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• **Supplementary File** •

Stability of networked control system subject to Denial-of-Service

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Appendix A Proof of Theorem 1

Before providing the proof of Theorem 1, we firstly introduce the following lemma about ms-ISS of the system without DoS attacks.

Lemma 1: Consider the closed-loop control system (7) without DoS attacks. Suppose that there exist a constant $\upsilon \in (0,1)$, a positive scalar κ and a symmetric positive definite matrix P satisfying the following inequality:

$$\begin{bmatrix} A_1^T P A_1 - \upsilon P & -A_1^T P \Lambda_2 \\ * & \Lambda_2^T P \Lambda_2 - \kappa I \end{bmatrix} < 0. \tag{A1}$$

If the parameter $\sigma \in (0, \sqrt{(1-\upsilon)\lambda_{min}(P)/\kappa})$ in the triggering condition (3), then the following inequality holds:

$$E[V(k+1)|k] \leqslant (1-\pi)E[V(k)] + \theta^2.$$

Hence the system is ms-ISS, where the Lyapunov function is $V(k) = z_k^T P z_k$, and $\pi = 1 - \upsilon - \frac{\kappa\sigma^2}{\lambda_{min}(P)}$, $\theta^2 = \left\|\Lambda_1^T P \Lambda_1\right\|$.

Proof of Lemma 1:

In case of $k \neq k_i$,

$$\begin{aligned} V(k+1) &= z_{k+1}^T P z_{k+1} \\ &= ({z_k}^T {A_1}^T + (\tilde{x}_k - x_k)^T {\Lambda_2}^T + {w_k}^T {\Lambda_1}^T) P (A_1 z_k + \Lambda_2(\tilde{x}_k - x_k) + \Lambda_1 w_k) \\ &= {z_k}^T {A_1}^T P A_1 z_k + (\tilde{x}_k - x_k)^T {\Lambda_2}^T P \Lambda_2 (\tilde{x}_k - x_k) + {w_k}^T {\Lambda_1}^T P \Lambda_1 w_k \\ &\quad + 2{z_k}^T {A_1}^T P \Lambda_2 (\tilde{x}_k - x_k) + 2{z_k}^T {A_1}^T P \Lambda_1 w_k + 2(\tilde{x}_k - x_k)^T {\Lambda_2}^T P \Lambda_1 w_k. \end{aligned}$$

According to $E[w_k] = 0$ and $E[{w_k}^T w_k] = 1$, taking mathematical expectation of both sides leads to:

$$\begin{aligned} E[V(k+1)] \leqslant\ & E[{z_k}^T {A_1}^T P A_1 z_k] + E[(\tilde{x}_k - x_k)^T {\Lambda_2}^T P \Lambda_2 (\tilde{x}_k - x_k)] \\ & + E[2{z_k}^T {A_1}^T P \Lambda_2 (\tilde{x}_k - x_k)] + \left\|{\Lambda_1}^T P \Lambda_1\right\|. \end{aligned}$$

Then,

$$E[\Delta V(k)] \leqslant [{z_k}^T \ \ -(\tilde{x}_k - x_k)^T] \begin{bmatrix} {A_1}^T P A_1 - P & -{A_1}^T P \Lambda_2 \\ * & {\Lambda_2}^T P \Lambda_2 \end{bmatrix} \begin{bmatrix} z_k \\ -(\tilde{x}_k - x_k) \end{bmatrix} + \theta^2, \tag{A2}$$

where $\theta^2 = \left\|{\Lambda_1}^T P \Lambda_1\right\|$.

