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Slow state estimation for singularly perturbed systems with discrete measurements

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Dear editor,

Many real-world systems exhibit a two-time-scale behavior, such as Hindmarsh-Rose oscillators, DC motors, power networks, and other related systems. Such systems can be modeled using singularly perturbed systems (SPSs). To alleviate the difficulties of high dimensionality and stiffness, Kokotović et al. [1] developed the singular perturbation method for reducing the model order, whereby the full system design is divided into designing separate controllers for fast and slow subsystems. In this regard, O'Reilly [2] proposed a composite full-order observer design method for linear SPSs through observer design in two separate time-scales. Later, the two-time-scale observer design was extended to nonlinear systems [3, 4]. However, in many engineering systems with two-time scales, the slow subsystems usually represent the "dominant" parts of the plants, while the fast subsystems represent the "parasitic" parts. For example, in an instrumented system, the slow and fast subsystems represent the process dynamics and the sensor dynamics, respectively, while the perturbation parameter represents a measure of the relative speed/time constant of the sensor dynamics. For these systems, fast subsystems are typically considered as unmodeled dynamics, and could be neglected if they are asymptotically stable. Consequently, questions related to robustness arise in relation to the design of control and estimation algorithms based on the approximate slow dynamics. In [5–7], slow reduced-order observers for linear and nonlinear SPSs were designed, and the observation errors generated by neglecting the unmodeled fast dynamics were obtained. Recently, feedback control based on slow dynamics was studied in [8, 9]. We note that the reduced-order observer designs considered in [5–7] are based on the assumption that the output measurements are continuously available. However, in the context of digital control, the measurements are generally available at discrete instants of time due to the use of digital sensors. Therefore, it is important to expand the observer design approach as mentioned above to the discrete time measurement case. This research studies the observer design problem for a class of SPSs with sampled measurements. Our objective is to design continuous-discrete time observers of slow states based only on an approximate model of the slow dynamics. To the best of the authors' knowledge, this is the first attempt to develop a slow state estimation algorithm for SPSs via sampled measurements. The main contribution of this research is with regard to two aspects. First, a method is proposed for using a single continuous-discrete time observer for slow state estimation in a singularly perturbed system with discrete measurements. Second, the estimation on the observation error is obtained by applying the singularly perturbed theory and the time-dependent Lyapunov functional method, which characterizes the effects of the singular perturbation parameter and the sampling period on the observation accuracy. In particular, it is proved that for a small enough sampling period, the observation error decays exponentially to $O(\varepsilon)$.

Notation. For two symmetric matrices P and Q, $P > Q$ means that $P - Q$ is positive-definite. I_{n_1} and 0_{n_1} denote the $n_1 \times n_1$ identity matrix and zero matrix, respectively. $\mathcal{I}_1 = [I_{n_1} \quad 0_{n_1}]$, $\mathcal{I}_2 = [0_{n_1} \quad I_{n_1}]$, and $\mathcal{I}_3 = \mathcal{I}_1 - \mathcal{I}_2$. For a real square matrix A, $\mathrm{He}(A)$ denotes $A + A^{\mathrm{T}}$. $\mathbb{N}$ stands for the set of nonnegative integers. $\|\cdot\|$ denotes the spectral norm for matrices and the Euclidean norm for vectors.

Consider the class of nonlinear SPSs described as follows:

$$\begin{aligned} &\dot{x}(t) = A_{11}x(t) + A_{12}z(t) + f(x(t)) + Bu(t),\\ &\varepsilon\dot{z}(t) = A_{21}x(t) + A_{22}z(t),\\ &y_k = C_1x(t_k) + C_2z(t_k),\ k \in \mathbb{N}, \end{aligned} \tag{1}$$

where $\varepsilon > 0$ is the singular perturbation parameter, $x \in \mathbb{R}^{n_1}$, $z \in \mathbb{R}^{n_2}$, and $u \in \mathbb{R}^p$ denote the slow state, fast state, and known bounded input, respectively. $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $C_i \in \mathbb{R}^{q \times n_i}$, $i, j \in \overline{1, 2}$, and $B \in \mathbb{R}^{n \times p}$, are constant matrices. $y_k \in \mathbb{R}^q$ denotes the measurement output, which is available at sampling instants t_k, satisfying $0 < h_0 \leqslant t_{k+1} - t_k \leqslant h$ for $k \in \mathbb{N}$. It is assumed that A_{22} is Hurwitz, and the nonlinear function $f(x) : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ satisfies the Lipschitz condition: $\|f(x_1) - f(x_2)\| \leqslant \|G(x_1 - x_2)\|$ with $G \in \mathbb{R}^{n_1}$. By setting $\varepsilon = 0$ in system (1), the reduced-order model on the slow manifold $\bar{z} = -A_{22}^{-1}A_{21}\bar{x}$ is obtained. This is given as follows:

