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• **Supplementary File** •

A Novel Opinion Model for Complex Macro-Behaviors of Mass Opinion

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Appendix A Model definition

Let $\mathcal{V} = \{1, 2, \dots, n\}$ be the set of agents and $\mathcal{E}$ be the set of edges that represent the interactions among the agents. Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ defines the social network, where weight matrix W is a stochastic matrix adapted to $\mathcal{G}$. That is, the row sums of W are all equal to 1; the entries of W are nonnegative; $w_{ij} > 0 \Leftrightarrow (i, j) \in \mathcal{E}$. Denote the opinion of agent i at time t by $x_i(t)$, and let $x(t)$ be the opinion vector at time t.

The DeGroot model [1] updates as follows:

$$x(t+1) = Wx(t), \quad t \geqslant 0.$$

In the DeGroot model, every individual updates its opinion with a convex combination of neighbors' opinions. When W is irreducible and aperiodic, the model will converge and reach a consensus [1].

Suppose that some agent $i \in \mathcal{V}$ is influenced by its initial view. The Friedkin-Johnson (F-J) model [1] can be defined as below:

$$x(t+1) = \Lambda W x(t) + (I_n - \Lambda)u, \quad t \geqslant 0,$$

where Λ is a diagonal matrix and $u = x(0)$. When W is irreducible and aperiodic, the F-J model will converge but not necessarily reach a consensus [1].

The update of our model has two stages. The first one is actually a DeGroot rule:

$$s_i(t) = \sum_{j \in \mathcal{N}_i} \frac{1}{|\mathcal{N}_i|} x_j(t), \quad t \geqslant 1,$$

where $\mathcal{N}_i$ is the neighbor set of agent i and contains i itself. The above equation can also be written as

$$s(t) = Wx(t),$$

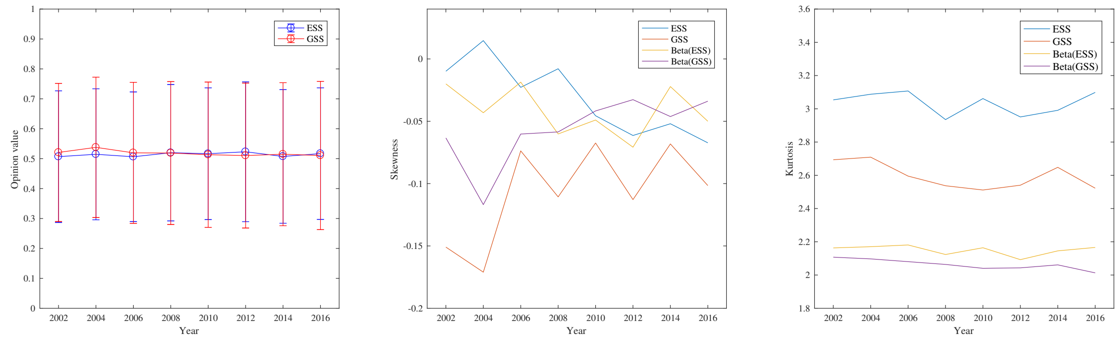
where W is the weight matrix of $\mathcal{G}$. The entry $w_{ij} = \dfrac{1}{|\mathcal{N}_i|}$ if $(i, j) \in \mathcal{E}$; $w_{ij} = 0$ if $(i, j) \notin \mathcal{E}$.

Let $\mathcal{I}_0(i) := [x_i(0) - c_i, x_i(0) + c_i]$, $c_i \geqslant 0$, be the confidence interval that controls the second stage of updating. The discrete-time opinion update rule of agent i is as follows:

$$x_i(t+1) = \begin{cases} s_i(t), & s_i(t) \in \mathcal{I}_0(i), \\ (1 - h_i)s_i(t) + h_i x_i(0), & s_i(t) \notin \mathcal{I}_0(i), \end{cases} \tag{A1}$$

where $h_i \in [0, 1]$ measures the stubbornness of agent i, and the initial value $x_i(0)$ represents the personal bias of i, $i \in \mathcal{V}$. For simplicity, here all c_i are equal and so are h_i, that is, we adopt a homogeneous model.

In general, not all $x_i(0)$ are equal. Therefore, let $\max_i\{x_i(0)\}$ be 1 and $\min_i\{x_i(0)\}$ be 0, which can be done via a coordinate transformation. Without loss of generality, suppose $c \in [0, 1]$. Model (A1) becomes a DeGroot model provided that $h = 0$ or $c = 1$. When $c = 0$, the model becomes a F-J model with $\Lambda = hI_n$, since $s_i(t) \in \mathcal{I}_i(0) \Leftrightarrow s_i(t) = x_i(0)$.



(a) Mean values and standard deviations

(b) Skewness

(c) Kurtosis

Figure B1 The descriptive statistics of the empirical data.

Appendix B The descriptive statistics of empirical data

In this letter, we use the data of political views in European Social Survey (ESS) [2–9] and General Social Survey (GSS) [10] for discussions. These surveys contain a series of standard questions and have been conducted for every two years in Europe and American. The raw data is given in Appendix F.

