SCIENCE CHINA
Information Sciences

• **Supplementary File** •

Bipartite Consensus Problem on Matrix-valued Weighted Directed Networks

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Appendix A An example of matrix-valued weighted directed graph

For instance, the node set and edge set of the positive-negative directed spanning tree $\mathcal{T}$ of $\mathcal{G}$ in Figure A1 are $\mathcal{V} = \{1, 2, \ldots, 10\}$ and $\mathcal{E} = \{(2,1),(1,3),(3,4),(4,6),(3,5),(5,7),(7,8),(5,9),(9,10)\}$, respectively.

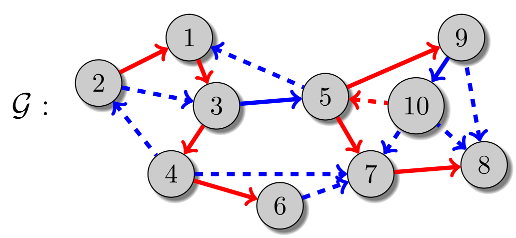

Figure A1 A matrix-valued weighted directed graph $\mathcal{G}$. The positive definite and the positive semi-definite matrix-valued weights are illustrated by solid lines and dashed lines in blue, respectively; the negative definite and the negative semi-definite matrix-valued weights are illustrated by solid lines and dashed lines in red, respectively.

Appendix B Proof of Theorem 1

The following lemma is important related to the null space of the matrix-valued weighted Laplacian matrix.

Definition 1. Define $\mathcal{R} = \textbf{range}\{D^*(\mathbf{1} \otimes I_d)\}$ as the *bipartite consensus subspace.*

Note that the matrix-valued weighted Laplacian L^s has at least d zero eigenvalues. Let $\lambda_1 \leqslant \cdots \leqslant \lambda_{dn}$ be the eigenvalues of L^s, then $0 = \lambda_1 = \cdots = \lambda_d \leqslant \lambda_{d+1} \leqslant \cdots \leqslant \lambda_{dn}$.

Lemma 1. A digon sign-symmetric matrix-valued weighted directed network $\mathcal{G}$ is structurally balanced if and only if $\mathcal{G}_u$ is structurally balanced.

Lemma 2. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a directed graph with corresponding undirected mirror graph $\mathcal{G}_u = (\mathcal{V}, \mathcal{E}_u, A_u)$. If $\mathcal{G}$ is digon sign-symmetric and balanced, then $L^s(\mathcal{G}_u) = (L^s(\mathcal{G}) + L^s(\mathcal{G})^T)/2$.

Lemma 3. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a matrix-valued weighted undirected network. If $\mathcal{G}$ is structurally balanced and has a positive-negative spanning tree, then the multi-agent system (2) admits a bipartite consensus solution.

Lemma 4. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a matrix-valued weighted undirected network. Denote the Laplacian matrix of $\mathcal{G}$ as L^s and the algebraic multiplicity of eigenvalue zero of L^s as $\kappa \in \mathbb{N}$. If the edge weight matrix $A_{ij} \in \mathbb{R}^{d\times d}$ ($d \in \mathbb{N}$) is such that $|A_{ij}| \succ 0$ for all $(i,j) \in \mathcal{E}$, then $\mathcal{G}$ is structurally balanced if and only if $\kappa = d$.

We now provide the proof of Theorem 1 as follows.

Proof. (Necessity) Assume the multi-agent system (2) admits a bipartite consensus solution $\tilde{\boldsymbol{x}} = D^*(\mathbf{1} \otimes (\frac{1}{n}(\mathbf{1}^T \otimes I_d)D^*\boldsymbol{x}(0)))$ and $\textbf{null}(L^s) \neq \mathcal{R}$. Then, there exists an $\boldsymbol{x}' \in \mathbb{R}^{dn}$ such that $L^s\boldsymbol{x}' = 0$ and $\boldsymbol{x}' \notin \mathcal{R}$. Choose $\boldsymbol{x}(0) = \boldsymbol{x}'$ and note that $L^s\boldsymbol{x}' = 0$, then $\boldsymbol{x}(t) = \boldsymbol{x}'$ for all $t \geqslant 0$. However, the multi-agent system (2) converges to $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{x}} \in \mathcal{R}$, which establishes a contradiction. Therefore, $\textbf{null}(L^s) = \mathcal{R}$.