$$\begin{aligned} &\dot{\bar{x}}(t) = A_0\bar{x}(t) + f(\bar{x}(t)) + Bu(t),\\ &\bar{y}_k = C_0\bar{x}(t_k),\ k \in \mathbb{N}, \end{aligned} \tag{2}$$

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where $A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}$, and $C_0 = C_1 - C_2A_{22}^{-1}A_{21}$. On the basis of model (2), we propose a continuous-discrete time observer for system (1):

$$\dot{\hat{x}}(t) = A_0\hat{x}(t) + f(\hat{x}(t)) + L(y_k - C_0\hat{x}(t_k)) + Bu(t), \quad \text{for } t \in [t_k, t_{k+1}),\ k \in \mathbb{N}, \tag{3}$$

where $L \in \mathbb{R}^{n_1\times q}$ is the observer gain to be designed. Let $e(t) = x(t) - \hat{x}(t)$, $\rho(t) = t - t_k$, for $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. In view of Lemma 1 in [5], the error dynamics can be written as follows:

$$\dot{e}(t) = A_0e(t) - LC_0e(t-\rho(t)) + F(t) + \omega_\varepsilon(t), \tag{4}$$

where $F(t) = f(x(t)) - f(\hat{x}(t))$, $\omega_\varepsilon(t) = (A_{12}\mathrm{e}^{\frac{A_{22}(t-t_0)}{\varepsilon}} - LC_2\mathrm{e}^{\frac{A_{22}(t-\rho(t)-t_0)}{\varepsilon}})\tilde{x}(t_0) + O(\varepsilon)$, wherein $\tilde{x}(t_0) = (z(t_0) + A_{22}^{-1}A_{21}x(t_0))$.

The Hurwitz property of A_{22} ensures the existence of K and β_1 such that $\mathrm{e}^{A_{22}t} \leqslant K\mathrm{e}^{-\beta_1 t}$, $\forall\, t \geqslant 0$. Moreover, we have $t_k - t_0 \geqslant \upsilon_0(t-t_0)$ for $t \in [t_k, t_{k+1})$ with $k \geqslant 1$, where $\upsilon_0 = h_0/(2h)$. For any given $\alpha > 0$, direct computation yields

$$\int_{t_0}^{t} \mathrm{e}^{-2\alpha(t-s)}\|\omega_\varepsilon(s)\|^2\mathrm{d}s \leqslant E_\varepsilon(t) \triangleq \sum_{i=1}^{2}\frac{3(c_iK)^2\varepsilon}{2(\alpha\varepsilon-\beta_i)} \times\left[\mathrm{e}^{-(2\beta_i/\varepsilon)(t-t_0)} - \mathrm{e}^{-2\alpha(t-t_0)}\right] + 3(c_2K)^2(\mathrm{e}^{2\alpha h} - 1)(1/(2\alpha))\mathrm{e}^{-2\alpha(t-t_0)} + O(\varepsilon^2),\ \forall\ t \geqslant t_0, \tag{5}$$

wherein $\beta_2 = \upsilon_0\beta_1$, $c_1 = \|A_{12}\|\|\tilde{x}(t_0)\|$, and $c_2 = \|LC_2\|\|\tilde{x}(t_0)\|$. The following theorem presents this research's main result.