Based on (A1) and the triggering condition (3) , (A2) can be converted to :

$$\begin{aligned} E[\Delta V(k)] &\leqslant [{z_k}^T \ \ -(\tilde{x}_k - x_k)^T] \begin{bmatrix} \upsilon P - P & 0 \\ 0 & \kappa I \end{bmatrix} \begin{bmatrix} z_k \\ -(\tilde{x}_k - x_k) \end{bmatrix} + \theta^2 \\ &\leqslant -(1-\nu){z_k}^T P z_k + \kappa(\tilde{x}_k - x_k)^T(\tilde{x}_k - x_k) + \theta^2 \\ &\leqslant -(1-\nu){z_k}^T P z_k + \kappa\sigma^2 {x_k}^T x_k + \theta^2, \end{aligned}$$

where the first inequality is from the following fact:

$$\begin{bmatrix} A_1^T P A_1 - \upsilon P & -A_1^T P \Lambda_2 \\ * & \Lambda_2^T P \Lambda_2 - \kappa I \end{bmatrix} = \begin{bmatrix} A_1^T P A_1 - P & -A_1^T P \Lambda_2 \\ * & \Lambda_2^T P \Lambda_2 \end{bmatrix} + \begin{bmatrix} -\upsilon P + P & 0 \\ 0 & -\kappa I \end{bmatrix} < 0,$$

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then, $\begin{bmatrix} A_1^T P A_1 - P & -A_1^T P \Lambda_2 \\ * & \Lambda_2^T P \Lambda_2 \end{bmatrix} < \begin{bmatrix} \upsilon P - P & 0 \\ 0 & \kappa I \end{bmatrix}$.

Due to the definition of z_k, obviously, $\|x_k\| \leqslant \|z_k\|$, and hence,

$$\begin{aligned} E[\Delta V(k)] &\leqslant -(1-\nu){z_k}^T P z_k + \kappa\sigma^2 {z_k}^T z_k + \theta^2 \\ &\leqslant -(1-\nu){z_k}^T P z_k + \kappa\sigma^2 \|z_k\|^2 + \theta^2 \\ &\leqslant -(1-\nu-\frac{\kappa\sigma^2}{\lambda_{\min}(P)})V(k) + \theta^2 \\ &= -\pi V(k) + \theta^2 \end{aligned}$$

By condition that $\sigma \in (0, \sqrt{(1-\upsilon)\lambda_{min}(P)/\kappa})$, then , $\pi \in (0,1)$.

Thus, $E[V(k+1)] \leqslant (1-\pi)E[V(k)] + \theta^2$. Furthermore, by repeating the process above, one can obtain:

$$\begin{aligned} E({z_{k+1}}^T P z_{k+1}) &\leqslant (1-\pi)E({z_k}^T P z_k) + \theta^2 \\ &\leqslant (1-\pi)^2 E({z_{k-1}}^T P z_{k-1}) + (1-\pi)\theta^2 \\ &\ \ \cdots \\ &\leqslant (1-\pi)^{k+1} E({z_0}^T P z_0) + (1-\pi)^k \theta^2. \end{aligned}$$

There exist positive constants a and b such that the following inequalities hold:

$$aE(\|z_{k+1}\|^2) \leqslant E({z_{k+1}}^T P z_{k+1}), E({z_0}^T P z_0) \leqslant bE(\|z_0\|^2). \tag{A3}$$

Therefore, $E(\|z_{k+1}\|^2) \leqslant \frac{b}{a} E((1-\pi)^{k+1}\|z_0\|^2) + \frac{1}{a}(1-\pi)^k\theta^2$.

Notice that $z_k = [x_k^T, \tilde{x}_k^T]^T$, which leads to

$$\|x_k\| \leqslant \|z_k\|, \|z_0\| \leqslant 2\|x_0\|.$$

Thus,

$$E[\|x_{k+1}\|^2] \leqslant \frac{4b}{a}(1-\pi)^{k+1}E[\|x_0\|^2] + \frac{1}{a}(1-\pi)^k\theta^2,$$

which implies that the system (7) subject to DoS attacks is ms-ISS by Definition 1.

The proof of Lemma is completed.

Based on the Lemma 1, the proof of Theorem 1 is provided as follows.

