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SVD based scale transform invariant observable degree for LTI system

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Dear editor,

The basic concept of observability is used to express the possibility to recover the state from measurements in modern control theory [1–3]. It is a qualitative parameter, which cannot reflect the observable degree (OD) ability [4,5]. Therefore, the OD is presented to quantitatively obtain the function. The quantitative parameter is effective for indicating the exact degree of estimation ability. The OD is closely related to the estimation performance of filters [6,7]. It shows that the estimation and control theories are closely related.

There are two main analytical methods of OD: estimation error covariance (EEC) and observability matrix methods [1, 2]. The eigenvalues are applied to reflect the OD in the direction of the corresponding eigenvector in EEC method, whereas the observability matrix method uses the system observability matrix to compute the OD. One of the classical ways of analyzing the OD is to employ the singular value decomposition (SVD) [8]. For OD based on the SVD method (SVD-OD), the singular values of the observability matrix show the observable levels of state combinations.

The SVD-OD method has been extensively applied in inertial navigation [9]. The SVD-ODs of state components which only have the same physical meaning can be compared to each other because it is known that the SVD-OD is scale transform variant [8]. For example, in a linear tracking system, state components x_1 and x_2 denote distance and velocity, respectively. It is noteworthy that SVD-ODs of x_1 and x_2 in km and m/s differ from those of x_1 and x_2 in m and m/s. In modern control theory, the qualitative parameter matrices including controllability and observability have scale transform invariance. Thus, there is an invariance to scale transform. It means the results are fixed for two LTI systems with linear relations [5]. However, it is not known whether the scale transform invariance still exists for the quantitative parameter of the OD. The problem is challenging because of changing from qualitative to quantitative.

Problem formulation. The following two linear systems are discussed:

$$\begin{cases} \boldsymbol{x}_k^{(i)} = \boldsymbol{A}^{(i)}\boldsymbol{x}_{k-1}^{(i)} + \boldsymbol{w}_{k,k-1}^{(i)}, \\ \boldsymbol{z}_k = \boldsymbol{H}^{(i)}\boldsymbol{x}_k^{(i)} + \boldsymbol{v}_k, \end{cases} \tag{1}$$

where $\boldsymbol{x}_k^{(i)} \in \mathbb{R}^n$ and $\boldsymbol{z}_k \in \mathbb{R}^m$ $(i = 1, 2)$ denote state and measurement vectors at time k, respectively; $\boldsymbol{A}^{(i)} \in \mathbb{R}^{n\times n}$ denotes state transition matrix; $\boldsymbol{H}^{(i)} \in \mathbb{R}^{m\times n}$ denotes measurement matrix; $\boldsymbol{w}_{k,k-1}^{(i)} \in \mathbb{R}^n$ and $\boldsymbol{v}_k \in \mathbb{R}^m$ are zero white Gaussian process and measurement noises $\boldsymbol{Q}_{k,k-1}^{(i)}$ and $\boldsymbol{R}_k$, respectively. The initial state $\boldsymbol{x}_0^{(i)}$ is uncorrelated with the process and measurement noises with mean $\boldsymbol{x}_{0|0}^{(i)}$ and covariance $\boldsymbol{P}_0^{(i)}$, respectively.

Suppose states have linear relation given by

$$\boldsymbol{x}_k^{(2)} = \boldsymbol{N}\boldsymbol{x}_k^{(1)}, \tag{2}$$

where

$$\boldsymbol{N} = \begin{bmatrix} n_1 & 0 & \cdots & 0 \\ 0 & n_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n_n, \end{bmatrix}, \tag{3}$$

$$\boldsymbol{x}_{0|0}^{(2)} = \boldsymbol{N}\boldsymbol{x}_{0|0}^{(1)}, \quad \boldsymbol{P}_0^{(2)} = \boldsymbol{N}\boldsymbol{P}_0^{(1)}\boldsymbol{N}, \tag{4}$$

$$\begin{cases} \boldsymbol{A}^{(2)} = \boldsymbol{N}\mathbf{A}^{(1)}\boldsymbol{N}^{-1}, \quad \boldsymbol{H}^{(2)} = \boldsymbol{H}^{(1)}\boldsymbol{N}^{-1}, \\ \boldsymbol{w}_{k,k-1}^{(2)} = \boldsymbol{N}\boldsymbol{w}_{k,k-1}^{(1)}, \quad \boldsymbol{Q}_{k,k-1}^{(2)} = \boldsymbol{N}\boldsymbol{Q}_{k,k-1}^{(1)}\boldsymbol{N}^{\mathrm{T}}. \end{cases} \tag{5}$$

