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Bipartite consensus problem on matrix-valued weighted directed networks

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Dear editor,

Consensus turns out to be an important paradigm in coordination of multi-agent systems [1]. Previous studies largely concentrate on consensus problem over networks where the weights on edges are scalars, which cannot completely characterize the correlation of different dimensions corresponding to the state of an agent in the network. In fact, matrix-valued weights naturally arise when characterizing the interaction between a pair of agents in multi-agent networks, such as generalized electrical networks where currents, voltages, and resistances take matrix values [2, 3], opinion dynamics of multiple related topics in social network analysis [4], the observability problems for an array of coupled LC oscillators [5], and the consensus problems over matrix-valued weighted networks [6]. Therefore, the matrix-valued weighted networks provide a natural extension of scalar-valued weighted networks, allowing characterizing more elaborate mutual interactions.

In scalar-valued weighted networks, the negatively weighted edges, also referred to as antagonistic interactions, can destroy the inherent stability of the network [7]. To this end, a bipartite consensus protocol is employed and the connection between bipartite consensus and structural balance of the network has been established [8, 9]. In matrix-valued weighted networks, challenges arise in terms of the complexity of the consensus space. For instance, the network connectivity cannot completely guarantee the consensus in the matrix-valued weighted networks due to the existence of the positive semi-definite weighted edges [6]. Moreover, multi-agent coordination on networks with edge weights allowing negative definite or negative semi-definite matrices is not well understood. This study examines the bipartite consensus problem on directed networks whose edge weights allow both positive semi-definite/definite and negative semi-definite/definite matrices. It is shown that the structural balance of the underlying matrix-valued weighted network together with the definiteness of the matrix-valued weights play a central role in determining bipartite consensus and the positive-negative directed tree is introduced to examine the necessary and/or sufficient conditions of bipartite consensus under matrix-valued weighted directed multi-agent networks.

Notations. Let $\mathbb{R}$ and $\mathbb{N}$ be real and natural numbers, respectively. Define $\underline{n} = \{1, 2, \ldots, n\}$ for an $n \in \mathbb{N}$. A matrix $M \in \mathbb{R}^{n\times n}$ is symmetric if $M^{\mathrm{T}} = M$. A symmetric matrix $M \in \mathbb{R}^{n\times n}$ is positive definite (resp. negative definite), denoted by $M \succ 0$ (resp. $M \prec 0$), if $z^{\mathrm{T}}Mz > 0$ (resp. $z^{\mathrm{T}}Mz < 0$) for all $z \in \mathbb{R}^n$ and $z \neq \mathbf{0}$, and is positive (resp. negative) semi-definite, denoted by $M \succeq 0$ (resp. $M \preceq 0$), if $z^{\mathrm{T}}Mz \geqslant 0$ (resp. $z^{\mathrm{T}}Mz \leqslant 0$) for all $z \in \mathbb{R}^n$ and $z \neq \mathbf{0}$. $0_{d\times d}$ denotes the $d \times d$ matrix whose entries are all equal to 0. The absolute value of a matrix $M \in \mathbb{R}^{n\times n}$ is denoted by $|M|$ such that $|M| = M$ if $M \succ 0$ or $M \succeq 0$ and $|M| = -M$ if $M \prec 0$ or $M \preceq 0$. The null space of a matrix $M \in \mathbb{R}^{n\times n}$ is $\mathrm{null}(M) = \{z \in \mathbb{R}^n | Mz = 0\}$.

Main results. Consider a multi-agent network consisting of $n \in \mathbb{N}$ agents. The state of an agent $i \in \underline{n}$ is denoted by a vector $x_i(t) = [x_{i1}, x_{i2}, \ldots, x_{id}]^{\mathrm{T}} \in \mathbb{R}^d$, where $d \in \mathbb{N}$ is corresponding to the dimension of weight matrices. The state of the multi-agent network is denoted by $x(t) = [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t), \ldots, x_n^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{dn}$. The interaction topology of the multi-agent system is characterized by a matrix-valued weighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$. The node set and the edge set of $\mathcal{G}$ are denoted by $\mathcal{V} = \{1, 2, \ldots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, respectively. The matrix-valued weight is a matrix $A_{ij} \in \mathbb{R}^{d\times d}$ such that $|A_{ij}| \succeq 0$ or $|A_{ij}| \succ 0$ if $(j, i) \in \mathcal{E}$ and $A_{ij} = 0_{d\times d}$ otherwise, where the node i and j are referred to as head and tail, respectively. An edge $(j, i) \in \mathcal{E}$ is positive (resp. negative) definite or positive (resp. negative) semi-definite if the corresponding weight matrix A_{ij} is positive (resp. negative) definite or positive (resp. negative) semi-definite. Thereby, the matrix-valued weighted adjacency matrix $A = [A_{ij}] \in \mathbb{R}^{dn\times dn}$ is a block matrix such that the block located in the i-th row and the j-th column is A_{ij}. We assume that $A_{ii} = 0_{d\times d}$ for all $i \in \mathcal{V}$. The in-degree neighbor set of an agent $i \in \mathcal{V}$ is denoted by