Proof of Theorem 1

In case of $k = k_i$, the control systems are attacked. Then one can obtain

$$\begin{aligned} V(k+1) &= [(1-\alpha_{k_i}){z_{k_i}}^T {A_1}^T + \alpha_{k_i}{z_{k_i}}^T {A_2}^T + {w_{k_i}}^T {\Lambda_1}^T]P \\ &\quad \times [(1-\alpha_{k_i})A_1 z_{k_i} + \alpha_{k_i}A_2 z_{k_i} + \Lambda_1 w_{k_i}] \\ &= (1-\alpha_{k_i})^2 {z_{k_i}}^T {A_1}^T P A_1 z_{k_i} + \alpha_{k_i}^2 {z_{k_i}}^T {A_2}^T P A_2 z_{k_i} + {w_{k_i}}^T {\Lambda_1}^T P \Lambda_1 w_{k_i} \\ &\quad + 2(1-\alpha_{k_i}){z_k}^T {A_1}^T P \Lambda_1 w_{k_i} + 2\alpha_{k_i}{z_{k_i}}^T {A_2}^T P \Lambda_1 w_{k_i}, \end{aligned}$$

due to $\alpha_{k_i}(1-\alpha_{k_i}) = 0$.

Taking mathematical expectation of both sides, we have

$$\begin{aligned} E[V(k+1)] \leqslant &(1-\bar{\alpha})E[z_{k_i}^T A_1^T P A_1 z_{k_i}] + \bar{\alpha}E[z_{k_i}^T A_2^T P A_2 z_{k_i}] + \left\|\Lambda_1^T P \Lambda_1\right\| \\ = &E[z_{k_i}^T((1-\bar{\alpha})A_1^T P A_1 + \bar{\alpha}A_2^T P A_2)z_{k_i}] + \left\|\Lambda_1^T P \Lambda_1\right\|, \end{aligned} \tag{A4}$$

where the first inequality utilizes the facts of $E[(1-\alpha_{k_i})^2] = 1-\bar{\alpha}, E[\alpha_{k_i}^2] = \bar{\alpha}$.

Based on (10), it is obtained that

$$E[V(k_i+1)|k_i] \leqslant E[z_{k_i}^T(P-Q)z_{k_i}] + \theta^2,$$

where $\theta^2 = \left\|\Lambda_1^T P \Lambda_1\right\|$, which leads to

$$\begin{aligned} E[\Delta V(k_i)|k_i] \leqslant &-\lambda_{min}(Q)E[\left\|z_{k_i}\right\|^2] + \theta^2 \\ \leqslant &-\frac{\lambda_{min}(Q)}{\lambda_{max}(P)}E[V(k_i)] + \theta^2. \end{aligned}$$

Therefore,

$$\begin{aligned} E[V(k_i+1)|k_i] \leqslant &\left(1 - \frac{\lambda_{min}(Q)}{\lambda_{max}(P)}\right)E[z_{k_i}^T P z_{k_i}] + \theta^2 \\ = &\gamma E[V(k_i)] + \theta^2, \end{aligned}$$

where $\gamma = 1 - \frac{\lambda_{min}(Q)}{\lambda_{max}(P)}$.

On the other hand, for the case $k \neq k_i$, it follows from Lemma 1 that $E[V(k+1)|k] \leqslant (1-\pi)E[V(k)] + \theta^2$. Let $\delta = \max\{(1-\pi), \gamma\}$, therefore, for two cases,

$$E[V(k+1)|k] \leqslant \delta E[V(k)] + \theta^2.$$

By iterations, it is easy to obtain that

$$E[V(k)] \leqslant \delta^k E[V(0)] + \frac{1-\delta^k}{1-\delta}\theta^2. \tag{A5}$$

Then, similar to the proof of Lemma 1, (A5) can lead to

$$E[\|z_k\|^2] \leqslant \frac{4b}{a}\delta^k E[\|z_0\|^2] + \frac{(1-\delta^k)}{a(1-\delta)}\theta^2. \tag{A6}$$

i.e., the system (7) is ms-ISS.