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The observability matrices of systems $i = 1, 2$ are

$$\boldsymbol{G}^{(i)} = \begin{bmatrix} \boldsymbol{H}^{(1)}\boldsymbol{N}^{-(1-i)} \\ \boldsymbol{H}^{(1)}\boldsymbol{A}^{(1)}\boldsymbol{N}^{-(1-i)} \\ \vdots \\ \boldsymbol{H}^{(1)}(\boldsymbol{A}^{(1)})^{n-1}\boldsymbol{N}^{-(1-i)} \end{bmatrix}. \tag{6}$$

Motivation. There are some invariant properties for two systems with linear relation for the traditional modern control theory, for example, controllability, observability, eigenvalue, and transfer function. Moreover, we want to know whether the OD has the invariant property. Herein, the OD is considered by applying the SVD (OD-SVD). For the SVD-OD method, the singular values of the observability matrices represent the ODs along the directions of the eigenvectors of $\boldsymbol{G}^{(s)\mathrm{T}}\boldsymbol{G}^{(s)}$ $(s = 1, 2)$. The OD can be analyzed by the SVD of $\boldsymbol{G}$ as $\boldsymbol{G}^{(s)} = \boldsymbol{U}\Sigma\boldsymbol{V}^{\mathrm{T}}$ [8, 9], where $\Sigma = \mathrm{diag}(S, 0)$ and $S = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$. S is a diagonal matrix with the singular values of $\boldsymbol{G}^{(s)}$ along the diagonal, and one has $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_r \geqslant 0$ and $r = \mathrm{rank}(\boldsymbol{G}^{(s)})$. Both $\boldsymbol{U}$ and $\boldsymbol{V}$ are orthogonal matrices, where $\boldsymbol{U} = [\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n]$ and $\boldsymbol{V} = [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n]$, respectively. We remark that $\boldsymbol{G}^1$ and $\boldsymbol{G}^2$ are different observability matrices. It means that the ODs computed by the SVD is different for the two linearly related systems due to scale transform variant property of the SVD [8]. The SVD-ODs of state components which have the same physical meaning are comparable, whereas SVD-ODs of state components which have different physical meanings are not comparable. This is because SVD-OD is scale transform variant. Obviously, it does not coincide with the current results in the modern control theory. Thus, it is believed that the OD also has the scale transform invariant property. Consequently, a scale transform invariant OD-SVD method is proposed by designing an adaptive adjustment factor effectively in this study. This ensures that state components having different physical meanings are comparable.

Scale transform invariant OD-SVD for linear systems. For the traditional OD-SVD, the state components having different physical meanings are not comparable. Therefore, it is necessary to design a transform to achieve scale invariance. Our method is to use the standard deviation of the square roots of a pseudo-covariance from $k = 1$ to τ to obtain an adaptive adjustment factor, which is directly proportional to scale transform. Because the observability matrix is inversely proportional to the scale transform, the introduction of the adaptive adjustment factor will effectively eliminate the influence from scale transform. A pseudo-covariance propagation model for the computation is successfully designed as

$$\boldsymbol{C}_{k+1} = \boldsymbol{A}\boldsymbol{C}_k\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{Q}_{k,k-1}, \tag{7}$$

where $\boldsymbol{C}_0 = \boldsymbol{P}_0$. It is noteworthy that Eq. (7) is not the formula for computing traditional EEC. For system 1, let

$$\boldsymbol{C}_k^{(1)} = \begin{bmatrix} c_{11,k}^{(1)} & \cdots & c_{1n,k}^{(1)} \\ \vdots & & \vdots \\ c_{n1,k}^{(1)} & \cdots & c_{nn,k}^{(1)} \end{bmatrix}, \tag{8}$$

where the adaptive adjustment factor starts from $k = 1$ to $k = \tau$. Usually, τ is greater than 50. For time τ, we have $\boldsymbol{C}_1^1, \boldsymbol{C}_2^1, \ldots, \boldsymbol{C}_\tau^1$. A type of adaptive adjustment factor can be given by

$$\Delta_j^{(1)} = \sqrt{\frac{\sum_{k=1}^{\tau}\left(\sqrt{c_{jj,k}^{(1)}} - \bar{\gamma}_j^{(1)}\right)^2}{\tau - 1}},\ j = 1, 2, \ldots, n, \tag{9}$$

where

$$\bar{\gamma}_j^{(1)} = \frac{\sum_{k=1}^{\tau}\sqrt{c_{jj,k}^{(1)}}}{\tau}. \tag{10}$$

Then, we have that

$$\varphi^{(1)} = \mathrm{diag}(\Delta_1^{(1)}, \Delta_2^{(1)}, \ldots \Delta_n^{(1)}). \tag{11}$$

Accordingly, the scale transform invariant observability matrix is given by

$$\overline{\boldsymbol{G}}^{(1)} = \begin{bmatrix} \boldsymbol{H}^{(1)}\boldsymbol{\varphi}^{(1)} \\ \vdots \\ \boldsymbol{H}^{(1)}\boldsymbol{A}^{(1)}\varphi^{(1)} \\ \boldsymbol{H}^{(1)}(\boldsymbol{A}^{(1)})^{n-1}\varphi^{(1)} \end{bmatrix}. \tag{12}$$

