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On dimensions of dimension-bounded linear systems

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Dear editor,

Dimensional-varying dynamic systems, also called cross-dimensional systems [1], are often met in actual life, such as electric power generators [2], spacecraft [3], biological systems [4], and internet networks [5, 6]. However, until now there is little literature about dimension-varying systems [7]. The main reason is that dimensional-varying dynamic systems are too complex to find proper mathematic tool to model them.

As succeeding mathematic concepts of the well known concept of semi-tensor product [8] (also called M-product), Cheng [1] recently proposed V-product, M-addition and V-addition of matrices or vectors, which could be key tools to investigate dimensional-varying dynamic systems. Additionally, using these new tools, Cheng et al. [5, 6] have modelled and analyzed dimensional-varying dynamic linear systems.

There are two classes of dimensional-varying dynamic linear systems, which are dimension-bounded systems and dimension-unbounded ones [5, 6]. For the former, although dimensions of a trajectory vary with time, after a certain time the trajectory enters to an invariant space. However, for the latter, after a certain time dimensions of a trajectory not only increase with time but also go to infinity. Furthermore, for a dimension-bounded system, Ref. [5] presented a recursive formula to calculate dimensions of its trajectories at any time. However, we cannot directly derive the invariant space of a dimension-bounded system with an initial value.

This study first presents a necessary and sufficient condition of invariant vector space for a dimension-bounded matrix. Then, for a dimension-bounded discrete system, this study investigates the dimension relationship of its invariant space, initial value, and system matrix. Finally, an algorithm is proposed, via which the dimension of invariant space can be computed directly using the dimensions of initial value and system matrix.

Notations. $\mathcal{M}_{m\times n}$ and $\mathcal{V}_r$ represent the set of $m \times n$ dimensional real matrices and that of r dimensional real vectors, respectively. Symbol $\otimes$ means Kronecker product of matrices [9]. One vector is $\mathbf{1}_n = [1, \ldots, 1]^{\mathrm{T}}$. (m, n) and $[m, n]$ denote the greatest common divisor and the least common multiple of m and n, respectively. $\mathbb{N}$ is the set of all positive integers. For $a \in \mathbb{N}$ and $b \in \mathbb{N}$, $a \mid b$ ($a \nmid b$) says that a is (is not) a divisor of b.

First we give some necessary concepts, including the concepts of vector product, invariant space, and dimension-bounded system.

Definition 1 ([1]). Given a matrix $A \in \mathcal{M}_{m\times n}$, a vector $x \in \mathcal{V}_r$, and $t = [n, r]$, the vector product (V-product) of A and x is defined as

$$A \vec{\ltimes} x = (A \otimes I_{t/n})(x \otimes \mathbf{1}_{t/r}). \tag{1}$$

Two aspects need to be noticed about Definition 1.

Remark 1. (1) For a given matrix $A \in \mathcal{M}_{m\times n}$ and any dimensional vector x, there exists an $l \in \mathbb{N}$, which depends on the dimension of vector x, such that the product $A \vec{\ltimes} x$ belongs to $\mathcal{V}_{lm}$. (2) In Definition 1, vector $x \in \mathcal{V}_r$ can be one dimensional vector, for example $A \vec{\ltimes} a = (A \otimes I_1)(a \otimes \mathbf{1}_n) \in \mathcal{V}_n$, where $A \in \mathcal{M}_{m\times n}$ and a is a scalar.

Definition 2 ([1]). Given a matrix $A \in \mathcal{M}_{m\times n}$, vector space $\mathcal{V}_r$ is called an invariant space of A if

$$A \vec{\ltimes} x \in \mathcal{V}_r, \ \forall x \in \mathcal{V}_r. \tag{2}$$

Given a matrix $A \in \mathcal{M}_{m\times n}$, consider the corresponding dimensional-varying system

$$x(t + 1) = A \vec{\ltimes} x(t), \ x(0) = x_0. \tag{3}$$

The dimensional-varying system above is called a time invariant discrete linear pseudo dynamic system [1].

Definition 3 ([1]). Considering system (3), matrix A is called a dimension-bounded operator, if for any $x(0) = x_0 \in \mathcal{V}_r$, there exist an $N \geqslant 0$ and an r^*, such that $x(t) \in \mathcal{V}_{r^*}$, $t \geqslant N$. Then $\mathcal{V}_{r^*}$ is called the invariant space of system (3). Additionally, dynamic system (3) is also called a dimension-bounded dynamic system.