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(Sufficiency) Consider a Lyapunov function $V(\boldsymbol{\delta}) = \frac{1}{2}\boldsymbol{\delta}^T\boldsymbol{\delta}$, where $\boldsymbol{\delta} = \boldsymbol{x} - \tilde{\boldsymbol{x}}$. Then $V(\boldsymbol{\delta})$ is positive semi-definite and $\dot{\boldsymbol{\delta}} = \dot{\boldsymbol{x}} = -L^s\boldsymbol{x} = -L^s\boldsymbol{\delta}$. Therefore,

$$\begin{aligned}\dot{V}(\boldsymbol{\delta}) &= \boldsymbol{\delta}^T\dot{\boldsymbol{\delta}}\\ &= \boldsymbol{\delta}^T\dot{\boldsymbol{x}}\\ &= -\boldsymbol{\delta}^T L^s\boldsymbol{x}\\ &= -\boldsymbol{\delta}^T L^s\boldsymbol{\delta}\\ &= -\boldsymbol{\delta}^T(\frac{1}{2}(L^s + {L^s}^T))\boldsymbol{\delta}\\ &= -\boldsymbol{\delta}^T L_u\boldsymbol{\delta}\\ &\leqslant -\lambda_2(\mathcal{G}_u)\boldsymbol{\delta}^T\boldsymbol{\delta}\\ &\leqslant 0.\end{aligned}$$

Moreover, $\dot{V}(\boldsymbol{\delta}) = 0$ if and only if $\boldsymbol{\delta} = \boldsymbol{0}$. Thus, the multi-agent system (2) admits a bipartite consensus solution $\tilde{\boldsymbol{x}}$.

Appendix C Proof of Theorem 2

Proof. Assume $\mathcal{G}$ is structurally balanced and has a positive-negative directed spanning tree, but the multi-agent system (2) can not admit a bipartite consensus solution $\tilde{\boldsymbol{x}} = D^*(\boldsymbol{1} \otimes (\frac{1}{n}(\boldsymbol{1}^T \otimes I_d)D^*\boldsymbol{x}(0)))$. According to Theorem 1, it is satisfied that $\mathbf{null}(L^s) \neq \mathcal{R}$ and there exists an $\boldsymbol{x}' \in \mathbb{R}^{dn}$ such that $L^s\boldsymbol{x}' = 0$ and $\boldsymbol{x}' \notin \mathcal{R}$. Then, considering the undirected matrix-valued weighted network $\mathcal{G}_u$,

$$\boldsymbol{x}'^T L^s(\mathcal{G}_u)\boldsymbol{x}' = \boldsymbol{x}'^T(\frac{1}{2}(L^s + (L^s)^T))\boldsymbol{x}' = 0.$$

Therefore, $\mathbf{null}(L^s(\mathcal{G}_u)) \neq \mathcal{R}$. However, $\mathcal{G}$ is structurally balanced and has a positive-negative directed spanning tree, then $\mathcal{G}_u$ is structurally balanced and has a positive-negative spanning tree. According to Lemma 3, $\mathbf{null}(L^s(\mathcal{G}_u)) = \mathcal{R}$, which establishes a contradiction. Therefore, $\mathbf{null}(L^s) = \mathcal{R}$ and the multi-agent system (2) admits a bipartite consensus solution.

Appendix D Proof of Theorem 3

Proof. (Necessity) Since $|A_{ij}| \succ 0$ for all $(j, i) \in \mathcal{E}$ and the matrix-valued weighted directed graph $\mathcal{G}$ is strongly connected, then $\mathcal{G}$ has a positive-negative directed spanning tree. If $\mathcal{G}$ is structurally balanced, then according to the Theorem 1, L^s has zero eigenvalue and the null space of L^s satisfies $\mathbf{null}(L^s) = \mathcal{R}$. Consequently, the algebraic multiplicity of zero eigenvalue of L^s equals to d which is the dimension of the matrix-valued weights.

