SCIENCE CHINA
Information Sciences

• LETTER •

April 2021, Vol. 64 149202:1–149202:3
https://doi.org/10.1007/s11432-018-9763-7

Finite element approach to continuous potential games

Yaqi HAO[1*] & Daizhan CHENG[2]

[1]*School of Management Science and Engineering, Shandong Normal University, Ji'nan* 250014*, China;*
[2]*Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing* 100190*, China*

Received 17 November 2018/Accepted 27 December 2018/Published online 14 April 2020

Citation Hao Y Q, Cheng D Z. Finite element approach to continuous potential games. Sci China Inf Sci, 2021, 64(4): 149202, https://doi.org/10.1007/s11432-018-9763-7

Dear editor,

The concept of potential games was proposed for the first time by Rosenthal in 1973 [1]. The seminal study by Monderer and Shapley [2] presented systematic investigation and fundamental results for this kind of games. Some of its recent developments can be found in [3]. Because of some nice properties, potential games have been applied to many control and optimization problems, for instance the existence of Nash equilibria [4] and game theoretic control [5].

In either theoretical investigations or applications, two fundamental issues should be solved, which are (i) verifying whether a game is potential and (ii) calculating potential function.

These two problems are closely related. Unfortunately, even for finite games, verifying whether they are potential is difficult. In 2011, a new algorithm [6] has been proposed and compared with several early studies. It is also mentioned in [6] that "It is not easy, however, to verify whether a given game is a potential game".

In 2014, Ref. [7] proposed a set of linear equations, called potential equation, to verify whether a finite game is potential. Meanwhile, a formula was also presented to calculate potential function for a potential game. Using potential equation, Ref. [8] proposed a minimum number of equations to verify it. These studies provided a satisfactory solution to verify finite potential games [9].

As for continuous games, Ref. [2] provides the following result for verifying whether a game is potential.

Theorem 1. Let $G = (N, S, C)$ be a continuous game, where S_i, $i = 1, \dots, n$, are intervals, and c_i, $i = 1, \dots, n$, are of class $C^2(S)$. Then

(1) G is an exact potential game, if and only if

$$\frac{\partial^2 c_j}{\partial x_i \partial x_j} = \frac{\partial^2 c_i}{\partial x_i \partial x_j}, \quad \forall i, j \in N; \tag{1}$$

(2) If Eq. (1) holds and $x_0 \in S$ is an arbitrary (but fixed) profile, then, an exact potential for G is given by

$$P(x) := \int_0^t \sum_{j=1}^n \frac{\partial^2 c_j}{\partial x_j}(\gamma(t))\gamma_j'(t)\mathrm{d}t, \tag{2}$$

where $\gamma : [0, 1] \to S$ is a piecewise continuously differentiable path such that $\gamma(1) = x$ and $\gamma(0) = x_0$.

This study considers the continuous game $G = (N, S, C)$, where $C = (c_1, c_2, \dots, c_n)$ and $c_i \in C^0(S)$, $i = 1, \dots, n$. Our goal is to provide a numerical algorithm to calculate the potential functions for continuous potential games. Since $c_i \notin C^2(S)$, Eq. (1) cannot be used to verify whether G is potential. Similarly, Eq. (2) cannot be used to calculate the potential function. To our best knowledge, there is no existing result to solve this problem.

Moreover, even $c_i \in C^2(S)$, $\forall i$, Eq. (2) is very complicated in computation. So our results are also useful for even $c_i \in C^2(S)$.

The main idea for solving this problem is to use finite element approach to solve continuous problem. Precisely speaking, the strategies $S_i = I_i$, where $I_i \subset \mathcal{R}$ is a closed interval, is quantized into a finite set for each $i \in N$. Restricting the continuous game on the finite set $A = \prod_{i=1}^n I_i$ yields a finite sub-game. Then the formula for finite game is used to get a potential function for the sub-game. A linear interpolation of this potential function is constructed to approximate the real potential function. The convergence of this algorithm is proved as the number of sampling points increases. Moreover, to avoid computational complexity, another algorithm is proposed to combine two finite potential functions together for improving the accuracy of the constructed approximated potential functions.

Finite element solution. Consider a continuous game $G = (N, S, C)$. Without loss of generality, we assume $S_i = [a_i, b_i], \forall i$, being compact sets. Let $\{x^i\} := \{x^i_1, x^i_2, \dots, x^i_{n_i}\}$ be the set of points selected from S_i, satisfying $a_i = x^i_1 < x^i_2 < \cdots < x^i_{n_i} = b_i$, denoted as $A_i := \{x^i\}$. It is clear that $\widetilde{G} = (N, A, \widetilde{C})$ is a finite sub-game of G, where $N = \{1, 2, \dots, n\}$, $A = \prod_{i=1}^n A_i \subset S$, $\widetilde{c}_i = c_i\big|_A : A \to \mathcal{R}$.