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Remark 2. Invariant space of system (3) only depends on the dimensions of A and $x(0)$, but not on their particular values. Note that the action of matrices on vectors satisfies the following semi-group property [1]: $(A\ltimes A)\vec{\ltimes}x = A\vec{\ltimes}(A\vec{\ltimes}x)$. Then (3) becomes a dynamic system.

Matrix $A \in \mathcal{M}_{m\times n}$ is dimension-bounded, if and only if there exists a k such that $n = km$ [1]. Thus, for dimension-bounded matrix A, it is reasonable that we assume $A \in \mathcal{M}_{m\times km}$.

From Definition 2, one necessary and sufficient condition of invariant spaces can be obtained.

Proposition 1. Given a matrix $A \in \mathcal{M}_{m\times mk}$, vector space $\mathcal{V}_r$ is an invariant space of A, if and only if there exists an l, such that $r = ml$ and $(l, k) = 1$.

Proof. (Sufficiency) Because $r = ml$ and $(l, k) = 1$, from Definition 1 we have $A\vec{\ltimes}x = (A \otimes I_l)(x \otimes \mathbf{1}_k) \in \mathcal{V}_r$ for any vector $x \in \mathcal{V}_r$. Thus, $\mathcal{V}_r$ is an invariant space of A.

(Necessity) Because $\mathcal{V}_r$ is an invariant space of A, from Remark 1, there exists an l, such that $r = ml$. We only need to prove $(l, k) = 1$.

Assume $(l, k) = s \neq 1$. Then there exist two integers l_1 and k_1, such that $l = sl_1$ and $k = sk_1$. Take any $x \in \mathcal{V}_r$, and then $A\vec{\ltimes}x = (A\otimes I_{l_1})(x\otimes\mathbf{1}_{k_1}) \in \mathcal{V}_{ml_1}$. Because $r = ml = msl_1$ with $s \neq 1$, $r \neq ml_1$, which is contradictory with the invariance of $\mathcal{V}_r$, the proof is completed.

Corollary 1. (1) For matrix $A \in \mathcal{M}_{m\times mk}$, the set of all invariant spaces of A can be derived as $\mathrm{Inv}(A) = \{\mathcal{V}_{ml} : (l, k) = 1\}$. (2) The invariant space of A with the least dimension is $\mathcal{V}_m$.

From item (1) of Corollary 1, it is easy to have that any matrix has countable infinite invariant spaces, which is just Corollary 5.6 of [1].

Now we discuss dimension relationship of invariant space, initial value and system matrix. For a given dimension-bounded matrix $A \in \mathcal{M}_{m\times mk}$, the dimension of invariant space of system (3) depends on that of initial value x_0. For example, assume the dimension-bounded dynamic system (3) is with $A_{3\times 6}$. When $x_0 \in \mathcal{V}_6$, the dimension of invariant space is $r^* = 3$, while the dimension of invariant space is $r^* = 15$ when $x_0 \in \mathcal{V}_5$. It is obvious that the dimension of invariant space depends on that of initial value for a given dimension-bounded dynamic system. That is, the dimension r^* of invariant space is a function of that of initial value. Hence, we denote r^* as $r^*(p)$ with p being the dimension of initial value.

Theorem 1 is about the dimension relationship of invariant space, initial value and system matrix.

Theorem 1. For a given dimension-bounded dynamic system (3) with $A \in \mathcal{M}_{m\times mk}$ and initial value $x(0) \in \mathcal{V}_p$, the following statements hold.

(1) If p is a divisor of km, then $r^*(p) = m$.

(2) If $p = k_1^\alpha m_1$, where α is an arbitrary positive integer, and k_1 and m_1 are divisors of k and m, then $r^*(p) = m$.

(3) If $p = k_1 m_1^\alpha$ and $k_1 m_1^\alpha \nmid km$, where $\alpha > 1$, $m_1 \neq 1$ is a prime, $(m_1, k) = 1$, and k_1 and m_1 are divisors of k and m respectively, then $r^*(p) = mm_1^{\alpha-\alpha_0} \neq m$ with α_0 satisfying $k_1 m_1^{\alpha_0} \mid km$, but $k_1 m_1^{\alpha_0+1} \nmid km$.