(Sufficiency) Note that zero is an eigenvalue of L^s with algebraic multiplicity d. Let $\lambda_1 = \cdots = \lambda_d = 0$. Denote the non-zero vector $\boldsymbol{w} = [\boldsymbol{w}_1^T, \boldsymbol{w}_2^T, \ldots, \boldsymbol{w}_n^T,]^T \in \mathbb{R}^{dn}$ as the eigenvector corresponding to the zero eigenvalue of L^s where $\boldsymbol{w}_i \in \mathbb{R}^d$ for all $i \in \underline{n}$. Clearly, $L^s\boldsymbol{w} = \boldsymbol{0}$ and $\boldsymbol{w}^T{L^s}^T = \boldsymbol{0}$. Thus

$$\frac{1}{2}\boldsymbol{w}^T(L^s + {L^s}^T)\boldsymbol{w} = \boldsymbol{w}^T L^s(\mathcal{G}_u)\boldsymbol{w} = 0, \tag{D1}$$

where $L^s(\mathcal{G}_u)$ is the Laplacian matrix of the undirected matrix-valued weighted network $\mathcal{G}_u$ corresponding to $\mathcal{G}$. From D1, we have 0 is an eigenvalue of $L^s(\mathcal{G}_u)$, therefore $\mathcal{G}_u$ is structurally balanced, then, we obtain $\mathcal{G}$ is structurally balanced.

A notable corollary of the Theorem 3 is presented below.

Corollary 1. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a matrix-valued weighted strongly connected directed balanced network, let D^* be a matrix-valued gauge transformation such that $D^*AD^* = [|A_{ij}|]$. If the edge weight matrix $A_{ij} \in \mathbb{R}^{d\times d}$ ($d \in \mathbb{N}$) is such that $|A_{ij}| \succ 0$ for all $(j, i) \in \mathcal{E}$. Then for the initial value $\boldsymbol{x}(0)$ satisfying $(\boldsymbol{1}^T \otimes I_d)D^*\boldsymbol{x}(0) \neq \boldsymbol{0}$, the multi-agent network (2) admits a bipartite consensus solution if and only if $\mathcal{G}$ is structurally balanced.

Corollary 2. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a matrix-valued weighted strongly connected directed balanced network. If the edge weight matrix $A_{ij} \in \mathbb{R}^{d\times d}$ ($d \in \mathbb{N}$) is such that $|A_{ij}| \succ 0$ for all $(j, i) \in \mathcal{E}$, then $\mathcal{G}$ is structurally imbalanced if and only if all eigenvalues of $L^s(\mathcal{G})$ are positive.

Proof. (Necessity) Assume zero is an eigenvalue of $L^s(\mathcal{G})$, then we have zero is an eigenvalue of $L^s(\mathcal{G}_u)$, then we obtain $\mathcal{G}_u$ is structurally balanced, thus $\mathcal{G}$ is structurally balanced, which establishes a contradiction. Therefore, all eigenvalues of $L^s(\mathcal{G})$ are positive.

(Sufficiency) This is an immediate result of the Theorem 3.

Corollary 3. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a matrix-valued weighted strongly connected directed balanced network and $\mathcal{G}$ is structurally imbalanced. If the edge weight matrix $A_{ij} \in \mathbb{R}^{d\times d}$ ($d \in \mathbb{N}$) is such that $|A_{ij}| \succ 0$ for all $(j, i) \in \mathcal{E}$, then the states of all the agents converge to zero.

Appendix E Simulations

In this section, we provide simulation examples regarding to the matrix-valued weighted directed networks in Figure E1 to demonstrate the theoretical results in this letter. Firstly, we shall examine two examples in which case the positive (semi-)definite and negative (semi-)definite edge weights are allowed, and subsequently examine another two examples where the edges are either positive/negative definite or null.

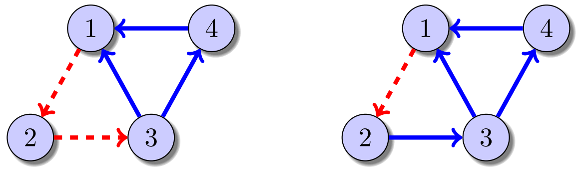

Figure E1 A structurally balanced matrix-valued weighted directed network (left) and a structurally imbalanced matrix-valued weighted directed network (right). The positive definite and the positive semi-definite matrix-valued weights are illustrated by solid lines in blue; the negative definite and the negative semi-definite matrix-valued weights are illustrated by dashed lines in red.