* Corresponding author (email: hoayaqi@outlook.com)

For the finite game $\widetilde{G} = (N, A, \widetilde{C})$, we can use potential equation in [7] to verify whether it is potential. If it is, its potential function $\widetilde{P} : A \to \mathcal{R}$ can be calculated via [7]. In the following, we assume the continuous game $G = (N, S, C)$ is a potential one with its continuous potential function $P : S \to \mathcal{R}$. To approximate P by using data obtained from sub-games, an n-linear interpolation can be used as follows.

Step 1. Let $x = \{x_1, x_2, \ldots, x_n\} \in S$. For each x_i, there exist two points $x^i_{m_i}, x^i_{m_i+1} \in A_i$, such that

$$x^i_{m_i} \leqslant x_i \leqslant x^i_{m_i+1},$$
$$m_i = 1, \ldots, n_i - 1, \quad i = 1, 2, \ldots, n.$$

Then, for each x_i, there is a unique nonnegative real number $\mu_i \in [0, 1]$ satisfying

$$\begin{aligned} x_i &:= \mu_i x^i_{m_i} + (1 - \mu_i) x^i_{m_i+1} \\ &= \omega^1_i x^i_{m_i} + \omega^2_i x^i_{m_i+1}, \end{aligned} \tag{3}$$

where $\omega^1_i = \mu_i$, $\omega^2_i = 1 - \mu_i$, and ω^j_i, $j = 1, 2$, are called the weights with respect to $x^i_{m_i}$ and $x^i_{m_i+1}$, respectively.

Step 2. Define the set of weight-pairs and point-pairs, respectively, as

$$\begin{aligned} W :=& \{(\omega^1_1, \omega^2_1), (\omega^1_2, \omega^2_2), \ldots, (\omega^1_n, \omega^2_n)\}, \\ D :=& \{(x^1_{m_1}, x^1_{m_1+1}), (x^2_{m_2}, x^2_{m_2+1}), \ldots, \\ & (x^n_{m_n}, x^n_{m_n+1})\} \\ :=& \{(x^{(1,1)}, x^{(1,2)}), (x^{(2,1)}, x^{(2,2)}), \ldots, \\ & (x^{(n,1)}, x^{(n,2)})\}. \end{aligned}$$

Define a function $f : S \to R$ as

$$\begin{aligned} f(x_1, \ldots, x_n) := \sum_{t_i \in \{1,2\}} \prod_{j=1}^{n} \omega^{t_j}_j \widetilde{P}\big(x^{(1,t_1)}, \\ \ldots, x^{(n,t_n)}\big), \end{aligned}$$

which is called the n-linear interpolated potential function with respect to the continuous potential game $G = (N, S, C)$. We can verify that it satisfies

$$\begin{gathered} f(x^1, x^2, \ldots, x^n) = \widetilde{P}(x^1, x^2, \ldots, x^n), \\ x^i \in A_i,\ i = 1, 2, \ldots, n; \\ \sum_{t_i \in \{1,2\}} \omega^{t_1}_1 \omega^{t_2}_2 \cdots \omega^{t_n}_n = 1. \end{gathered}$$

Step 3. Arrange the superscripts of $(\omega^{t_1}_1, \omega^{t_2}_2, \ldots, \omega^{t_n}_n)$ as $(t_1, t_2, \ldots, t_n)$ and set

$$T := \{(t_1, t_2, \ldots, t_n) \mid t_i \in \{1, 2\},\ i = 1, 2, \ldots, n\}.$$

Arrange the elements in T in the alphabetic order as

$$\begin{aligned} T &:= \{(1, \ldots, 1, 1), (1, \ldots, 1, 2), \ldots, (2, \ldots, 2, 2)\} \\ &:= \{l_1, l_2, \ldots, l_{2^n}\}, \end{aligned}$$

where $l_j := (l^1_j, l^2_j, \ldots, l^n_j)$.

Step 4. Let

$$\begin{aligned} \omega_{l_j} &:= \omega^{l^1_j}_1 \omega^{l^2_j}_2 \cdots \omega^{l^n_j}_n, \\ x_{l_j} &:= \left(x^{(1,l^1_j)}, x^{(2,l^2_j)}, \ldots, x^{(n,l^n_j)}\right). \end{aligned} \tag{4}$$

Using (4), $f(x_1, x_2, \ldots, x_n)$ can be rewritten as

$$f(x_1, x_2, \ldots, x_n) := \sum_{j=1}^{2^n} \omega_{l_j} \widetilde{P}(x_{l_j}), \tag{5}$$

where

$$\sum_{j=1}^{2^n} \omega_{l_j} = 1. \tag{6}$$

The n-linear approximation potential function f obtained by using n-linear approximation algorithm can be proved to be convergent to the accurate potential function P.

Merge interpolation. As the number of sampling points increases, the approximation accuracy increases. This is an obvious fact. But when the number of sampling points for each player increases to twice (precisely speaking, from n_i to $2n_i$), the computational complexity will increase to 2^n. In the following, we propose a method, called the data merge, to reduce the computational complexity to 2 only.