The respondents were asked to report their political positions by answering an eleven-point (0-10) or seven-point (0-6) scale. In these scales, point 0 represents left or extremely liberal. Point 10 in the eleven-point scale and 6 in the seven-point scale represent right or extremely conservative. For better comparison between the mathematical models and the empirical data, we translate the scores of these scales isometrically into the interval $[0, 1]$ such that 0 represents extremely left, and 1 represents extremely right.

The mean values, standard deviations, skewness and kurtosis of the data are shown in Fig. B1. The results suggest that the mean values of the political landscapes are around 0.5, which is the moderate value. It can be observed that the skewness values of the ESS attitude distributions are around zero, but those of the GSS are slightly negative skew, indicating that there are a little more individuals placing themselves on the right. The kurtosis values of the ESS data are around 3; those of the GSS are less than 3. To test whether the landscapes can be explained by Beta distributions, we compute the skewness and kurtosis of corresponding Beta distributions whose mean values and standard deviations are estimated based on those of the empirical data. Fig. B1(c) shows that the kurtosis values of the corresponding Beta distributions are far less than those of the empirical data. This provides the negative answer.

It is known that the skewness and kurtosis of any normal distribution is 0 and 3, but the following findings suggest that the shape of the ESS data cannot be explained by normal distributions either. Recall the three main properties of the political landscapes described in the main text:
(i) a large part of the population hold moderate or neutral views;
(ii) on both sides of the middle peak there are two clusters with non-extreme opinions;
(iii) two small groups of individuals hold oppositely extreme views.

For an opinion vector x, let

$$P_1(x) := \frac{1}{|\mathcal{V}|}|\{i \in \mathcal{V} : x_i \in [0.4, 0.6]\}|,$$

$$P_2(x) := \frac{1}{|\mathcal{V}|}|\{i \in \mathcal{V} : x_i \in [0.1, 0.4) \cup (0.6, 0.9]\}|,$$

and

$$P_3(x) := \frac{1}{|\mathcal{V}|}|\{i \in \mathcal{V} : x_i \in [0, 0.1) \cup (0.9, 1]\}|,$$

be the proportions of people who hold moderate, non-extreme, and extreme views, respectively [11]. The notation $|\cdot|$ is the cardinality of a set. Fig. B2 gives P_1, P_2 and P_3 of the ESS and GSS data and those of the normal distributions whose mean values and standard deviations are estimated based on the survey data. As shown in Fig. B2, the values of P_1 for the empirical data are much greater than those for the normal distributions; at the same time, the P_2 of the empirical data are much smaller. This indicates that the attitude distributions may not be normal.

Appendix C Macro-behaviors of the proposed model

In this section, the properties of macro-behaviors, that is, the opinion distributions of our proposed model will be discussed via numerical simulations. We consider four topologies: (i) the Facebook dataset with 4039 nodes and 88324 edges [12], (ii) a Watts–Strogatz graph with 4039 nodes, degree 44, and rewiring probability 0.01, (iii) an Erdős–Rényi graph with 4039 nodes and 88324 edges, and (iv) a Barabási–Albert graph with 4039 nodes where each new node is connected to 22 existing nodes [13].

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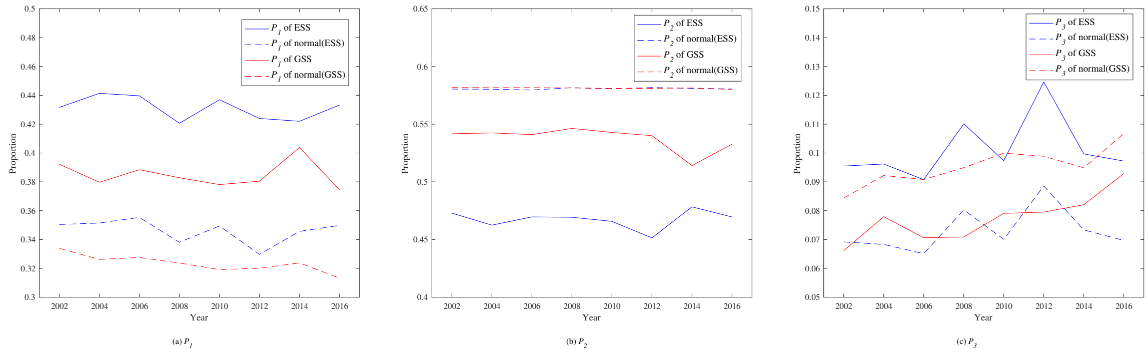

Figure B2 The macro-behaviors of the empirical data.

