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Stabilization analysis for Markov jump systems with multiplicative noise and indefinite weight costs

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Abstract This paper mainly discusses the stabilization problem for discrete-time Markov jump linear systems (MJLSs) involving multiplicative noise with an infinite horizon. The cost weighting matrices are generalized to be indefinite. To the best of our knowledge, this paper is novel and unlike most previous studies, it provides the necessary and sufficient conditions that stabilize the MJLSs in the mean square sense with indefinite weighting matrices.

Keywords stabilization, indefinite, multiplicative noise, Markov jump linear system

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1 Introduction

Recently, the Markov jump linear system (MJLS) model has found extensive applications; hence, research on the linear-quadratic (LQ) optimal control problem has been of tremendous interest to many researchers. Firstly, the discrete-time MJLSs were introduced by Costa et al. [1]. For the case whereby the transition probabilities are partly unknown, Zhang and Boukas [2] focused on the H_∞ control problem for the MJLSs, etc. (see [3–7] and references therein). The above-mentioned results were derived on the premise of non-negative even positive definite cost weighting matrices. However, the stochastic LQ problem without the constraints of weighting matrices (only symmetry matrices) may be still well-posed which was first considered by Chen et al. [8]. This case differs from the conventional problem and it is named as indefinite stochastic problem. Furthermore, the indefinite stochastic problem finds extensive applications in the field of financial portfolio.

Many research and studies have been conducted on indefinite problems. Li et al. [9] focused on the indefinite LQ control problem for MJLSs with an infinite horizon. For the discrete case, Ma and Boukas [10] developed the guaranteed indefinite cost control problem for singular MJLSs with uncertain parameters. Costa and de Paulo [11] studied the indefinite optimal control problem for MJLSs with multiplicative noise, in which the quadratic and linear part simultaneously exist in the index functional. Then, the generalized coupled algebraic Riccati equations associated with indefinite optimal control problems for discrete-time MJLSs with multiplicative noise in infinite horizon case were analyzed by Costa and de Paulo [12]. Notice that the results obtained in [13] were based on the condition of mean-square stabilizablity.

Significantly, previous studies on the indefinite case only considered the optimal control problem, while the stabilization problem for discrete-time MJLSs with indefinite weighting matrices in cost function has received negligible attention with limited publications. However, in most cases, stability is a vital precondition for a control system. Factually, optimal control studies are worthwhile when the system is stable (see [13–17]). The stabilization problem is a matter of serious concern due to its importance. Ref. [18] discussed the optimal LQ control problem with irregular performance in the finite-horizon

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and infinite-horizon cases, respectively. Li et al. [19] studied the LQ optimal control and stabilization problem for discrete-time networked control systems simultaneous with input delay and Markovian packet dropout. The stabilization problem for stochastic systems with multiplicative noises and input delay in discrete and continuous cases was considered by Zhang et al. [20], Zhang and Xu [21], respectively. To proceed, Zhang et al. [22] provided the stabilization for discrete-time mean-field systems. Controllers of cost weighting matrices in [18–22] are not required to be positive definite. Because of this, in this study, we mainly consider the indefinite LQ optimal control and stabilization problem for discrete-time MJLSs with multiplicative noise.

In this study, under the preconditions that a set satisfying some linear matrix inequality and kernel restriction is nonempty and the system is exactly observable, we derive the existence of the maximum solution to generalized algebraic Riccati equations with Markov jump (GARE-MJ) by discussing the convergence of the associated generalized difference Riccati equations with Markov jump (GDRE-MJ). Another contribution is that the conclusion of stabilization for discrete-time MJLSs involving multiplicative noise with indefinite weighting matrices is provided for the first time under the basic assumption of exact observablity, which is different from previous studies.

The structure of this paper is as follows. The problem statement and preliminaries are introduced in Section 2. Section 3 gives the main result of stabilization. A numerical example is given in Section 4 to further validate the conclusion. Section 5 summarizes the paper.

2 Problem statement and preliminaries

For readability, we first define some notations. $\mathbb{R}^m$ denotes the m-dimensional Euclidean space; M' denotes the transposition of M; $M > 0$ ($M \geqslant 0$) represents that the symmetric matrix $M \in \mathbb{R}^{m\times m}$ is positive definite (positive semi-definite); $M^\dagger$ denotes the Moore-Penrose pseudo-inverse of M; $\mathrm{Ker}(M)$ means the kernel of a matrix M; $(\Omega, \mathcal{G}, \mathcal{G}_k, \mathcal{P})$ is a complete probability space with $\mathcal{G}_k$ generated by $\{x_0, \theta_0, \ldots, x_k, \theta_k\}$; $\mathrm{E}[\cdot|\mathcal{G}_k]$ represents the conditional expectation with respect to $\mathcal{G}_k$ and $\mathcal{G}_{-1}$ is understood as $\{\emptyset, \Omega\}$.

Considering the following discrete-time MJLS with multiplicative noise:

$$x_{k+1} = (A_{\theta_k} + B_{\theta_k}\omega_k)x_k + (C_{\theta_k} + D_{\theta_k}\omega_k)u_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the control process. $\{\omega_k\}_{k\geqslant 0}$ denotes scalar-valued Gaussian white noise with $\mathrm{E}[\omega_k] = 0$ and covariance σ^2. θ_k denotes a discrete-time Markov chain with finite state space $\{1, 2, \ldots, L\}$ and transition probability $\rho_{i,j} = P(\theta_{k+1} = j|\theta_k = i)$ $(i, j = 1, 2, \ldots, L)$. A_i, B_i, C_i, D_i $(i = 1, \ldots, L)$ are constant matrices. The known initial value x_0 is independent of θ_k.