If disturbances are absent ($w_k = 0$), according to the analysis above, one can easily obtain that

$$E[\|x_k\|^2] \leqslant \frac{4b}{a}\delta^k E[\|x_0\|^2],$$

which implies that $E[x_k] \to 0$ as $k \to \infty$. Thus, the event-triggered control systems subject to DoS attacks is mean-square globally asymptotically stable when $w_k = 0$.

Appendix B Proof of Theorem 2

The proof is provided by showing that for any $\bar{\alpha} \in (0, \alpha_{\max})$, (11) and (12) are always satisfied.

According to (11), $A_1^T P A_1 - \upsilon P < 0$ is not violated, dut to $\upsilon \in (0,1)$. And the following inequality holds always:

$$\alpha_{\max} {A_2}^T P A_2 + (1-\alpha_{\max}) {A_1}^T P A_1 - P < 0. \tag{B1}$$

Let ${A_1}^T P A_1 - P =: G < 0$ and ${A_2}^T P A_2 - {A_1}^T P A_1 =: F$ with $\alpha_{\max} F + G < 0$.

For any $\bar{\alpha} < \alpha_{\max}$, we have:

$$\begin{aligned} &\bar{\alpha}({A_2}^T P A_2 - {A_1}^T P A_1) + {A_1}^T P A_1 - P \\ &= \frac{\bar{\alpha}}{\alpha_{\max}}(\alpha_{\max} F + G) + \frac{\alpha_{\max} - \bar{\alpha}}{\alpha_{\max}} G \\ &= \frac{\bar{\alpha}}{\alpha_{\max}}(\alpha_{\max} F + G) + (1 - \frac{\bar{\alpha}}{\alpha_{\max}}) G \\ &< 0. \end{aligned} \tag{B2}$$

It is obviously that $0 < \frac{\bar{\alpha}}{\alpha_{\max}} < 1$. Therefore, there exists a symmetric positive definite matrix $\bar{Q}$ such that

$$\bar{\alpha} F + G + \bar{Q} = 0.$$

Based on the above, (10) is satisfied.

Applying the Schur Complement to (9) can obtain the following matrix inequalities:

$$A_1^T P A_1 - \upsilon P < 0, \tag{B3}$$

and

$$\Lambda_2^T P \Lambda_2 - \kappa I - \Lambda_2^T P A_1 (A_1^T P A_1 - \upsilon P)^{-1} A_1^T P \Lambda_2 < 0. \tag{B4}$$

Obviously, there always exist a symmetric positive matrix P and a constant υ such that (B3) holds if $A + B_1 K$ is Schur stable and there is a large enough constant κ satisfying (B4). Therefore, (9) is satisfied.

Thus, $\alpha_{\max}$ is the solution to this optimization problem by Definition 2. And it is obvious that for any $\bar{\alpha} \in (0, \alpha_{\max})$, (11) and (12) are always satisfied with some positive definite matrix P.

Appendix C Corollary 1: Result of time-triggered control

To compare the robustness between event-triggered control system and time-triggered control system, we give the following corollary. Time-triggered control system can be written:

$$z_{k+1} = (1 - \alpha_{k_i}) A_1 z_k + \alpha_{k_i} A_2 z_k + \Lambda_1 \omega_k,$$

where,

$$A_1 = \begin{bmatrix} A + B_1 K & 0_{n\times n} \\ I_{n\times n} & 0_{n\times n} \end{bmatrix}, \Lambda_1 = \begin{bmatrix} B_2 \\ 0_{n\times m} \end{bmatrix}, A_2 = \begin{bmatrix} A & B_1 K \\ 0_{n\times n} & I_{n\times n} \end{bmatrix}, \Lambda_2 = \begin{bmatrix} B_1 K \\ I_{n\times n} \end{bmatrix}.$$

Corollary 1 Assuming the attack probability $\bar{\alpha}$ is known. Then the time-triggered control system is ms-ISS stable, if for any given symmetric positive definite matrix Q, there is a symmetric positive definite solution to the equation

$$\bar{\alpha} {A_2}^T P A_2 + (1 - \bar{\alpha}) {A_1}^T P A_1 + Q = P. \tag{C1}$$

The proof of Corollary 1 is omitted, which is similar to Theorem 1.