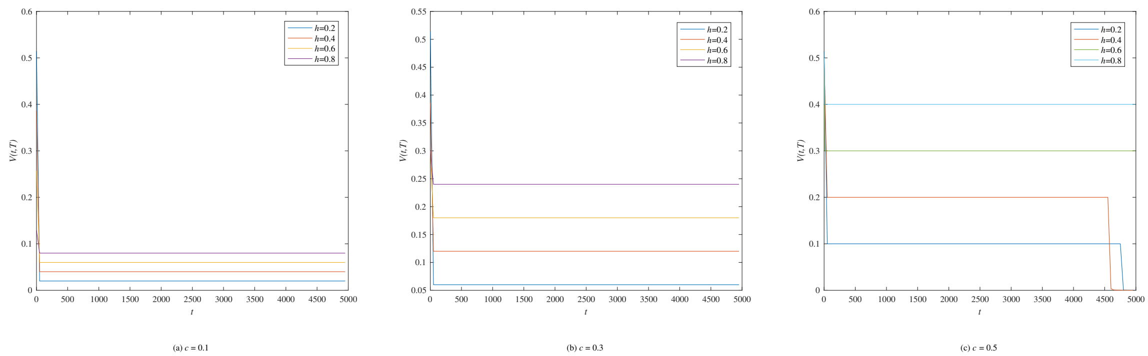

Figure C1 The maximum variations for $c = 0.1$, 0.3, and 0.5, where $T = 5000$.

It is worth noting that the proposed model may not converge after a given iteration time T; instead the opinions keep fluctuating. So we define the variation of an agent i's opinion from time t to T, $0 \leqslant t \leqslant T$, as:

$$V_i(t,T) := \max_{k\in[t,T]} x_i(k) - \min_{k\in[t,T]} x_i(k),$$

and define the maximum variation of opinions from time t to T as:

$$V(t,T) := \max_{i\in\mathcal{V}} V_i(t,T).$$

The simulation (Fig. C1) shows that the maximum variation decreases when t increases. Moreover, the maximum variation is constant for large t for $c = 0.1$ or 0.3. It is less than 0.2 for almost all h, but greater than 0.3 for $c = 0.5$ and $h = 0.6$ or 0.8. This indicates a vigorous fluctuation. The maximum variation tends to zero, however, for small h and $c = 0.5$, implying that the model reaches a consensus. From the analysis in Appendix D and Appendix E, we know that the model may converge to some period orbit for certain c and h. Therefore, considering the limitation of computation time, we set the iteration time of simulations below to be $T = 1000$. If the model does not converge, then the final opinion distribution is defined as the average of the last 50-time-step distributions.

To compare the influence of topology, we carried out simulations for twenty different initial vectors. The initial values are selected independently from uniform distribution on $[0,1]$ and then rescaled such that $\max_i\{x_i(0)\} = 1$ and $\min_i\{x_i(0)\} = 0$. The parameters c and h are set to be $c = 0.3$ and $h = 0.6$, with which the model provides an acceptable approximation of the empirical opinion distributions (Fig. 1 in the main text). As shown in Fig. C2, the mean values, skewness and kurtosis of opinion distributions of different topologies are similar to each other and are also similar to the GSS data. The value of P_1 (P_2) for the Facebook network, however, is a little smaller (greater) than those for the other three graphs (Fig. C2(d)(e)). This indicates the underlying difference between real networks and artificially generated ones. The values of P_3 for all four topologies are zero, but the approximation of P_3 can be improved by some heterogeneous models. In such models, the Agreeableness parameter h_i of some agents holding extreme initial values are close to 1 (as discussed below and in the main text).

We discuss the influence of parameters c and h by setting the topology to be the Facebook network. Numerical simulations and analytical results show that the model becomes the DeGroot model (Appendix D and Appendix E) when c is larger than 0.5. Therefore, c is set to be 0(homogeneous F-J model), $0.1, 0.2, 0.3$, and 0.4; h is set to be $0.2, 0.4, 0.6$, and 0.8. As before, twenty initial vectors are selected independently from uniform distribution on $[0,1]$ and then rescaled.

The following can be observed in Fig. C3(a)(b)(c): the mean values are around 0.5 for all parameter pairs; the skewness values are around 0 for almost all (c,h); the kurtosis values are monotonously increasing functions of parameter c and are almost the same for different h. From Fig. C3(d)(e)(f), we know that greater c makes P_1 become larger, while greater h