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$\mathcal{N}_i = \{j \in \mathcal{V} \,|\, (j,i) \in \mathcal{E}\}$. The out-degree neighbor set of an agent $i \in \mathcal{V}$ is denoted by $\mathcal{N}'_i = \{j \in \mathcal{V} \,|\, (i,j) \in \mathcal{E}\}$. Let $C_r = \mathrm{diag}\{C_{r1}, C_{r2}, \ldots, C_{rn}\} \in \mathbb{R}^{dn}$ denote the matrix-valued weighted row degree matrix of a graph where $C_{ri} = \sum_{j\in\mathcal{N}_i} |A_{ij}| \in \mathbb{R}^{d\times d}$. Let $C_c = \mathrm{diag}\{C_{c1}, C_{c2}, \ldots, C_{cn}\} \in \mathbb{R}^{dn}$ denote the matrix-valued weighted column degree matrix of a graph where $C_{ci} = \sum_{j\in\mathcal{N}'_i} |A_{ji}| \in \mathbb{R}^{d\times d}$. If $C_r = C_c$, the matrix-valued weighted directed graph $\mathcal{G}$ is balanced. The matrix-valued weighted Laplacian matrix of a matrix-valued weighted directed graph $\mathcal{G}$ is defined as $L^s = C_r - A$. Analogously, a directed path in a matrix-valued weighted directed graph $\mathcal{G}$ is a sequence of directed edges in the form of $(i_1, i_2), (i_2, i_3), \ldots, (i_{p-1}, i_p)$ where nodes $i_1, i_2, \ldots, i_p \in \mathcal{V}$ are all distinct. A matrix-valued weighted directed graph $\mathcal{G}$ is strongly connected if there exists a directed path between any two distinct nodes in $\mathcal{G}$. A directed cycle in a matrix-valued weighted directed graph $\mathcal{G}$ is a directed path starting and ending with the same node. A matrix-valued weighted directed graph is acyclic if it contains no directed cycles. A directed tree in a matrix-valued weighted directed graph $\mathcal{G}$ is an acyclic directed graph with the following property: there exists a node, called the root, such that any other nodes in $\mathcal{G}$ can be reached by one and only one directed path starting at the root. A positive-negative directed path in a matrix-valued weighted directed graph $\mathcal{G}$ is a directed path such that every edge in this directed path is positive definite or negative definite. A positive-negative directed tree in a matrix-valued weighted directed graph $\mathcal{G}$ is a directed tree such that every edge in this directed tree is positive definite or negative definite. A positive-negative directed spanning tree of a matrix-valued weighted directed graph $\mathcal{G}$ is a positive-negative directed tree containing all nodes in $\mathcal{G}$. The positive-negative directed spanning tree in the matrix-valued weighted directed networks is essentially more complicated than the directed spanning tree in the scalar-valued weighted directed networks. An example of a matrix-valued weighted directed network is shown in Figure A1. The matrix-sign function $\mathrm{sgn}(\cdot) : \mathbb{R}^{d\times d} \mapsto \{0, -1, 1\}$ for the matrix-valued weight A_{ij} of an edge $(j,i) \in \mathcal{E}$ satisfies that $\mathrm{sgn}(A_{ij}) = 1$ if $A_{ij} \succeq 0$ or $A_{ij} \succ 0$, $\mathrm{sgn}(A_{ij}) = -1$ if $A_{ij} \preceq 0$ or $A_{ij} \prec 0$, and $\mathrm{sgn}(A_{ij}) = 0$ if $A_{ij} = 0_{d\times d}$. A pair of edges in a directed graph sharing the same nodes is referred to as a digon.

Assumption 1. This study assumes that $\mathrm{sgn}(A_{ij})\mathrm{sgn}(A_{ji}) \geqslant 0$ for all $i, j \in \underline{n}$, which is referred to as the digon sign-symmetry.

Under Assumption 1, a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ admits an undirected mirror graph $\mathcal{G}_u = (\mathcal{V}, \mathcal{E}_u, A_u)$ where $A_u = (A(\mathcal{G}) + A(\mathcal{G})^{\mathrm{T}})/2$.