The following cost functional is introduced:

$$J = \mathrm{E}\left\{\sum_{k=0}^{\infty}[x_k' Q_{\theta_k} x_k + u_k' R_{\theta_k} u_k]\right\}, \tag{2}$$

where Q_{θ_k}, R_{θ_k} are just symmetric matrices.

The following problem will be mainly discussed in this paper.

Problem 1. Find a $\mathcal{G}_k$-measurable controller u_k to stabilize (1) while minimizing (2). Let $A = (A_1, \ldots, A_L)$, $B = (B_1, \ldots, B_L)$. For brevity, we usually say that the pair (A, B) is mean-square stabilizable if system (1) is mean-square stabilizable.

Now we define the following GARE-MJ for $i = 1, \ldots, L$ as

$$\begin{cases} P_i = A_i'\left(\sum_{j=1}^{L}\rho_{i,j}P_j\right)A_i + \sigma^2 B_i'\left(\sum_{j=1}^{L}\rho_{i,j}P_j\right)B_i + Q_i - M_i'\Upsilon_i^\dagger M_i, \\ \Upsilon_i\Upsilon_i^\dagger M_i - M_i = 0, \\ \Upsilon_i \geqslant 0, \end{cases} \tag{3}$$

in which

$$\Upsilon_i = C_i'\left(\sum_{j=1}^{L}\rho_{i,j}P_j\right)C_i + \sigma^2 D_i'\left(\sum_{j=1}^{L}\rho_{i,j}P_j\right)D_i + R_i, \tag{4}$$

$$M_i = C_i'\left(\sum_{j=1}^{L}\rho_{i,j}P_j\right)A_i + \sigma^2 D_i'\left(\sum_{j=1}^{L}\rho_{i,j}P_j\right)B_i. \tag{5}$$

For illustrating, establish the following set which is inspired by [23],

$$\mathcal{S} \triangleq \left\{\begin{array}{c}\tilde{P} = \tilde{P}'|Z \geqslant 0,\\ \mathrm{Ker}\left(C_i'\left(\sum_{j=1}^{L}\rho_{i,j}\tilde{P}_j\right)C_i + \sigma^2 D_i'\left(\sum_{j=1}^{L}\rho_{i,j}\tilde{P}_j\right)D_i + R_i\right) \subseteq (\mathrm{Ker}C_i \cap \mathrm{Ker}D_i)\end{array}\right\},$$

where $\tilde{P} = (\tilde{P}_1, \ldots, \tilde{P}_L)$,

$$Z = \begin{bmatrix} A_i'\left(\sum_{j=1}^{L}\rho_{i,j}\tilde{P}_j\right)A_i + \sigma^2 B_i'\left(\sum_{j=1}^{L}\rho_{i,j}\tilde{P}_j\right)B_i + Q_i - \tilde{P}_i & A_i'\left(\sum_{j=1}^{L}\rho_{i,j}\tilde{P}_j\right)C_i + \sigma^2 B_i'\left(\sum_{j=1}^{L}\rho_{i,j}\tilde{P}_j\right)D_i \\ C_i'\left(\sum_{j=1}^{L}\rho_{i,j}\tilde{P}_j\right)A_i + \sigma^2 D_i'\left(\sum_{j=1}^{L}\rho_{i,j}\tilde{P}_j\right)B_i & C_i'\left(\sum_{j=1}^{L}\rho_{i,j}\tilde{P}_j\right)C_i + \sigma^2 D_i'\left(\sum_{j=1}^{L}\rho_{i,j}\tilde{P}_j\right)D_i + R_i \end{bmatrix}.$$

To simplify notation in the sequel, for any $\tilde{P} \in \mathcal{S}$, we define

$$\begin{cases} \tilde{Q}_i = A_i'\left(\sum_{j=1}^{L}\rho_{i,j}\tilde{P}_j\right)A_i + \sigma^2 B_i'\left(\sum_{j=1}^{L}\rho_{i,j}\tilde{P}_j\right)B_i + Q_i - \tilde{P}_i,\\ \tilde{L}_i = A_i'\left(\sum_{j=1}^{L}\rho_{i,j}\tilde{P}_j\right)C_i + \sigma^2 B_i'\left(\sum_{j=1}^{L}\rho_{i,j}\tilde{P}_j\right)D_i,\\ \tilde{R}_i = C_i'\left(\sum_{j=1}^{L}\rho_{i,j}\tilde{P}_j\right)C_i + \sigma^2 D_i'\left(\sum_{j=1}^{L}\rho_{i,j}\tilde{P}_j\right)D_i + R_i. \end{cases} \tag{6}$$

Remark 1. Obviously, we have $\left[\begin{smallmatrix}\tilde{Q}_i & \tilde{L}_i\\ \tilde{L}_i' & \tilde{R}_i\end{smallmatrix}\right] \geqslant 0$. From Theorem 1 in [24], it yields that

$$\tilde{R}_i \geqslant 0, \quad \tilde{Q}_i - \tilde{L}_i\tilde{R}_i^{\dagger}\tilde{L}_i' \geqslant 0, \quad \tilde{L}_i(I - \tilde{R}_i\tilde{R}_i^{\dagger}) = 0. \tag{7}$$

For discussing, some associated definitions and assumption will be given in the following.