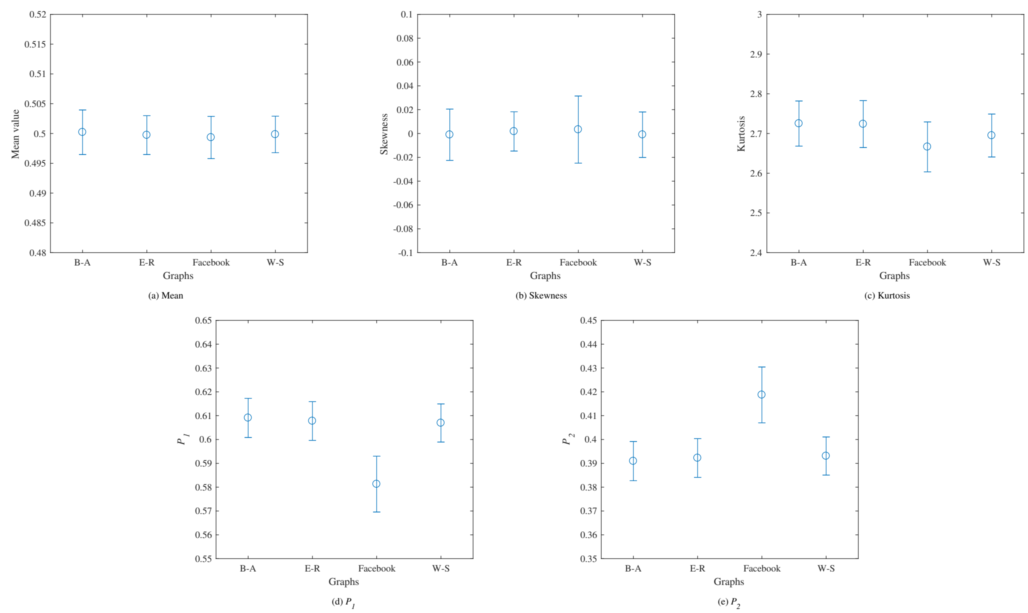

Figure C2 The macro-behaviors of different topologies.

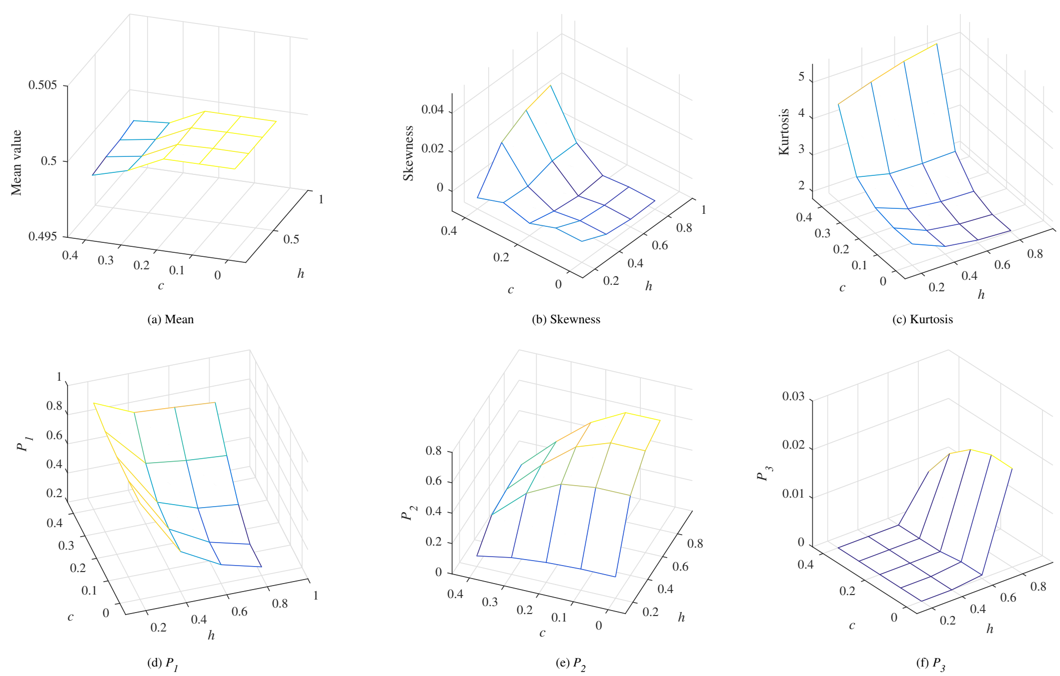

Figure C3 The macro-behaviors of different parameters.

makes P_2 and P_3 become larger. Note that P_3 is the proportion of individuals holding extreme opinions. This indicates that if some agents, $i_1, \cdots, i_m$, holding extreme views have sufficiently large h_k, $k \in \{i_1, \cdots, i_m\}$, then the heterogeneous model will provide better approximation for P_3, which is shown in Fig. 1 of the main text.

For the homogeneous F-J model, i.e., the proposed model with $c = 0$, the kurtosis of the opinion distribution is around 2. This may be like the shape of Beta distributions. For the DeGroot model, the P_1 of the final opinion distribution may be equal to 1 with a large probability, since the group will finally reach a consensus. This can also validate the monotonously increasing relationship between P_1 and c.

Recall that the Agreeableness parameter h measures one's tendency to defer to others. As individuals holding extreme views may not change their opinions readily, they may have large h. This heterogeneous model explains the approximation of P_2 and P_3. On the contrary, for individuals holding non-extreme views, proper openness makes them influenced by both positive and negative sides. Thus, proper c (0.3 in the main text) provides an acceptable estimation of P_1.