Similar to linear system 1, the associated scale transform invariant observability matrix for the linear system 2 is given by

$$\overline{\boldsymbol{G}}^{(2)} = \begin{bmatrix} \boldsymbol{H}^{(2)}\varphi^{(2)} \\ \vdots \\ \boldsymbol{H}^{(2)}\boldsymbol{A}^{(2)}\varphi^{(2)} \\ \boldsymbol{H}^{(2)}(\boldsymbol{A}^{(2)})^{n-1}\varphi^{(2)}, \end{bmatrix}, \tag{13}$$

where the adaptive adjustment matrix

$$\varphi^{(2)} = \mathrm{diag}(\Delta_1^{(2)}, \Delta_2^{(2)}, \ldots \Delta_n^{(2)}), \tag{14}$$

$$\Delta_j^{(2)} = \sqrt{\frac{\sum_{k=1}^{\tau}\left(\sqrt{c_{jj,k}^{(2)}} - \bar{\gamma}_j^{(2)}\right)^2}{\tau - 1}},\ j = 1, \ldots, n, \tag{15}$$

$$\bar{\gamma}_j^{(2)} = \frac{\sum_{k=1}^{\tau}\sqrt{c_{jj,k}^{(2)}}}{\tau}, \tag{16}$$

$$\boldsymbol{C}_k^{(2)} = \begin{bmatrix} c_{11,k}^{(2)} & \cdots & c_{1n,k}^{(2)} \\ \vdots & & \vdots \\ c_{n1,k}^{(2)} & \cdots & c_{nn,k}^{(2)} \end{bmatrix}. \tag{17}$$

Using (2), (6) and (8)–(17), we obtain the following relation:

$$\varphi^{(2)} = \mathrm{diag}(\Delta_1^{(2)}, \Delta_2^{(2)}, \ldots, \Delta_n^{(2)}) = \boldsymbol{N} \cdot \varphi^{(1)}. \tag{18}$$

Main result.

Theorem 1. By applying the adaptive adjustment factor for the two systems with linear relation, the scale transform invariant observability matrices coincide, and thus

$$\overline{\boldsymbol{G}}^{(2)} = \overline{\boldsymbol{G}}^{(1)}. \tag{19}$$

Proof. Using (5), (12), (13), and (18), we have

$$\overline{\boldsymbol{G}}^{(2)} = \begin{bmatrix} \boldsymbol{H}^{(2)}\varphi^{(2)} \\ \boldsymbol{H}^{(2)}\boldsymbol{A}^{(2)}\varphi^{(2)} \\ \vdots \\ \boldsymbol{H}^{(2)}(\boldsymbol{A}^{(2)})^{n-1}\varphi^{(2)} \end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} \boldsymbol{H}^{(1)}\boldsymbol{N}^{-1}\varphi^{(2)} \\ \boldsymbol{H}^{(1)}\boldsymbol{A}^{(1)}\boldsymbol{N}^{-1}\varphi^{(2)} \\ \vdots \\ \boldsymbol{H}^{(1)}(\boldsymbol{A}^{(1)})^{n-1}\boldsymbol{N}^{-1}\varphi^{(2)} \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{H}^{(1)}\boldsymbol{N}^{-1}\boldsymbol{N}\varphi^{(1)} \\ \boldsymbol{H}^{(1)}\boldsymbol{A}^{(1)}\boldsymbol{N}^{-1}\boldsymbol{N}\varphi^{(1)} \\ \vdots \\ \boldsymbol{H}^{(1)}(\boldsymbol{A}^{(1)})^{n-1}\boldsymbol{N}^{-1}\boldsymbol{N}\varphi^{(1)} \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{H}^{(1)}\varphi^{(1)} \\ \boldsymbol{H}^{(1)}\boldsymbol{A}^{(1)}\varphi^{(1)} \\ \vdots \\ \boldsymbol{H}^{(1)}(\boldsymbol{A}^{(1)})^{n-1}\varphi^{(1)} \end{bmatrix} \\
&= \overline{\boldsymbol{G}}^{(1)}. \qquad (20)
\end{aligned}
$$

This completes the proof.

Theorem 1 shows that for the linear systems 1 and 2, both the observability matrices achieve the scale transform invariance by applying the adaptive adjustment factor. Consequently, the OD-SVD based on the observability matrices with scale transform invariance is also invariant.

Conclusion and future work. The invariance of the OD based on the SVD is successfully discussed herein. The essence of this study is to look for an adaptive adjustment factor. The study makes the quantitative OD coincide with the current result on the scale transform invariance of the observability, and a proof is given to validate the invariance. Moreover, the invariance should be further studied on other OD computation methods. In the future, we hope to extend this study to multisensor systems and nonlinear systems.

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