It can be seen from above, the stability of ETC system under DoS attacks need to satisfy (9) and (10), while the time-triggered control system only need to satisfy (C1). Hence, the robustness of time-triggered control system is better than ETC system.

Appendix D Corollary 2: The method of designing feedback gain matrix K

Theorem 1 only supports the sufficient conditions. Thus, if the conditions of Theorem 2 are not satisfied for a pre-specific K, whether the control gain matrix K can be designed to guarantee the stability of the system (7). The following Corollary is obtained.

Corollary 2 Assume that the attack probability $\bar{\alpha}$, positive constants χ and υ are given. Then system (7) is ms-ISS if there exist a symmetric positive definite matrix M, a matrix K and $\kappa > 0$ satisfying the linear matrix inequality:

$$\begin{bmatrix} -2\upsilon\chi I + \upsilon\chi^2 M & * & * & * & * & * \\ 0 & -\kappa I & * & * & * & * \\ \begin{bmatrix} A+B_1K & 0 \\ I & 0 \end{bmatrix} & \begin{bmatrix} B_1K \\ I \end{bmatrix} & -M & * & * & * \\ 0 & 0 & 0 & -2\chi I + \chi^2 M & * & * \\ 0 & 0 & 0 & \begin{bmatrix} A & B_1K \\ 0 & I \end{bmatrix} & -\alpha^{-1}M & * \\ 0 & 0 & 0 & \begin{bmatrix} A+B_1K & 0 \\ I & 0 \end{bmatrix} & 0 & -(1-\alpha)^{-1}M \end{bmatrix} < 0. \tag{D1}$$

Proof of Corollary 2

Due to $(P^{\frac{1}{2}} - \chi P^{-\frac{1}{2}})^2 \geqslant 0$, then $-P \leqslant -2\chi I + \chi^2 P^{-1}$. From this and (D1), it can be obtained:

$$\begin{bmatrix} -\upsilon M^{-1} & * & * \\ 0 & -\kappa I & * \\ \begin{bmatrix} A+B_1K & 0 \\ I & 0 \end{bmatrix} & \begin{bmatrix} B_1K \\ I \end{bmatrix} & -M \end{bmatrix} < 0, \tag{D2}$$

$$\begin{bmatrix} -M^{-1} & * & * \\ \begin{bmatrix} A & B_1K \\ 0 & I \end{bmatrix} & -\alpha^{-1}M & 0 \\ \begin{bmatrix} A+B_1K & 0 \\ I & 0 \end{bmatrix} & 0 & -(1-\alpha)^{-1}M \end{bmatrix} < 0. \tag{D3}$$

Equation (9) equals to $\begin{bmatrix} -\upsilon P & 0 \\ 0 & -\kappa I \end{bmatrix} - \begin{bmatrix} A_1^T \\ \Lambda_2^T \end{bmatrix} (-P) \begin{bmatrix} A_1 & -\Lambda_2 \end{bmatrix} < 0$. Applying Schur Complement to this inequality, it follows that:

$$\begin{bmatrix} -\upsilon P & * & * \\ 0 & -\kappa I & * \\ A_1 & \Lambda_2 & -P^{-1} \end{bmatrix} < 0. \tag{D4}$$

(D4) is equivalent to (D2).