Appendix D Behaviors of the proposed model

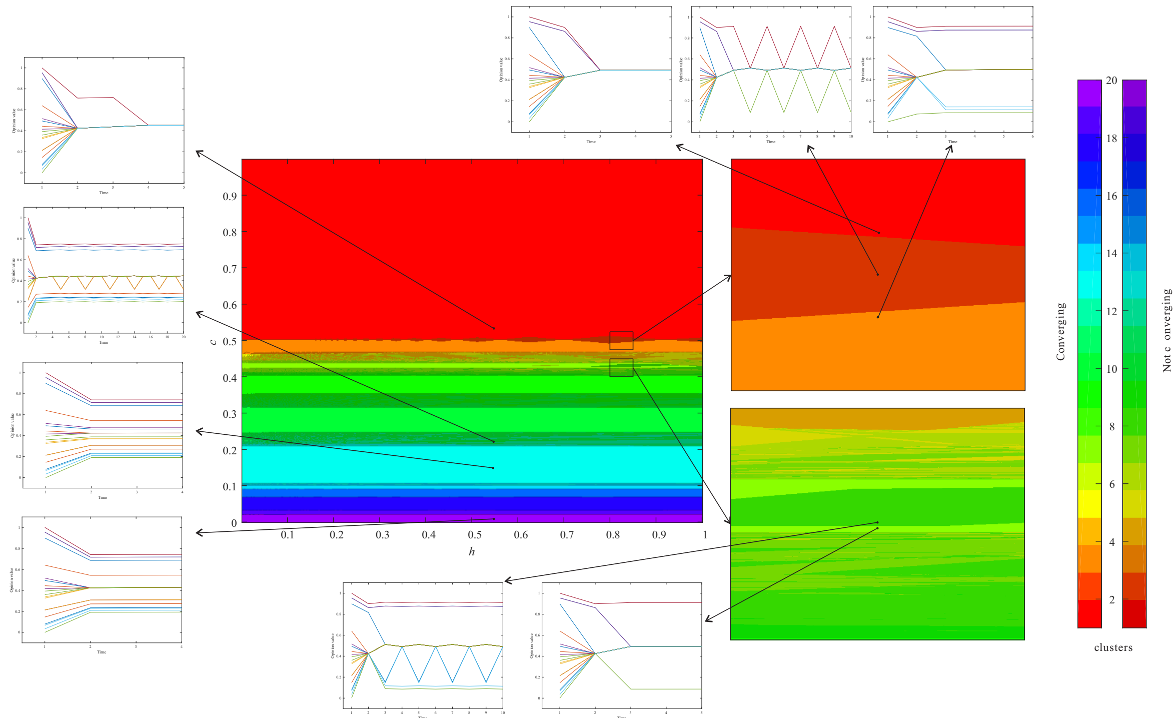

Figure D1 The behaviors of the proposed model.

We discuss the convergence condition of the system by numerical simulations in this section. The underlying network is set to be a complete graph with twenty agents. The initial value is selected from the uniform distribution on $[0, 1]$ and then rescaled.

The result is presented in Fig. D1. Each point (h_0, c_0) in the coordinate system represents the convergence situation of the system with $h = h_0$ and $c = c_0$. The opinions of agents may fall into different clusters when the system converges. The number of clusters is shown in different colors, which is demonstrated by colorbars in the right side of the figure. If the system, however, does not converge for a preset iterative time, $T = 10^5$, the corresponding point will be covered by gray shade. The clustering number of the fluctuating systems is defined as the maximum number of different opinions per time slot after $T_1 = 5 \times 10^4$. Two more detailed regions and the behaviors of several particular systems with certain (h, c) are illustrated in the figure too.

For c greater than 0.5, which represents that people are acceptive towards distinct views, the group will reach a consensus finally although some may have different backgrounds. Conversely, when people are less open to new ideas, which means that c is relatively small, they will end in disagreement related to their biases. These two findings are compatible with the DeGroot and F-J models.

More complex phenomena can emerge when c is neither too large nor too small. In fact, numerical simulations illustrate that the convergence of system mainly depends on the value of c, and the situation may be slightly different for different h. Systems that do not converge for a given iterative time, which lie in the shade areas of Fig. D1, may end in fluctuating periodically. This complexity shows that, the opinion fluctuations in society [14, 15] may be caused by the distrust between individuals, rather than the existence of stubborn agents [15]. It can also be observed that, for fixed h, the number of clusters decreases as c increasing from 0 to 1, which means that high openness boosts consensus.

Appendix E Analytical results

Several special cases of the model (A1) are analyzed in this section. The results suggest that the model behaves like the Friedkin-Johnson model for a sufficiently small c, while its behavior is similar to that of the DeGroot model for a sufficiently large c. The existence of some period orbits, which is uncommon for classic deterministic opinion models, is also verified for two-island networks with a special initial condition.

In the following texts, the DeGroot model whose weight matrix and initial vector coincide with W and with $x(0)$ of model (A1), is called the corresponding DeGroot model of (A1). The F-J model whose weight matrix, initial vector and stubbornness matrix coincide with W, with $x(0)$ of model (A1) and with hI_n, is referred to as the corresponding F-J model of (A1).