Definition 1. $P_{\max}$ is called a maximal solution to the GARE-MJ (3), if

$$P_{\max} \geqslant \tilde{P}, \quad \forall \tilde{P} \in \mathcal{S}, \tag{8}$$

where $P_{\max} = (P_{\max_1}, \ldots, P_{\max_L})$.

Definition 2. Consider the following MJLS with multiplicative noises

$$\begin{cases} x_{k+1} = (A_{\theta_k} + \omega_k B_{\theta_k})x_k,\\ y_k = \tilde{Q}_{\theta_k}^{\frac{1}{2}} x_k. \end{cases} \tag{9}$$

$(A, B, \tilde{Q}^{\frac{1}{2}})$ is exactly observable, if for any $N \geqslant 0$,

$$y_k = 0, \quad \text{a.s.}, \quad \forall k \in [0, N] \Rightarrow x_0 = 0,$$

where $\tilde{Q}^{\frac{1}{2}} = (\tilde{Q}_1^{\frac{1}{2}}, \ldots, \tilde{Q}_L^{\frac{1}{2}})$.

Assumption 1. $(A, B, \tilde{Q}^{\frac{1}{2}})$ is exactly observable, in which $\tilde{Q} = (\tilde{Q}_1, \ldots, \tilde{Q}_L)$ defined as in (6).

3 Main results

The stabilization analysis will be given next. Based on the aforementioned preliminaries, the existence of the solution to GARE-MJ (3) will be investigated.

Theorem 1. The GARE-MJ (3) exists the maximal solution P_i, if the following two conditions are true: (i) $\mathcal{S} \neq \emptyset$; (ii) the system (1) is mean-square stabilizable.

Proof. In view of $\mathcal{S} \neq \emptyset$, take any $\tilde{P} \in \mathcal{S}$ and consider a new GDRE-MJ with $\theta_k = i$:

$$\begin{cases} X_i(k) = A_i'\left(\sum_{j=1}^{L}\rho_{i,j}X_j(k+1)\right)A_i + \sigma^2 B_i'\left(\sum_{j=1}^{L}\rho_{i,j}X_j(k+1)\right)B_i + \tilde{Q}_i - \tilde{M}_i'(k)\tilde{\Upsilon}_i^{\dagger}(k)\tilde{M}_i(k), \\ \tilde{\Upsilon}_i(k)\tilde{\Upsilon}_i^{\dagger}(k)\tilde{M}_i(k) - \tilde{M}_i(k) = 0, \\ \tilde{\Upsilon}_i(k) \geqslant 0, \end{cases} \tag{10}$$

in which

$$\tilde{\Upsilon}_i(k) = C_i'\left(\sum_{j=1}^{L}\rho_{i,j}X_j(k+1)\right)C_i + \sigma^2 D_i'\left(\sum_{j=1}^{L}\rho_{i,j}X_j(k+1)\right)D_i + \tilde{R}_i, \tag{11}$$

$$\tilde{M}_i(k) = C_i'\left(\sum_{j=1}^{L}\rho_{i,j}X_j(k+1)\right)A_i + \sigma^2 D_i'\left(\sum_{j=1}^{L}\rho_{i,j}X_j(k+1)\right)B_i + \tilde{L}_i, \tag{12}$$

with its terminal values $X_i(N+1) = 0$ for $i = 1, \ldots, L$ and $\tilde{Q}_i$, $\tilde{L}_i$, $\tilde{R}_i$ are defined as in (6). Firstly, we will prove that Eqs. (10)–(12) indeed has a solution. It is easy to see that the iterative sequence

$$X_i(k) = A_i'\left(\sum_{j=1}^{L}\rho_{i,j}X_j(k+1)\right)A_i + \sigma^2 B_i'\left(\sum_{j=1}^{L}\rho_{i,j}X_j(k+1)\right)B_i + \tilde{Q}_i - \tilde{M}_i'(k)\tilde{\Upsilon}_i^{\dagger}(k)\tilde{M}_i(k) \tag{13}$$

has a solution $X_i(k)$ with terminal values $X_i(N+1) = 0$.

Further, $\tilde{\Upsilon}_i(k) \geqslant 0$ will be shown. To this end, we first illustrate that $X_i(k)$ is semi-definite positive. Considering the following formula

$$\tilde{M}_i'(k)\tilde{\Upsilon}_i^{\dagger}(k)\tilde{M}_i(k) = -\tilde{M}_i'(k)\tilde{F}_i(k) - \tilde{F}_i'(k)\tilde{M}_i(k) - \tilde{F}_i'(k)\tilde{\Upsilon}_i(k)\tilde{F}_i(k), \tag{14}$$

in which $\tilde{F}_i(k) = -\tilde{\Upsilon}_i^{\dagger}(k)\tilde{M}_i(k)$, thus (13) can be rewritten as

$$X_i(k) = \bar{A}_i'(k)\left(\sum_{j=1}^{L}\rho_{ij}X_j(k+1)\right)\bar{A}_i(k) + \bar{B}_i'(k)\left(\sum_{j=1}^{L}\rho_{ij}X_j(k+1)\right)\bar{B}_i(k) + \bar{Q}_i(k), \tag{15}$$

where

$$\begin{aligned} &\bar{A}_i(k) = A_i + C_i\tilde{F}_i(k), \quad \bar{B}_i(k) = B_i + D_i\tilde{F}_i(k), \\ &\bar{Q}_i(k) = \tilde{Q}_i + \tilde{L}_i'\tilde{F}_i(k) + \tilde{F}_i'(k)\tilde{L}_i + \tilde{F}_i'(k)\tilde{R}_i\tilde{F}_i(k). \end{aligned} \tag{16}$$