Equation (10) equals to $-P + \bar{\alpha} A_2^T P A_2 + (1-\bar{\alpha}) A_1^T P A_1 = -Q < 0$. Applying Schur Complement to this inequality, it follows that:

$$\begin{bmatrix} -P & {A_2}^T & {A_1}^T \\ * & -\alpha^{-1}P^{-1} & 0 \\ * & * & -(1-\alpha)^{-1}P^{-1} \end{bmatrix} < 0, \tag{D5}$$

where $M = P^{-1}$. And (D5) is equivalent to (D3).

Therefore, (D1) can deduce that (9) and (10) hold. Then the system (7) is ms-ISS according to Theorem 1.

Appendix E Simulations

An numerical example is shown to demonsatrate the effectiveness of the proposed results in this letter. Consider the linear plant:

$$A = \begin{bmatrix} 1 & 0.1 \\ -0.5 & 1.1 \end{bmatrix}, B_1 = \begin{bmatrix} 0.1 \\ 1.2 \end{bmatrix}, B_2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \tag{E1}$$

and the initial condition $x_0 = \begin{bmatrix} -2 & 3 \end{bmatrix}$. The feedback gain of the controllers is $K = \begin{bmatrix} -0.1314 & -0.4819 \end{bmatrix}$.

According to Theorem 2, we can obtain that $\alpha_{max} = 0.4528$. And we pick for $\bar{\alpha} = 0.2$ and $\bar{\alpha} = 0.5$ respectively.

Obviously, the system with attack probabilities $\bar{\alpha} = 0.2 < \alpha_{max} = 0.4528$ is ms-ISS from Theorem 1. Figure E1 depicts the evolution of $\|x_k\|$, where $'+'$ denote DoS attacks on Channel SC.

When $\bar{\alpha} = 0.5$, it can be seen from Figure E2 that the system under DoS attacks is still stable, although $\bar{\alpha} = 0.5 > 0.4528$. This indicates that the estimation of α_{max} is conservative. Thus, Theorem 2 could estatimate the robustness of systems under the DoS attacks obeying Bernoulli distribution although with some degree of conservatism.

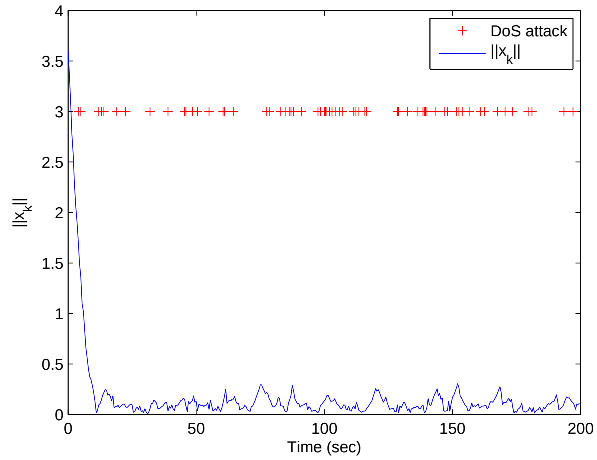

Figure E1 Simulation example for event-triggered control system with $\bar{\alpha} = 0.2$.

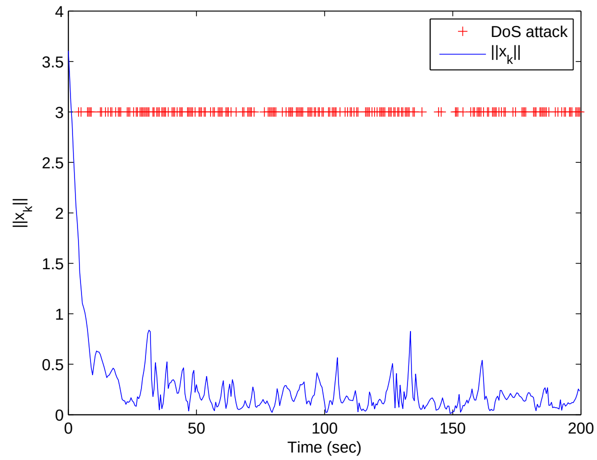

Figure E2 Simulation example for event-triggered control system with $\bar{\alpha} = 0.5$.