Theorem 1. Suppose that $\mathcal{G}$ is connected, then system (A1) reaches a consensus and converges to the limit point of the corresponding DeGroot model for c satisfying $1-\dfrac{1}{n^d}\min\{(1-M_2(0)),m_2(0)\}\leqslant c\leqslant 1$, where d is the diameter of $\mathcal{G}$.

Proof. For any $i\in\mathcal{V}$ such that not all $x_j(0)=1$ and not all $x_j(0)=0$, where $j\in\mathcal{N}_i$, we have that

$$\begin{aligned} s_i(0) &\leqslant \frac{1}{|\mathcal{N}_i|}M_2(0)+\frac{|\mathcal{N}_i|-1}{|\mathcal{N}_i|}\\ &=1-\frac{1}{|\mathcal{N}_i|}(1-M_2(0))\\ &\leqslant 1-\frac{1}{n}(1-M_2(0)),\end{aligned}$$

$$s_i(0)\geqslant \frac{1}{|\mathcal{N}_i|}m_2(0)\geqslant\frac{1}{n}m_2(0),$$

since $\mathcal{G}$ is connected. Thus, for c in the assumption and the above agent i, $s_i(0)\in\mathcal{I}_0(i)$ and $x_i(1)=S_i(0)$. Therefore, $M_2(1)\leqslant 1-\dfrac{1}{n}(1-M_2(0))$ and $m_2(1)\geqslant\dfrac{1}{n}m_2(0)$. Inductively, we have that $M_2(t+1)\leqslant 1-\dfrac{1}{n}(1-M_2(t))$ and $m_2(t+1)\geqslant \dfrac{1}{n}m_2(t)$, if the condition for c holds. Also note that when $t=d$, $s_i(d)\in\mathcal{I}_0(i)$ for all $i\in\mathcal{V}$, which implies that (A1) actually follows the DeGroot update rule. Thus the conclusion follows.

From the proof of Theorem 1, we have that:

Corollary 1. Suppose that $\mathcal{G}$ is connected and there are only two agents with initial value 1 and 0, respectively. Then system (A1) reaches a consensus and converges to the limit point of the corresponding DeGroot model for c satisfying $1-\dfrac{1}{n}\min\{(1-M_2(0)),m_2(0)\}\leqslant c\leqslant 1$.

Theorem 2. Suppose that $\mathcal{G}$ is a complete graph.
i) System (A1) reaches a consensus and converges to the limit point of the corresponding DeGroot model for $c_1\leqslant c\leqslant 1$, where $c_1:=\max\{\frac{1}{n}\sum_{i=1}^n x_i(0),1-\frac{1}{n}\sum_{i=1}^n x_i(0)\}$.
ii) If $0<h<1$ and $\min_i\{|\frac{1}{n}\sum_{i=1}^n x_i(0)-x_i(0)|\}>0$, then system (A1) converges to the limit point of the corresponding F-J model for $0\leqslant c<c_2$, where $c_2:=\min_i\{|\frac{1}{n}\sum_{i=1}^n x_i(0)-x_i(0)|\}$.

Proof. Because $s(1)=\frac{1}{n}\sum_{i=1}^n x_i(0)\mathbf{1}$ and $c\geqslant c_1$, $s_i(1)\in\mathcal{I}_0(i)$, $1\leqslant i\leqslant n$. Thus $x(1)=s(1)=\frac{1}{n}\sum_{i=1}^n x_i(0)$.

When $0\leqslant c<c_2$, $s(1)=\frac{1}{n}\sum_{i=1}^n x_i(0)\notin\mathcal{I}_0(i)$, $1\leqslant i\leqslant n$. So $x(1)=(1-h)S(1)+hx(0)$, and $s(2)=\frac{1}{n}\mathbf{1}\mathbf{1}^Tx(1)=s(1)$. Thus, $x(2)=(1-h)S(2)+hx(0)=x(1)$. By induction, $x(t)$ follows the F-J update rule for $t\geqslant 0$.

Remark 1. The bound c_1 is a better one for complete graphs, since $1-\frac{1}{n}\sum_{i=1}^n x_i(0)<1-\frac{1}{n}m_2(0)$ and $\frac{1}{n}\sum_{i=1}^n x_i(0)=1-\frac{1}{n}\sum_{i=1}^n(1-x_i(0))<1-\frac{1}{n}(1-M_2(0))$.