By the Schur complementary, and in view of $\tilde{Q}_i \geqslant 0$, $\tilde{R}_i \geqslant 0$, we have

$$\begin{aligned} \bar{Q}_i(k) &= \tilde{Q}_i + \tilde{L}_i'\tilde{F}_i(k) + \tilde{F}_i'(k)\tilde{L}_i + \tilde{F}_i'(k)\tilde{R}_i\tilde{F}_i(k) \\ &\geqslant \tilde{L}_i'\tilde{R}_i^{\dagger}\tilde{L}_i + \tilde{L}_i'\tilde{R}_i^{\dagger}\tilde{R}_i\tilde{F}_i(k) + \tilde{F}_i'(k)\tilde{R}_i\tilde{R}_i^{\dagger}\tilde{L}_i + \tilde{F}_i'(k)\tilde{R}_i\tilde{R}_i^{\dagger}\tilde{R}_i\tilde{F}_i(k) \\ &= (\tilde{L}_i + \tilde{R}_i\tilde{F}_i(k))'\tilde{R}_i^{\dagger}(\tilde{L}_i + \tilde{R}_i\tilde{F}_i(k)) \geqslant 0, \end{aligned} \tag{17}$$

and on the ground of $X_i(N+1) = 0$, $i = 1, \ldots, L$, it yields that $X_i(N) \geqslant 0$ and by induction, it is not difficult to verify that $X_i(k) \geqslant 0$, for $0 \leqslant k \leqslant N$. Thus, from (11), $\tilde{\Upsilon}_i(k) \geqslant 0$ is established.

Next we will investigate $\tilde{\Upsilon}_i(k)\tilde{\Upsilon}_i^{\dagger}(k)\tilde{M}_i(k) - \tilde{M}_i(k) = 0$. From $\tilde{\Upsilon}_i(k) \geqslant 0$, it yields that

$$\tilde{\Upsilon}_i^{\dagger}(k) = V_i(k)\begin{bmatrix} T_i^{-1}(k) & 0 \\ 0 & 0 \end{bmatrix}V_i'(k), \tag{18}$$

where $T_i(k) > 0$ has same dimension with the rank of $\tilde{\Upsilon}_i(k)$ and $V_i(k)$ is an orthogonal matrix. Now, let $V_i(k)$ decompose as $[V_i^1(k)\ V_i^2(k)]$ where the columns of the matrix $V_i^2(k)$ form a basis of $\mathrm{Ker}(\tilde{\Upsilon}_i(k))$. The positive semi-definite of matrices $\tilde{R}_i, \sum_{j=1}^L \rho_{i,j} X_j(k+1)$ yields that $\mathrm{Ker}(\tilde{\Upsilon}_i(k)) \subseteq \mathrm{Ker}(\tilde{R}_i)$. A simple calculation yields that

$$\begin{aligned}&\left[A_i'\left(\sum_{j=1}^L \rho_{i,j} X_j(k+1)\right)C_i + \sigma^2 B_i'\left(\sum_{j=1}^L \rho_{i,j} X_j(k+1)\right)D_i + \tilde{L}_i'\right][I - \tilde{\Upsilon}_i(k)\tilde{\Upsilon}_i^\dagger(k)]\\ &= \left[A_i'\left(\sum_{j=1}^L \rho_{i,j}(X_j(k+1)+\tilde{P}_j)\right)C_i + \sigma^2 B_i'\left(\sum_{j=1}^L \rho_{i,j}(X_j(k+1)+\tilde{P}_j)\right)D_i\right]V_i^2(k)(V_i^2(k))'. \quad (19)\end{aligned}$$

On the ground of $\tilde{\Upsilon}_i(k)V_i^2(k)(V_i^2(k))' = 0$, it is easy to verify $\tilde{R}_i V_i^2(k)(V_i^2(k))' = 0$. And further considering the condition of $\mathrm{Ker}(C_i'(\sum_{j=1}^L \rho_{i,j}\tilde{P}_j)C_i + \sigma^2 D_i'(\sum_{j=1}^L \rho_{i,j}\tilde{P}_j)D_i + R_i) \subseteq (\mathrm{Ker}C_i \cap \mathrm{Ker}D_i)$, we have $\tilde{\Upsilon}_i(k)\tilde{\Upsilon}_i^\dagger(k)\tilde{M}_i(k) - \tilde{M}_i(k) = 0$.

Up to now, we know that for (10) there exists a positive semi-definite solution with terminal values $X_i(N+1) = 0$ for $i = 1,\dots,L$. For discussion, we rewrite $X_i(k)$ as $X_i^N(k)$ to express its time horizon N clearly. Now we will prove the convergence of $X_i^N(k)$ when $N \to \infty$. Considering the following cost function

$$\tilde{J}_N = \mathrm{E}\left\{\sum_{k=0}^N \begin{bmatrix} x_k \\ u_k \end{bmatrix}' \begin{bmatrix} \tilde{Q}_{\theta_k} & \tilde{L}_{\theta_k}' \\ \tilde{L}_{\theta_k} & \tilde{R}_{\theta_k} \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}\right\}, \quad (20)$$

from Remark 1, we know that $\tilde{J}_N \geqslant 0$. Based on the existence of solution to Riccati Eqs. (10)–(12) and Theorem 1 in [25], the optimal controller and cost value of (20) subject to (1) are $u^*(k) = -\tilde{\Upsilon}_i^\dagger(k)\tilde{M}_i(k)x_k$ and $\tilde{J}_N^* = \mathrm{E}[x_0' X_{\theta_0}^N(0)x_0]$, respectively. Then, following Theorem 2 in [20], the convergence of $X_i^N(k)$ can be deduced in a similar way. Let $\lim_{N\to\infty} X_i^N(k) = X_i, \lim_{N\to\infty}\tilde{\Upsilon}_i^N(k) = \tilde{\Upsilon}_i, \lim_{N\to\infty}\tilde{M}_i^N(k) = \tilde{M}_i$, $i = 1,\dots,L$.