Proof. Because item (1) is trivial, we only give the proof for items (2) and (3) here.

Proof of item (2). For the case of $p = k_1^\alpha m_1$ with $k_1^\alpha m_1$ being a divisor of km, the proof is trivial, so it is omitted here. Thus, we only need to prove the case of $k_1^\alpha m_1$ not being a divisor of km. Assume that $k_1^{\alpha_0} m_1 \mid km$, but $k_1^{\alpha_0+1} m_1 \nmid km$.

First, we prove the result for the case of $k_1^{\alpha_0+1} m_1$. Because $k_1^{\alpha_0} m_1 \mid km$, there exists an integer φ such that $km = k_1^{\alpha_0} m_1 \varphi$. Take any initial value $x(0) \in \mathcal{V}_{k_1^{\alpha_0+1} m_1}$, and then $x(1) = A\vec{\ltimes}x(0) = (A \otimes I_{k_1})(x(0) \otimes \mathbf{1}_\varphi) \in \mathcal{V}_{k_1 m}$, which means that $r^*(k_1^{\alpha_0+1} m_1) = r^*(k_1 m)$. From item (1), we get $r^*(k_1^{\alpha_0+1} m_1) = r^*(k_1 m) = m$. Similarly, we can prove that $r^*(k_1^{\alpha_0+\eta} m_1) = r^*(k_1^\eta m) = m$, here $\eta < \alpha_0$.

Thus, for any integer α, assuming that $\alpha = \xi\alpha_0 + \eta$ with $\eta < \alpha_0$, we have $r^*(k_1^{\xi\alpha_0+\eta} m_1) = r^*(mk_1^{(\xi-1)\alpha_0+\eta}) = \cdots = r^*(k_1^\eta m) = m$.

Proof of item (3). For any initial value $x(0) \in \mathcal{V}_{k_1 m_1^\alpha}$, because $(m_1, k) = 1$, we have $x(1) = A\vec{\ltimes}x(0) = (A \otimes I_{m_1^{\alpha-\alpha_0}})(x(0) \otimes \mathbf{1}_{mk/(k_1 m_1^{\alpha_0})}) \in \mathcal{V}_{mm_1^{\alpha-\alpha_0}}$. Because $k_1 m_1^\alpha \nmid km$, it is obvious that $\alpha - \alpha_0 \geqslant 1$. Noting $(m_1, k) = 1$, we get $x(2) = A\vec{\ltimes}x(1) = (A \otimes I_{m_1^{\alpha-\alpha_0}})(x(1) \otimes \mathbf{1}_k) \in \mathcal{V}_{mm_1^{\alpha-\alpha_0}}$, which means $r^*(k_1 m_1^\alpha) = mm_1^{\alpha-\alpha_0} \neq m$.

Results of Theorem 1 can be extended to more complex cases.

Theorem 2. For a given dimension-bounded dynamic system (3) with $A \in \mathcal{M}_{m\times mk}$ and initial value $x(0) \in \mathcal{V}_p$, the following statements hold.

(1) If $p = \tau\xi$ with $\tau \mid km$ and $(\xi, km) = 1$, then $r^*(p) = r^*(\xi) = \xi m$.

(2) If $p = k_1^\alpha m_1 \xi$ and $(\xi, km) = 1$, where $\alpha \in \mathbb{N}$ is arbitrary, and k_1 and m_1 are divisors of and k and m, then $r^*(p) = r^*(\xi) = \xi m$.

(3) If $p = k_1 m_1^\alpha \xi$, $(\xi, km) = 1$ and $k_1 m_1^\alpha \nmid km$, where $\alpha > 1$, $m_1 \neq 1$ is a prime, $(m_1, k) = 1$, and k_1 and m_1 are divisors of k and m respectively, then $r^*(p) = mm_1^{\alpha-\alpha_0}\xi \neq r^*(\xi)$ with α_0 satisfying $k_1 m_1^{\alpha_0} \mid km$, but $k_1 m_1^{\alpha_0+1} \nmid km$.