Example 1. Consider the structurally balanced matrix-valued weighted directed network in the left panel in Figure E1. Let

$$A_{21} = -\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

$$A_{32} = -\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

$$A_{14} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

$$A_{13} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$A_{43} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Note that $A_{32} \prec 0$, $A_{21} \prec 0$, $A_{14} \succ 0$, $A_{13} \succeq 0$ and $A_{43} \succ 0$ and the network has a positive-negative directed spanning tree in this case, then the multi-agent network (2) admits a bipartite consensus solution in this example as shown in Figure E2.

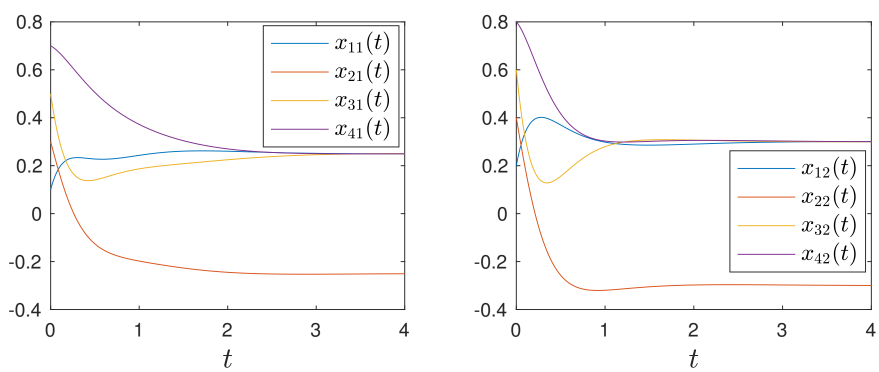

Figure E2 The trajectory of multi-agent system (2) under the structurally balanced matrix-valued weighted directed network in the left panel in Figure E1.

Example 2. Consider the structurally balanced matrix-valued weighted directed network in the left panel in Figure E1. Let

$$A_{21} = -\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix},$$

$$A_{32} = -\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix},$$

$$A_{14} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix},$$

$$A_{13} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix},$$

$$A_{43} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

Note that $A_{32} \preceq 0$, $A_{21} \preceq 0$, $A_{14} \succeq 0$, $A_{13} \succeq 0$ and $A_{43} \succeq 0$. Examining the dimension of the null space of the matrix-valued weighted Laplacian matrix

$$L^s = \begin{bmatrix} 2 & 4 & 0 & 0 & -1 & -2 & -1 & -2 \\ 4 & 8 & 0 & 0 & -2 & -4 & -2 & -4 \\ 2 & 4 & 2 & 4 & 0 & 0 & 0 & 0 \\ 4 & 8 & 4 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 4 & 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 8 & 4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 & -2 & -4 & 2 & 4 \end{bmatrix} \tag{E1}$$

yields that $\mathbf{dim}(\mathbf{null}(L^s)) = 5$, then $\mathbf{null}(L^s) \neq \mathcal{R}$ implying that the multi-agent network (2) cannot achieve a bipartite consensus solution as shown in Figure E3.

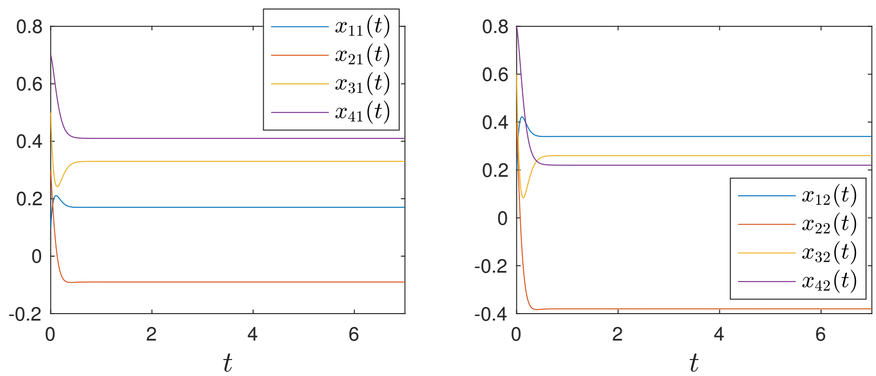

Figure E3 The trajectory of multi-agent system (2) under the structurally balanced matrix-valued weighted directed network in the left panel in Figure E1.