One of the networks in which researchers are interested is the two-island network [16, 17]. The definition is as follows:

Definition 1. Given integers n_1, $n_2\geqslant 0$ and real numbers p_s, $p_d\in(0,1)$, a (n_1,n_2,p_s,p_d)-two-island network is a weighted undirected graph $\mathcal{G}=(\mathcal{V}_1\cup\mathcal{V}_2,\mathcal{E})$, where
i) $|\mathcal{V}_1|=n_1$, $|\mathcal{V}_2|=n_2$, and $\mathcal{V}_1\cap\mathcal{V}_2=\emptyset$.
ii) each node $i\in\mathcal{V}_1$ has n_1p_s neighbors (including itself) in $\mathcal{V}_1$ and n_2p_d neighbors in $\mathcal{V}_2$.
iii) each node $i\in\mathcal{V}_2$ has n_2p_s neighbors (including itself) in $\mathcal{V}_2$ and n_1p_d neighbors in $\mathcal{V}_1$.
iv) $p_s>p_d$.

For simplicity, assume that the quantities n_1p_s, n_2p_s, n_1p_d and n_2p_d are all integers. For such network, we are able to show that there exist period orbits for a special initial vector.

Theorem 3. Let $\mathcal{G}=(\mathcal{V}_1\cup\mathcal{V}_2,\mathcal{E})$ be a (n,n,p_s,p_d)-two-island network. Further assume $x_i(0)=1$ for all $i\in\mathcal{V}_1$ and $x_i(0)=0$ for all $i\in\mathcal{V}_2$. Then
i) for $(h,c)\in R_D$, the system reaches a consensus, where $R_D:=[0,1]\times[\frac{1}{2},1]\cup\{0\}\times[0,1]$;
ii) for $(h,c)\in R_F$, the system converges to the limit point of the corresponding F-J model, where $R_F:=\{(h,c)\in[0,1]\times[0,1]:(1-w)-2(1-w)c-(2w-1)ch\geqslant 0\}\setminus\{(1,1-w)\}$;
iii) for $(h,c)\in[0,1]\times[0,1]\setminus(R_D\cup R_F)$, either the system converges to a period orbit or the ω-limit set of (A1) is a Cantor set.

Proof. It is easy to see from induction that the update rules of agents in $\mathcal{V}_1$ are the same, and so are those of agents in $\mathcal{V}_2$. Hence system (A1) can be reduced into a two-dimension system:

$$x_i(t+1)=\begin{cases}S_i(t), & \text{if } S_i(t)\in\mathcal{I}_0(i),\\ (1-h)S_i(t)+hx_i(0), & \text{if } S_i(t)\notin\mathcal{I}_0(i),\end{cases}\tag{E1}$$

where $x_1(0)=1$, $x_2(0)=0$, and $S_i(t)=wx_i(t)+(1-w)x_{3-i}(t)$, with $w=\dfrac{p_s}{p_s+p_d}\in(\frac{1}{2},1)$, $i=1,2$. The trajectories of $x_i(t)$ and every agents in $\mathcal{V}_i$ are the same for $i=1,2$.

By induction, one can prove that $x_1(t)+x_2(t)=S_1(t)+S_2(t)\equiv 1$. Thus, to analyze (E1), it suffices to study the behavior of $y(t)=S_1(t)-S_2(t)\in[0,1]$. The initial value is $y(0)=2w-1$, and

$$y(t+1)=\begin{cases}(2w-1)y(t), & \text{if } y(t)\geqslant 1-2c,\\ (2w-1)(1-h)y(t)+(2w-1)h, & \text{if } y(t)<1-2c.\end{cases}\tag{E2}$$

Suppose that $\frac{1}{2} \leqslant c \leqslant 1$. Then $1-2c \leqslant 0 \leqslant y(t)$. Therefore, $y(t+1) = (2w-1)y(t)$ and $\lim_{t\to\infty} y(t) = 0$, which implies that $\lim_{t\to\infty} x_i(t) = \frac{1}{2}$, $i = 1, 2$. (This conclusion can also be obtained by Theorem 1.) Under this condition, the whole network reaches a consensus.

To prove ii), note that equation

$$y = (2w-1)(1-h)y + (2w-1)h,$$

has solution $y^* = \dfrac{(2w-1)h}{2(1-w)+(2w-1)h} \in (0, 2w-1]$. When $0 \leqslant c < \dfrac{1-y^*}{2} = \dfrac{1-w}{2(1-w)+(2w-1)h} := c_F(h)$, we have that $1-2c > y^*$. If $y(t) \geqslant 1-2c$, then $y(t+1) = (2w-1)y(t)$. Thus, there must be an integer t_0 such that, for the first time, $y(t_0) < 1-2c$. Because

$$\begin{aligned} y(t_0+1) - y^* &= (2w-1)(1-h)y(t_0) + (2w-1)h - y^* \\ &= (2w-1)(1-h)(y(t_0) - y^*), \end{aligned}$$

and $(2w-1)(1-h) \geqslant 0$, it follows that $|y(t_0+1) - y^*| < |y(t_0) - y^*|$ and $(y(t_0+1) - y^*)(y(t_0) - y^*) \geqslant 0$. Therefore, $y(t_0+1) < 1-2c$. Hence by induction, $y(t) < 1-2c$, $t \geqslant t_0$, and $\lim_{t\to\infty} y(t) = y^*$, which implies that $\lim_{t\to\infty} x_1(t) = \dfrac{1-w+wh}{2(1-w)+(2w-1)h}$, and $\lim_{t\to\infty} x_2(t) = \dfrac{(1-w)(1-h)}{2(1-w)+(2w-1)h}$. This means that the opinions of the two factions converge to different positions.