The bipartite consensus protocol for each agent $i \in \mathcal{V}$ in a matrix-valued weighted directed network is dictated by

$$\dot{x}_i(t) = -\sum_{j\in\mathcal{N}_i} |A_{ij}|(x_i(t) - \mathrm{sgn}(A_{ij})x_j(t)), \tag{1}$$

and the resultant dynamics of the multi-agent system is characterized by

$$\dot{x}(t) = -L^s x(t). \tag{2}$$

In signed networks, the concept of structural balance turns out to be an important graph-theoretic object playing a critical role in bipartite consensus problems [8]. Here, we extend this concept to the matrix-valued weighted networks.

Definition 1. A bipartition of node set $\mathcal{V}$ of matrix-valued weighted network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ is a pair of two subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ such that $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$.

Definition 2. A matrix-valued weighted network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ is structurally balanced if there exists a bipartition of the node set $\mathcal{V}$, $\mathcal{V}_1$ and $\mathcal{V}_2$, such that the matrix weights on the edges within each subset are positive definite or positive semi-definite, but negative definite or negative semi-definite for the edges between the two subsets; $\mathcal{G}$ is structurally imbalanced if it is not structurally balanced.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a structurally balanced matrix-valued weighted directed network with a node bipartition $\mathcal{V}_1$ and $\mathcal{V}_2$ and $d \in \mathbb{N}$ representing the dimension of edge weight. The matrix-valued gauge transformation for $\mathcal{G}$ is performed by the diagonal matrix $D^* = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ where $\sigma_i = I_d$ if $i \in \mathcal{V}_1$ and $\sigma_i = -I_d$ if $i \in \mathcal{V}_2$.

Definition 3. The multi-agent system (2) admits a bipartite consensus solution if $\lim_{t\to\infty}|x_i(t)| = \lim_{t\to\infty}|x_j(t)| \neq \mathbf{0}$ for all $i, j \in \mathcal{V}$.

Theorem 1. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a structurally balanced matrix-valued weighted directed network and D^* be a matrix-valued gauge transformation such that $D^*AD^* = [|A_{ij}|]$. For the initial value $x(0)$ satisfying $(\mathbf{1}^{\mathrm{T}} \otimes I_d)D^*x(0) \neq \mathbf{0}$, the multi-agent network (2) admits a bipartite consensus solution if and only if $\mathrm{null}(L^s) = \mathcal{R}$, and the bipartite solution of (2) is $\tilde{x} = D^*(\mathbf{1}\otimes(\frac{1}{n}(\mathbf{1}^{\mathrm{T}}\otimes I_d)D^*x(0)))$.

Theorem 2. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a structurally balanced matrix-valued weighted directed network. If $\mathcal{G}$ has a positive-negative directed spanning tree, let D^* be a gauge transformation such that $D^*AD^* = [|A_{ij}|]$. For the initial value $x(0)$ satisfying $(\mathbf{1}^{\mathrm{T}} \otimes I_d)D^*x(0) \neq \mathbf{0}$, the multi-agent network (2) admits a bipartite consensus solution.

Theorems 1 and 2 provide the conditions for the bipartite consensus from the algebraic and graph-theoretic perspectives, respectively. Next, we shall examine conditions for the bipartite consensus under the matrix-valued weighted directed networks with only positive definite and negative definite edges.

Theorem 3. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a matrix-valued weighted strongly connected directed balanced network. Let L^s denote the Laplacian matrix of $\mathcal{G}$ and $\kappa \in \mathbb{N}$ denote the algebraic multiplicity of eigenvalue zero of L^s. If the edge weight matrix $A_{ij} \in \mathbb{R}^{d\times d}$ satisfies $|A_{ij}| \succ 0$ for all $(j,i) \in \mathcal{E}$, then $\mathcal{G}$ is structurally balanced if and only if $\kappa = d$.

Note that for a matrix-valued weighted strongly connected directed balanced network, the structural balance is not a sufficient condition for bipartite consensus. However, the structural balance is necessary and sufficient for bipartite consensus when the matrix-valued weights are either positive/negative definite or null. It turns out that the semi-definiteness of matrix-valued weights plays a negative role in achieving a bipartite consensus solution to multi-agent system (2).

Conclusion. This study investigates the bipartite consensus problem on matrix-valued weighted directed networks. Necessary and/or sufficient conditions for bipartite consensus are provided in terms of the null space of matrix-valued weighted Laplacian matrix and the positive-negative directed spanning tree. Simulation examples are given in Appendix E to illustrate the theoretical results.

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Supporting information Appendixes A–E. The supporting information is available online at info.scichina.com and link.

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