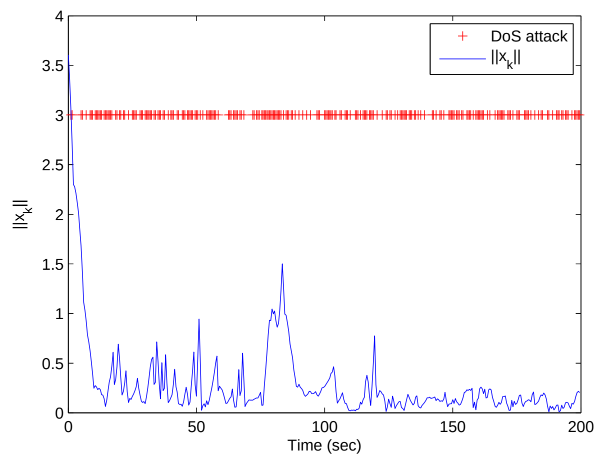

Figure E3 Simulation example for event-triggered control system with $\bar{\alpha} = 0.6636$.

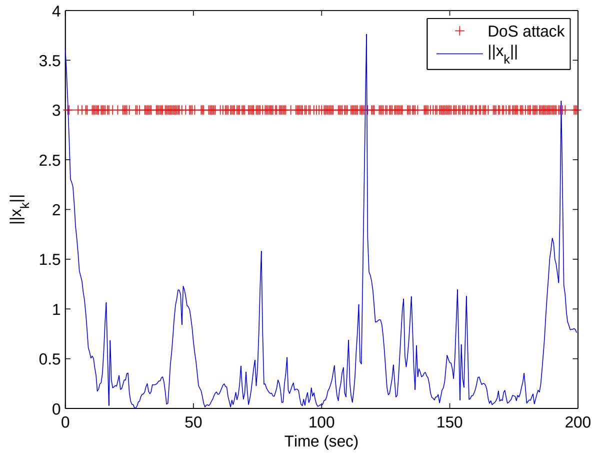

Figure E4 Simulation example for event-triggered control system with $\bar{\alpha} = 0.6637$.

Table E1 The relationship between σ and $\bar{\alpha}$

$\bar{\alpha}$	0.1	0.2	0.3	0.4
σ	0.16	0.15	0.13	0.09

Further, the upper bound of $\bar{\alpha}$ could be obtained in simulation by linear search method. When $\bar{\alpha} \leqslant 0.6636$, see Figure E3, the system under DoS attacks is stable. However, when $\bar{\alpha} \geqslant 0.6637$ the system under DoS attacks is unstable, which can be seen from Figure E4. Through series of simulations, we can obtain that $\bar{\alpha} = 0.2$ verifies the feasibility of the proposed method in this letter, $\bar{\alpha} = 0.5$ shows the conservatism of the proposed method and $\bar{\alpha} = 0.6636$ along with $\bar{\alpha} = 0.6637$ explains the existence of the upper bound of $\bar{\alpha}$ when the ms-ISS stability of system is always preserved.

The value of α_{max} of time-triggered control systems is 0.4581, which is obtaind by Lemma 5 in [1]. Thus, the robustness of the time-triggered control systems is slightly better than the event-triggered control systems under DoS attacks.

It is noted that the range of σ in triggering condition parameter (3) is not involved in Theorem 2. Hence, to further demonstrate the effect of σ on systems, the relationship between σ and attack probability $\bar{\alpha}$ is indicated by simulaion, see Table E1. Table E1 implies that $\bar{\alpha}$ becomes greater as σ decreases. In other words, a smaller inter-sampling is beneficial to the ms-ISS of systems. Although a bigger σ can save more communication resources, the robustness of system subject to DoS attacks is receding.

References

1 Hu S, Yan W. Y. Stability robustness of networked control systems with respect to packet loss, Automatica, 2007,43(7): 1243-1248.