When $0 < h < 1$, the above conclusion still holds for $1-2c = y^*$. That is because $(2w-1)(1-h) > 0$. For $h = 1$ and $1-2c = y^*$, however, we have that $y(2t) = 2w-1$ and $y(2t+1) = (2w-1)^2$, $t \in \mathbb{N}$.

Apropos of iii), when $\dfrac{1-y^*}{2} < c < \dfrac{1}{2}$ and $h > 0$, (E2) has no equilibrium and the dynamics are attracted into $[f(1-2c), f((1-2c)-)]$. Since

$$\begin{aligned} f(f(1-2c)) &= (2w-1)^2(1-h)(1-2c) + (2w-1)h \\ &> (2w-1)^2(1-h)(1-2c) + (2w-1)^2 h \\ &= f(f((1-2c)-)), \end{aligned}$$

function (E2) on $[f(1-2c), f((1-2c)-)]$ can be transformed to a circle map (Definition 2 below) via a change of variables. The conclusion follows from Lemmas 1 and 2 below.

Definition 2. [Definition 20 in [18]] We say that

$$f : S^1 \to S^1,$$

$S^1 = \mathbb{R}/\mathbb{Z}$, is an orientation preserving circle map if there exists a unique $c \in [0, 1]$ (where $[0, 1]$ is identified with the circle S^1) such that the following hold:
i) f is C^0 in $(0, c)$ and $(c, 1)$;
ii) f is increasing in $[0, c)$ and $(c, 1]$;
iii) $f(c-) = 1$ and $f(c+) = 0$;
iv) $f(0+) \geqslant f(1-)$.

Define $F(y+n) = f(y) + n$, if $0 \leqslant y < c$, and $F(y+n) = f(y) + n + 1$, if $c \leqslant y < 1$. The rotation number of f is defined as

$$\rho(f) = \lim_{t\to\infty} \frac{F^n(y) - y}{n},$$

for any $y \in \mathbb{R}$. The rotation number $\rho(f)$ is independent of $y \in \mathbb{R}$. (Proposition 25 in [18])

Lemma 1. [Remark 34 in [18]] Let f be a circle map satisfying i)-iv) of Definition 2 and assume that $\rho(f) \in \mathbb{Q}$. Then if $f'(y) < 1$ for $y \in (0, c) \cup (c, 1)$, there exists an unique and attracting periodic orbit for system

$$y(t+1) = f(y(t)), \quad t \geqslant 0.$$

Lemma 2. [Proposition 39 in [18]] Let f be a circle map satisfying i)-iv) of Definition 2 and assume that $\rho(f) \in \mathbb{R}/\mathbb{Q}$. Then if $f'(y) < 1$ for $y \in (0, c) \cup (c, 1)$ and condition iv) in Definition 2 is a strict inequality, the ω-limit set of system

$$y(t+1) = f(y(t)), \quad t \geqslant 0,$$

is a Cantor set.

Appendix F Survey data

The placement on left right scale (ESS):
Literal question: In politics people sometimes talk of "left" and "right". Using this card, where would you place yourself on this scale, where 0 means the left and 10 means the right?

Table F1 The placement on left right scale.

Year	Left	1	2	3	4	5	6	7	8	9	Right
2002	1291	886	2140	3875	4041	12265	3494	3843	3014	965	1330
2004	1234	932	2226	3998	4188	13943	3677	4274	3256	1153	1619
2006	1196	869	2053	3743	3784	12213	3517	3740	2893	860	1214
2008	1589	1261	2590	4644	4695	15412	4498	4960	4102	1601	2217
2010	1418	1093	2338	4206	4206	14939	4257	4749	3722	1282	1665
2012	1946	1180	2462	3944	4269	15400	4323	4942	3987	1500	2504
2014	1357	907	2062	3788	3575	11557	3573	3790	2984	997	1265
2016	1040	622	1582	2840	2919	9826	3157	3279	2503	776	1147

Think of self as liberal or conservative (GSS):
We hear a lot of talk these days about liberals and conservatives.
I'm going to show you a seven-point scale on which the political views that people might hold are arranged from extremely liberal–point 1–to extremely conservative–point 7. Where would you place yourself on this scale?

Table F2 Think of self as liberal or conservative.

Year	Extremely liberal	Liberal	Slightly liberal	Moderate	Slightly conservative	Conservative	Extremely conservative
2002	47	143	159	522	209	210	41
2004	46	120	153	497	214	223	56
2006	139	524	517	1683	618	685	167
2008	69	240	221	740	268	327	68
2010	76	259	232	746	265	315	80
2012	81	244	208	713	268	292	68
2014	94	304	263	989	334	358	107
2016	136	350	310	1032	382	426	120

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