Therefore, X_i is a solution of the following new GARE-MJ (NGARE-MJ for short):

$$\begin{cases} X_i = A_i'\left(\sum_{j=1}^L \rho_{i,j}X_j\right)A_i + \sigma^2 B_i'\left(\sum_{j=1}^L \rho_{i,j}X_j\right)B_i + \tilde{Q}_i - \tilde{M}_i'\tilde{\Upsilon}_i^\dagger\tilde{M}_i, \\ \tilde{\Upsilon}_i\tilde{\Upsilon}_i^\dagger\tilde{M}_i - \tilde{M}_i = 0, \\ \tilde{\Upsilon}_i \geqslant 0, \end{cases} \quad (21)$$

in which

$$\tilde{\Upsilon}_i = C_i'\left(\sum_{j=1}^L \rho_{i,j}X_j\right)C_i + \sigma^2 D_i'\left(\sum_{j=1}^L \rho_{i,j}X_j\right)D_i + \tilde{R}_i, \quad (22)$$

$$\tilde{M}_i = C_i'\left(\sum_{j=1}^L \rho_{i,j}X_j\right)A_i + \sigma^2 D_i'\left(\sum_{j=1}^L \rho_{i,j}X_j\right)B_i + \tilde{L}_i. \quad (23)$$

Define $P_i^N(k) = X_i^N(k) + \tilde{P}_i$. It is easy to verify that $P_i^N(k)$ satisfies the GDRE-MJ (6) in [25] and monotonically increasing with respect to N and bounded. Therefore, there exists a constant P_i satisfying

$$P_i = \lim_{N\to\infty} X_i^N(k) + \tilde{P}_i = X_i + \tilde{P}_i.$$

Obviously, P_i satisfies GARE-MJ (3). Moreover, for the arbitrariness of $\tilde{P}_i$ and $X_i \geqslant 0$, we can obtain that $P_i \geqslant \tilde{P}_i$, i.e., P_i is the maximal solution to the GARE-MJ (3). The proof is complete.

Remark 2. From the process of the above proof, it yields that the solvability of the GARE-MJ (3) is equal to that of the NGARE-MJ (21).

Actually, under Assumption 1, the conclusion that the solution to NGARE-MJ (21) is strictly positive definite can be further illustrated.

Lemma 1. Let Assumption 1 be satisfied and $\mathcal{S} \neq \emptyset$. If the system (1) is mean-square stabilizable, then the solution $X = (X_1, \dots, X_L)$ to NGARE-MJ (21) is strictly positive definite, i.e., $X_i > 0$, $i = 1, \dots, L$.

Proof. From Theorem 1, we know that $X_i^N(k)$ is positive semi-definite, thus, its limit X_i is also positive semi-definite, i.e., $X_i \geqslant 0$. Now we verify $X_i > 0$. If not, there must exist nonzero vector x_0 such that $E[x_0' X_i x_0] = 0$.

Define the Lyapunov function as

$$V_X(k, x_k) = \mathrm{E}[x_k' X_{\theta_k} x_k], \quad k \geqslant 0. \tag{24}$$

So we have

$$\begin{aligned}
&\sum_{k=0}^{N} \big[V_X(k+1, x_{k+1}) - V_X(k, x_k)\big] \\
&= \mathrm{E}\big[x(N+1)' X_{\theta_k} x(N+1) - x_0' X_{\theta_k} x_0\big] \\
&= -\sum_{k=0}^{N} \mathrm{E}\big[x_k' \tilde{Q}_{\theta_k} x_k + x_k' \tilde{L}_{\theta_k}' u_k + u_k' \tilde{L}_{\theta_k} x_k + u_k' \tilde{R}_{\theta_k} u_k\big] \\
&= -\sum_{k=0}^{N} \mathrm{E}\big[x_k' \tilde{Q}_{\theta_k} x_k + x_k' \tilde{L}_{\theta_k}' \tilde{F}_{\theta_k} x_k + x_k' \tilde{F}_{\theta_k}' \tilde{L}_{\theta_k} x_k + x_k' \tilde{F}_{\theta_k}' \tilde{R}_{\theta_k} \tilde{F}_{\theta_k} x_k\big] \\
&= -\sum_{k=0}^{N} \mathrm{E}\big[x_k' (\tilde{Q}_{\theta_k} + \tilde{L}_{\theta_k}' \tilde{F}_{\theta_k} + \tilde{F}_{\theta_k}' \tilde{L}_{\theta_k} + \tilde{F}_{\theta_k}' \tilde{R}_{\theta_k} \tilde{F}_{\theta_k}) x_k\big] \\
&= -\sum_{k=0}^{N} \mathrm{E}\big[x_k' \bar{Q}_{\theta_k} x_k\big] \leqslant 0,
\end{aligned} \tag{25}$$

where $u_k = \tilde{F}_{\theta_k} x_k = -\tilde{\Upsilon}_{\theta_k}^{\dagger} \tilde{M}_{\theta_k} x_k$ is used in the above equations. Obviously,