Example 3. Consider the structurally balanced matrix-valued weighted directed network in the left panel in Figure E1. Let

$$A_{21} = -\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

$$A_{32} = -\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

$$A_{14} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

$$A_{13} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A_{43} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Note that $A_{32} \prec 0$, $A_{21} \prec 0$, $A_{14} \succ 0$, $A_{13} \succ 0$ and $A_{43} \succ 0$ and the multi-agent network (2) admits a bipartite consensus solution as shown in Figure E4.

Example 4. Consider the structurally imbalanced matrix-valued weighted directed network in the right panel in Figure E1. Let

$$A_{21} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

$$A_{32} = -\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

$$A_{14} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

$$A_{13} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

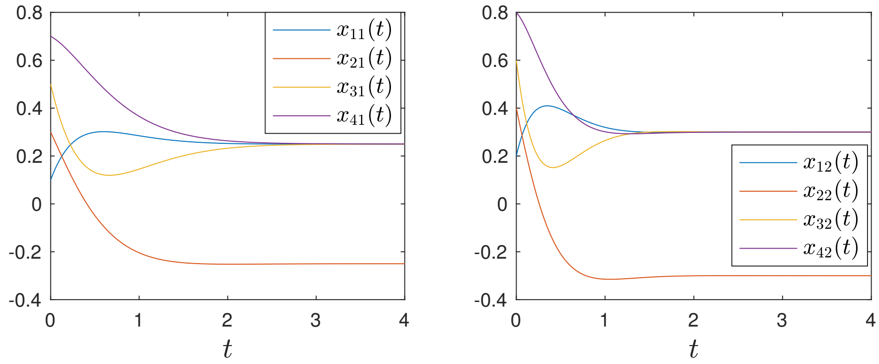

Figure E4 The trajectory of multi-agent system (2) under the structurally balanced matrix-valued weighted directed network in the left panel in Figure E1.

$$A_{43} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Note that $A_{32} \prec 0$, $A_{21} \succ 0$, $A_{14} \succ 0$, $A_{13} \succ 0$ and $A_{43} \succ 0$, and the multi-agent network (2) in this case admits a asymptotical stable solution ($\lim_{t\to\infty} \boldsymbol{x}_i(t) = \mathbf{0}$ for all $i \in \mathcal{V}$) as shown in Figure E5.

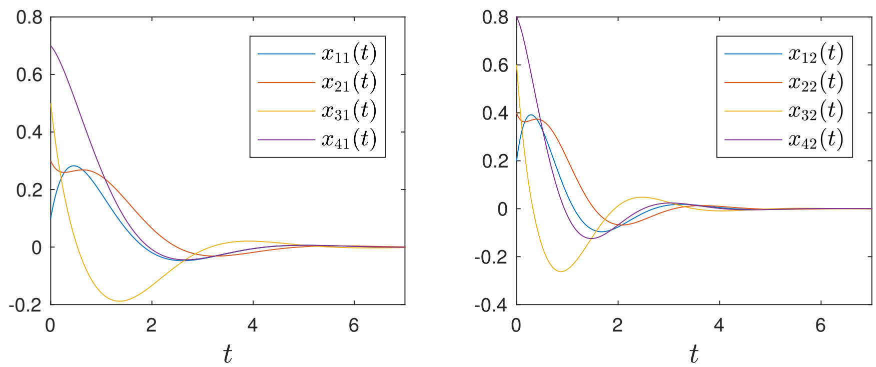

Figure E5 The trajectory of multi-agent system (2) under the structurally imbalanced matrix-valued weighted directed network in the right panel in Figure E1.