))^{\frac{1+\alpha}{2}}(\alpha \in (0,\ 1))$. From which and (A11), there exists some sufficiently small neighborhood $\hat\Omega \subset \Omega$ of the origin such that $-k_1m^{\frac{1+\alpha}{2}}(H(X))^{\frac{1+\alpha}{2}} + (\theta-\hat\theta)^T\Phi^T G^T(X)\nabla_X H(X)$ is negative definite, that is, $-k_1m^{\frac{1+\alpha}{2}}(H(X))^{\frac{1+\alpha}{2}} + (\theta-\hat\theta)^T\Phi^T G^T(X)\nabla_X H(X) = \Big(-k_1m^{\frac{1+\alpha}{2}} + M(\max\{\eta_1\alpha_1^{-1},\eta_2\alpha_2^{-1}\})^{\frac12}(\lambda_{\max}\{\chi\})^{\frac12}(H(X))^{\frac{1-\alpha}{2}}\Big)(H(X))^{\frac{1+\alpha}{2}} := m_2(H(X))^{\frac{1+\alpha}{2}}$, where $m_2 < 0$ holds on $\hat\Omega$.

Therefore, we obtain $\dot H(X) \leqslant m_2(H(X))^{\frac{1+\alpha}{2}}$ if $w = 0$, namely, X converges to 0 in a finite time for $X \in \hat\Omega$. Using Lemma 3, one can obtain the system (A1) is finite-time stable.

References

1 Boyd S, Ghaoui L E, Feron E, Balakrishnan V. Linear matrix inequalities in system and control theory. Philadelphia: SIAM, 1994.