$$0 \leqslant \sum_{k=0}^{N} \mathrm{E}\{x_k' \bar{Q}_i x_k\} = -\mathrm{E}[x(N+1)' X_i x(N+1)] \leqslant 0,$$

which implies that

$$\sum_{k=0}^{N} \mathrm{E}\{x_k' \bar{Q}_i x_k\} = 0,$$

i.e.,

$$\bar{Q}_i^{\frac{1}{2}} x_k = 0. \tag{26}$$

Following [26], we know that the exact observable of $(\bar{A}, \bar{B}, \bar{Q}^{\frac{1}{2}})$ can be deduced by the exact observable of $(A, B, \tilde{Q}^{\frac{1}{2}})$, in which $\bar{A} = A_i + C_i \tilde{F}_i$, $\bar{B} = B_i + D_i \tilde{F}_i$. Therefore, from (26), it yields that $x_0 = 0$ which is contrary with $x_0 \neq 0$. Hence, we have $X_i > 0$.

The above conclusion will be useful to illustrate the following stabilization result.

Theorem 2. Let Assumption 1 be satisfied and $\mathcal{S} \neq \emptyset$. Then the closed-loop system (1) is mean-square stabilizable if and only if the GARE-MJ (3) has a solution $P = (P_1, \dots, P_L)$, which is also the maximal solution to the GARE-MJ (3). On such condition, the optimal stabilizing solution can be derived as

$$u_k^* = F_i x_k, \quad i = 1, \dots, L, \tag{27}$$

where

$$F_i = -\left[C_i' \left(\sum_{j=1}^{L} \rho_{i,j} P_j\right) C_i + \sigma^2 D_i' \left(\sum_{j=1}^{L} \rho_{i,j} P_j\right) D_i + R_i\right]^{\dagger}$$

$$\times \left[C_i' \left(\sum_{j=1}^{L} \rho_{i,j} P_j \right) A_i + \sigma^2 D_i' \left(\sum_{j=1}^{L} \rho_{i,j} P_j \right) B_i \right]. \tag{28}$$

Moreover, the optimal cost functional is

$$J^* = \mathrm{E}[x_0' P_{\theta_0} x_0]. \tag{29}$$

Proof. ($\Longleftarrow$) For illustrating the mean-square stabilizablity of closed-loop system (1), take any $\tilde{P} \in \mathcal{S}$. In view of Remark 2, we have that the NGARE-MJ (21) has a positive definite solution $X = (X_1, \ldots, X_L)$, i.e., $X_i > 0$, $i = 1, \ldots, L$. And $P = X + \tilde{P}$. In the sequence, we will show that the following system

$$x_{k+1} = (A_{\theta_k} + C_{\theta_k} F_{\theta_k}) x_k + \omega_k (B_{\theta_k} + D_{\theta_k} F_{\theta_k}) x_k \tag{30}$$

is mean-square stabilizable. Considering the fact that $F_i = \tilde{F}_i$ $(i = 1, \ldots, L)$, it yields that the stabilization for the system (1) with $u_k = F_{\theta_k} x_k$ is equivalent to that with $u_k = \tilde{F}_{\theta_k} x_k$. Now, take consideration of the following Lyapunov function as

$$V_X(k, x_k) = \mathrm{E}[x_k' X_{\theta_k} x_k], \quad k \geqslant 0. \tag{31}$$

Since $X_{\theta_k} = X_i$ with $\theta_k = i$, $i = 1, \ldots, L$, then $V_X(k, x_k) \geqslant 0$. So we have

$$\begin{aligned} V_X(k+1, x_{k+1}) - V_X(k, x_k) &= \mathrm{E}\left\{ x_{k+1}' X_{\theta_{k+1}} x_{k+1} - x_k' X_{\theta_k} x_k \right\} \\ &= -\mathrm{E}\left\{ x_k' \tilde{Q}_i x_k + x_k' \tilde{L}_i' u_k + u_k' \tilde{L}_i x_k + u_k' \tilde{R}_i u_k \right\} \\ &= -\mathrm{E}\left\{ \begin{bmatrix} x_k \\ u_k \end{bmatrix}' \begin{bmatrix} \tilde{Q}_i & \tilde{L}_i' \\ \tilde{L}_i & \tilde{R}_i \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \right\} \\ &\leqslant 0, \end{aligned} \tag{32}$$

where $\tilde{Q}_i$, $\tilde{L}_i$, $\tilde{R}_i$ are defined in (6).

From Eq. (32), we can see that $V_X(k, x_k)$ is non-increasing with respect to k. That implies $V_X(k, x_k) \leqslant V_X(0, x_0)$, i.e., $V_X(k, x_k)$ is bounded. Therefore, $\lim_{k\to+\infty} V_X(k, x_k)$ exists.

Now for any integer $m \geqslant 0$, taking summation from $k = m$ to $k = m + N$ on both sides of the above formulation, we can obtain that

$$V_X(m+N+1, x(m+N+1)) - V_X(m, x(m)) = -\sum_{k=m}^{m+N} \mathrm{E}\left\{ \begin{bmatrix} x_k \\ u_k \end{bmatrix}' \begin{bmatrix} \tilde{Q}_i & \tilde{L}_i' \\ \tilde{L}_i & \tilde{R}_i \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \right\}. \tag{33}$$