Assume a set of points

$$X := \left\{\{x^i\} \mid i = 1, \ldots, n\right\},$$

where

$$X_i := \{x^i\} = \{x^i_1, x^i_2, \ldots, x^i_{n_i}\} \subset S_i,\ i = 1, \ldots, n$$

are chosen. Define the set of profiles as

$$A_X := \prod_{i=1}^{n} X_i,$$

and the finite potential function as $\widetilde{P}_X : A_X \to \mathcal{R}$. Using the n-linear interpolation, we can get an n-linear interpolation function $f_1(x)$, $x \in S$ based on $\widetilde{P}_X$. Similarly, another set of points

$$Z := \left\{\{z^j\} \mid j = 1, \ldots, n\right\},$$

where

$$\begin{aligned} & Z_j := \{z^j\} = \{z^j_1, z^j_2, \ldots, z^j_{m_j}\} \subset S_j, \\ & j = 1, \ldots, n, \end{aligned}$$

are used to construct another interpolation function $f_2(x)$, $x \in S$ based on $\widetilde{P}_Z : A_Z \to \mathcal{R}$.

Because for a potential game, its potential function is unique up to a constant number, we assume

$$P_Z - P_X = C, \tag{7}$$

where P_X and P_Z are the potential functions, satisfying

$$\begin{aligned} P_X(s) = \widetilde{P}_X(s), &\quad s \in A_X, \\ P_Z(s) = \widetilde{P}_Z(s), &\quad s \in A_Z. \end{aligned}$$

In fact, in each set of points, it is obvious that

$$x^i_1 = z^i_1 = a_i,\ x^i_{n_i} = z^i_{m_i} = b_i,\ i = 1, 2, \ldots, n.$$

Hence, there are at least two same profiles in A_X and A_Z, i.e., $a := (a_1, a_2, \ldots, a_n)$, $b := (b_1, b_2, \ldots, b_n)$. Therefore, using (7) yields

$$C = P_X(a(b)) - P_Z(a(b)) = \widetilde{P}_Z(a(b)) - \widetilde{P}_X(a(b)).$$

Put all the selected points together as

$$\Xi := \left\{\{\xi^1\}, \dots, \{\xi^n\}\right\}, \tag{8}$$

where

$$\Xi_i := \{\xi^i\} = (\{x^i\} \cup \{z^i\}) \subset S_i,\ i = 1, 2, \dots, n.$$

Denote the set of profiles determined by Ξ, and

$$\begin{aligned} A_\Xi &:= \textstyle\prod_{i=1}^n \Xi_i \\ &:= \{\xi_1, \xi_2, \dots, \xi_\alpha\}, \end{aligned}$$

where $\alpha = \Pi_{i=1}^n |\Xi_i|$.

Next, we consider any profile

$$\xi = (\zeta^1, \dots, \zeta^n) \in A_\Xi,$$

where

$$\zeta^k \in \left\{x_1^k, \dots, x_{n_k}^k; z_1^k, \dots, z_{m_k}^k\right\} = \{\xi^k\}.$$

Assume $\#(k \mid \zeta^k \in \{x^k\}) = \mu_\xi$, and then $\#(k \mid \zeta^k \in \{z^k\}) = n - \mu_\xi$. We define the interpolation value:

$$\widetilde{P}(\xi) := \frac{\mu_\xi}{n} f_1(\xi) + \left(1 - \frac{\mu_\xi}{n}\right)(f_2(\xi) - C). \tag{9}$$

Using Ξ, which is described by (8) as the selected finite points from S, and $\widetilde{P}(\xi)$, $\xi \in \Xi$, as the approximated values of the potential function P, we can construct the n-linear interpolation of P. It is called the merged interpolation.

The merge interpolation can also be applied to combine more than two potential functions together.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 61773371, 61733018).

References

1 Rosenthal R W. A class of games possessing pure-strategy Nash equilibria. Int J Game Theor, 1973, 2: 65–67
2 Monderer D, Shapley L S. Potential games. Games Econ Behav, 1996, 14: 124–143
3 González-Sánchez D, Hernández-Lerma O. A survey of static and dynamic potential games. Sci China Math, 2016, 59: 2075–2102
4 Cheung M W, Lahkar R. Nonatomic potential games: the continuous strategy case. Games Econ Behav, 2018, 108: 341–362
5 Marden J R, Arslan G, Shamma J S. Cooperative control and potential games. IEEE Trans Syst Man Cybern B, 2009, 39: 1393–1407
6 Hino Y. An improved algorithm for detecting potential games. Int J Game Theor, 2011, 40: 199–205
7 Cheng D. On finite potential games. Automatica, 2014, 50: 1793–1801
8 Liu X, Zhu J. On potential equations of finite games. Automatica, 2016, 68: 245–253
9 Wang Y, Liu T, Cheng D. From weighted potential game to weighted harmonic game. IET Control Theor Appl, 2017, 11: 2161–2169