Proof. The proof of items (2) and (3) is similar to that of Theorem 1, so it is omitted here. We only give the proof of item (1). Because $(\xi, km) = 1$, it is easy to prove that $r^*(\xi) = \xi m$. We only need to prove that $r^*(p) = \xi m$. Take any initial value $x(0) \in \mathcal{V}_{\tau\xi}$, and then we have $x(1) = A\vec{\ltimes}x(0) = (A \otimes I_\xi)(x(0) \otimes \mathbf{1}_{km/\tau}) \in \mathcal{V}_{\xi m}$.

For general cases, we have Corollary 2.

Corollary 2. For a given dimension-bounded dynamic system (3) with $A \in \mathcal{M}_{m\times mk}$ and initial value $x(0) \in \mathcal{V}_p$, the following statements hold.

(1) If $p = k_1^{\alpha_1} k_2^{\alpha_2} \cdots k_w^{\alpha_w} m_1 \xi$ and $(\xi, km) = 1$, where k_i and m_1 are divisors of k and m respectively, and $\alpha_i \in \mathbb{N}$, $i = 1, 2, \ldots, w$ are arbitrary, then $r^*(p) = r^*(\xi) = \xi m$.

(2) If $p = k_1 m_1^{\alpha_1} m_2^{\alpha_2} \cdots m_z^{\alpha_z} \xi$, $k_1 m_j^{\alpha_j} \nmid km$ and $(\xi, km) = 1$, where $\alpha_j > 1$, $m_j \neq 1$ is a prime, $(m_j, k) = 1$, and k_1 and m_j are divisors of k and m respectively, then $r^*(p) = m\xi \prod_{j=1}^{z} m_j^{\alpha_j - \alpha_{0j}} \neq r^*(\xi)$ with α_{0j} satisfying $k_1 m_j^{\alpha_{0j}} \mid km$, but $k_1 m_1^{\alpha_{0j}+1} \nmid km$, $j = 1, 2, \ldots, z$.

Finally, we derive the algorithm to compute the dimensions of invariant spaces.

Algorithm 1. For a given dimension-bounded dynamic system (3) with $A \in \mathcal{M}_{m\times mk}$ and initial value $x(0) \in \mathcal{V}_p$, dimension $r^*(p)$ of the corresponding invariant space can be computed by the following process:

(1) Factorize k as $k = k_1^{\alpha_1} k_2^{\alpha_2} \cdots k_w^{\alpha_w}$, where k_i, $i = 1, 2, \ldots, w$ are all different prime numbers.

(2) Write p in form of $p = k_1^{\beta_1} k_2^{\beta_2} \cdots k_w^{\beta_w} \bar{p}$, where $\bar{p}$ may not be a prime number, but $k_i \nmid \bar{p}$, $i = 1, 2, \ldots, w$.

(3) Factorize m as $m = m_1^{\gamma_1} m_2^{\gamma_2} \ldots m_\varpi^{\gamma_\varpi}$, where m_i, $i = 1, 2, \ldots, \varpi$ are all different prime numbers.

(4) Write $\bar{p}$ in form of $\bar{p} = m_1^{\theta_1} m_2^{\theta_2} \ldots m_\varpi^{\theta_\varpi} \tilde{p}$, where $\tilde{p}$ may not be a prime number, m_i may be a divisor of $\tilde{p}$, and

θ_i, $i = 1, 2, \ldots, \varpi$ take as large as possible, but $\theta_i \leqslant \gamma_i$, $i = 1, 2, \ldots, \varpi$.

(5) Finally, we derive $r^*(p) = r^*(\tilde{p}) = \tilde{p}m$.

Via this algorithm the invariant space dimension of a given dimension-bounded system can be derived directly from that of initial value and system matrix, which is shown by the following example.

Example 1. Given system (3) with $A \in \mathcal{M}_{m\times mk}$ and $x(0) \in \mathcal{V}_p$, the dimension $r^*(p)$ can be computed by Algorithm 1 directly:

(1) Take $m = 6$, $k = 2$. When $p = 5$ we have $\bar{p} = \tilde{p} = p = 5$. Then $r^*(5) = p \times m = 30$.

(2) Take $m = 6$, $k = 20$. When $p = 30$ we have $\bar{p} = 3$, $\tilde{p} = 1$. Then $r^*(30) = \tilde{p} \times m = 1 \times 6 = 6$.

(3) Take $m = 3$, $k = 2$. When $p = 18$ we have $\bar{p} = 9$, $\tilde{p} = 3$. Then $r^*(18) = \tilde{p} \times m = 3 \times 3 = 9$.

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