In view of the convergence of $V_X(k, x_k)$, when we take limitation of m on both sides of the aforementioned equation, the following result can be derived:

$$\begin{aligned} &-\lim_{m\to\infty} \sum_{k=m}^{m+N} \mathrm{E}\left\{ \begin{bmatrix} x_k \\ u_k \end{bmatrix}' \begin{bmatrix} \tilde{Q}_i & \tilde{L}_i' \\ \tilde{L}_i & \tilde{R}_i \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \right\} \\ &= \lim_{m\to\infty} [V_1(m+N+1, x(m+N+1)) - V_1(m, x(m))] \\ &= 0. \end{aligned} \tag{34}$$

Further considering that the optimal cost function of $\tilde{J}_N$ is $\mathrm{E}[x_0' X_{\theta_0}^N x_0] \geqslant 0$, via a time-shift of length of m, it yields that

$$\begin{aligned} 0 &= \lim_{m\to\infty} \sum_{k=m}^{m+N} \mathrm{E}\left\{ \begin{bmatrix} x_k \\ u_k \end{bmatrix}' \begin{bmatrix} \tilde{Q}_i & \tilde{L}_i' \\ \tilde{L}_i & \tilde{R}_i \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \right\} \\ &\geqslant \lim_{m\to\infty} \mathrm{E}[x_m' X_{\theta_m}^{m+N} x_m] \\ &= \lim_{m\to\infty} \mathrm{E}[x_m' X_{\theta_0}^{N} x_m] \\ &\geqslant 0. \end{aligned} \tag{35}$$

Obviously, it implies that $\lim_{m\to\infty} \mathrm{E}[x_m' X_{\theta_0}^N x_m] = 0$. On the ground of the positive definiteness of $X_{\theta_0}^N$, it is easy to verify that $\lim_{m\to\infty} \mathrm{E}[x_m' x_m] = 0$. That is to say that the controller $u_k = \tilde{F}_{\theta_k} x_k = F_{\theta_k} x_k$ stabilizes system (1) in the mean square sense.

Lastly, we show the optimal controller and optimal cost. Define

$$V_P(k, x_k) = \mathrm{E}[x_k' P_{\theta_k} x_k], \quad k \geqslant 0. \tag{36}$$

Therefore,

$$\begin{aligned}
&\sum_{k=0}^{N}[V_P(k+1, x_{k+1}) - V_P(k, x_k)] \\
&= \sum_{k=0}^{N} \mathrm{E}\big[(u_k + \Upsilon_{\theta_k}^{\dagger}(k) M_{\theta_k}(k) x_k)' \Upsilon_{\theta_k}(k)(u_k + \Upsilon_{\theta_k}^{\dagger}(k) M_{\theta_k}(k) x_k)\big] - \sum_{k=0}^{N} \mathrm{E}[x_k' Q_{\theta_k} x_k + u_k' R_{\theta_k} u_k],
\end{aligned} \tag{37}$$

that is,

$$J_N = \mathrm{E}[x_0' P_{\theta_0} x_0] + \sum_{k=0}^{N} \mathrm{E}\big[\big(u_k + \Upsilon_{\theta_k}^{\dagger}(k) M_{\theta_k}(k) x_k\big)' \Upsilon_{\theta_k}(k)\big(u_k + \Upsilon_{\theta_k}^{\dagger}(k) M_{\theta_k}(k) x_k\big)\big], \tag{38}$$

where $\Upsilon_{\theta_k}(k)$ and $M_{\theta_k}(k)$ are defined as (7) and (8) in [27].

Thus the infinite cost function can be obtained as the following form on account of the mean-square stabilizable of system (1):

$$J = \mathrm{E}[x_0' P_{\theta_0} x_0] + \sum_{k=0}^{\infty} \mathrm{E}\big[\big(u_k + \Upsilon_{\theta_k}^{\dagger} M_{\theta_k} x_k\big)' \Upsilon_{\theta_k}\big(u_k + \Upsilon_{\theta_k}^{\dagger} M_{\theta_k} x_k\big)\big]. \tag{39}$$

Hence, it is easy to conclude that the optimal controller is $u_k^* = -\Upsilon_{\theta_k}^{\dagger} M_{\theta_k} x_k$ and furthermore the corresponding optimal cost is $J^* = \mathrm{E}[x_0' P_{\theta_0} x_0]$.

($\Longrightarrow$) See the proof of Theorem 1.

Remark 3. With regard to the infinite case, some previous papers have obtained good results, such as [13], where they mainly discussed the existence of mean-square stabilizing solution to the generalized coupled algebraic Riccati equations, and moreover, gave a necessary and sufficient condition based on the assumption that the system is stabilizable. Compared with it, the above results discussed the stabilization of the system, which is another important point of view.

4 Numerical example

A numerical example will be given in this section to further illustrate our results. Now consider system (1) with the following coefficients:

$$\begin{aligned}
&A_1 = \frac{1}{2}, \quad B_1 = -\frac{1}{2}, \quad C_1 = \frac{1}{2}, \quad D_1 = -\frac{1}{2}, \quad Q_1 = -1, \quad R_1 = -3; \\
&A_2 = \frac{1}{4}, \quad B_2 = -\frac{1}{4}, \quad C_2 = \frac{1}{4}, \quad D_2 = -\frac{1}{4}, \quad Q_2 = 20, \quad R_2 = 0.
\end{aligned}$$

The transition probabilities of the Markov chain $\{\theta_k; k = 1, 2, \ldots\}$ taking values in $\{1, 2\}$ are $\rho_{11} = 0.2$ and $\rho_{22} = 0.6$. The variance of system noise is 1.

By simply computing we know that

$$\begin{aligned}
\tilde{Q}_1 &= -0.9\tilde{P}_1 + 0.4\tilde{P}_2 - 1, \quad \tilde{L}_1 = 0.1\tilde{P}_1 + 0.4\tilde{P}_2, \quad \tilde{R}_1 = 0.1\tilde{P}_1 + 0.4\tilde{P}_2 - 3, \\
\tilde{Q}_2 &= 0.05\tilde{P}_1 - 0.925\tilde{P}_2 + 20, \quad \tilde{L}_2 = 0.05\tilde{P}_1 + 0.075\tilde{P}_2, \quad \tilde{R}_2 = 0.05\tilde{P}_1 + 0.075\tilde{P}_2.
\end{aligned}$$

Firstly, we calculate the solution of set $\mathcal{S}$. In the case of $\tilde{R}_2 = 0$, i.e., $\tilde{P}_1 = -\frac{3}{2}\tilde{P}_2$, it yields that $\mathrm{Ker}(\tilde{R}_2) \subseteq (\mathrm{Ker}C_2 \cap \mathrm{Ker}D_2)$ is not satisfied. Hence, $\tilde{R}_2 \neq 0$. In this situation, by Schurs Lemma, the

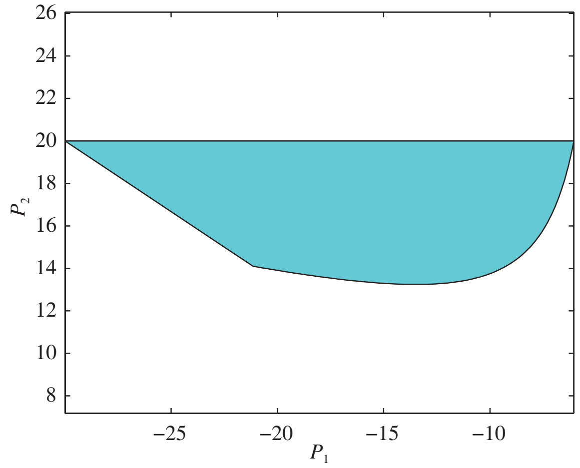

Figure 1 (Color online) The solution set.

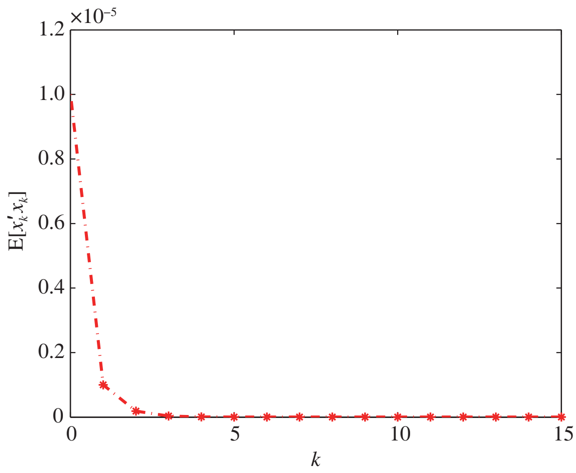

Figure 2 (Color online) Simulation for the state trajectory $\mathrm{E}[x_k'x_k]$.

matric inequality in set $\mathcal{S}$ can be written as

$$\begin{cases} -0.1\tilde{P}_1^2 + 2.6\tilde{P}_1 - 1.6\tilde{P}_2 - 0.4P_1\tilde{P}_2 + 3 \geqslant 0, \\ -0.075\tilde{P}_2^2 + \tilde{P}_1 + 1.5\tilde{P}_2 - 0.05\tilde{P}_1\tilde{P}_2 \geqslant 0, \\ 0.1\tilde{P}_1 + 0.4\tilde{P}_2 - 3 > 0, \\ 0.05\tilde{P}_1 + 0.075\tilde{P}_2 > 0, \\ -0.9\tilde{P}_1 + 0.4\tilde{P}_2 - 1 \geqslant 0, \\ 0.05\tilde{P}_1 - 0.925\tilde{P}_2 + 20 \geqslant 0. \end{cases} \tag{40}$$

We can obtain the solution set of the above inequalities shown as Figure 1 by using MATLAB tool. Further it is obvious to see that for any $\tilde{P} = (\tilde{P}_1, \tilde{P}_2)$ in above solution set, the condition of $\mathrm{Ker}(\tilde{R}_i) \subseteq (\mathrm{Ker}C_i \cap \mathrm{Ker}D_i)$, $i = 1, 2$ is satisfied. Therefore, $\mathcal{S} \neq \emptyset$ and the solution set is expressed as in Figure 1. And the condition of Assumption 1 can be easily tested. Further the GARE-MJ (3) can be solved as $P_{\max} = (\sqrt{103} - 11, 20)$. Therefore, $F_1 = -1.61$ and $F_2 = -1$. When $\theta_k = 2$, that is, the optimal controller is $u_k = -x_k$, in this case $x_k = 0$, and obviously the system is stabilized in the mean square sense. On condition of $\theta_k = 1$, the optimal controller can be given as $u_k = -1.61x_k$ and the simulation result is shown in Figure 2. It can be seen that the state x_k is stabilized with the optimal controller as expected.

5 Conclusion

In this paper, we discussed the stabilization problem for discrete-time MJLSs with multiplicative noise and indefinite weighting matrices. The necessary and sufficient conditions that stabilize the discrete-time MJLSs in the mean square sense with indefinite weighting matrices in the cost have been obtained firstly. Evidently, under the basic assumption that the system is exactly observable, based on linear matrix inequality and kernel restrictions, the stabilization of Markov jump systems is equivalent to the existence of the maximum solution to the GARE-MJ. Finally, an example was given to illustrate the correctness of the main results.

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