Theorem 1. Consider system (1). For a given scalar $\alpha > 0$, if there exist $n_1\times n_1$ matrices $P > 0$, and $R > 0$, $2n_1\times n_1$ matrix M, $n_1\times q$ matrix $\bar{L}$, and scalars $\sigma > 0$, $\gamma > 0$, α_i, and κ_i, $i = 1, 2$, such that the following matrix inequalities hold:

$$\begin{bmatrix} P & h\kappa_2P \\ * & \mathrm{e}^{-2\alpha h} + h(\kappa_1 - 2\kappa_2)P \end{bmatrix} > 0, \tag{6}$$

$$\begin{bmatrix} \Xi_i & h\Psi_i \\ * & h\Phi_i \end{bmatrix} < 0,\ i = 1, 2, \tag{7}$$

where $\Phi_1 = -R$, $\Phi_2 = -2\sigma P + \sigma^2R$,

$$\Psi_1^{\mathrm{T}} = \mathrm{e}^{\alpha h}[M^{\mathrm{T}}\ \ 0\ \ 0],\ \Psi_2^{\mathrm{T}} = [\bar{\mathcal{A}}^{\mathrm{T}}\ \ P\ \ P],$$

$$\Xi_i = \begin{bmatrix} \Upsilon_i + h\tilde{\Upsilon}_i & (\mathcal{I}_1 + h\mathcal{I}_{\kappa i})^{\mathrm{T}}P & (\mathcal{I}_1 + h\mathcal{I}_{\kappa i})^{\mathrm{T}}P \\ * & -\alpha_iI & 0 \\ * & * & -\gamma I \end{bmatrix},$$

whererin $\bar{\mathcal{A}} = PA_0\mathcal{I}_1 - \bar{L}C_0\mathcal{I}_3$, $\tilde{\Upsilon}_1 = 0$, $\mathcal{I}_{\kappa1} = 0$, $\mathcal{I}_{\kappa2} = \mathcal{I}_\kappa \triangleq \kappa_1\mathcal{I}_2 + \kappa_2\mathcal{I}_3$, and

$$\Upsilon_i = 2\alpha\mathcal{I}_1^{\mathrm{T}}P\mathcal{I}_1 - \kappa_1\mathcal{I}_2^{\mathrm{T}}P\mathcal{I}_2 + \alpha_i\mathcal{I}_1^{\mathrm{T}}G^{\mathrm{T}}G\mathcal{I}_1 + \mathrm{He}(\mathcal{I}_1^{\mathrm{T}}\bar{\mathcal{A}} - \kappa_2\mathcal{I}_3^{\mathrm{T}}\mathcal{I}_2 + M\mathcal{I}_2),$$

$$\tilde{\Upsilon}_2 = \mathrm{He}(\alpha(\kappa_1(\mathcal{I}_2 + 2\kappa_2\mathcal{I}_3)^{\mathrm{T}}P\mathcal{I}_2) + \mathcal{I}_\kappa\bar{\mathcal{A}}),$$

then the estimation error $e(t)$ induced by the continuous-discrete observer (3) with $L = P^{-1}\bar{L}$ satisfies the following inequality:

$$\|e(t)\| \leqslant c_0\mathrm{e}^{-\alpha(t-t_0)} + \bar{E}_\varepsilon(t),\ t \geqslant t_0,$$

where $c_0 > 0$ is a positive scalar, and $\lim_{t\to\infty}\bar{E}_\varepsilon(t) = O(\varepsilon)$.

Proof. Define $\psi(t) = t_{k+1} - t$, and $\tilde{e}(t) = e(t) - e(t_k)$, for $t \in [t_k, t_{k+1})$. Set $\eta(t) = \mathrm{col}(e(t), \tilde{e}(t))$, and $\xi(t) = \mathrm{col}(\eta(t), F(t), \omega_\varepsilon(t))$. Consider the following continuous Lyapunov functional for error dynamics (4) with $L = P^{-1}\bar{L}$:

$$V(t) = e^{\mathrm{T}}(t)Pe(t) + \psi(t)\int_{t-\rho(t)}^{t}\mathrm{e}^{-2\alpha(t-s)}\dot{e}^{\mathrm{T}}(s)R \times\dot{e}(s)\mathrm{d}s + \psi(t)(\kappa_1\tilde{e}(t) + 2\kappa_2(e(t) - \tilde{e}(t)))^{\mathrm{T}}P\tilde{e}(t).$$

From $\tilde{e}(t) = \int_{t_k}^{t}\dot{e}(s)\mathrm{d}s$, we have

$$0 \leqslant \eta^{\mathrm{T}}(t)(M\mathcal{I}_2 + \mathcal{I}_2^{\mathrm{T}}M^{\mathrm{T}} + (h - \psi(t))\mathrm{e}^{2\alpha h}MR^{-1} \times M^{\mathrm{T}})\eta(t) + \int_{t_k}^{t}\mathrm{e}^{-2\alpha(t-s)}\dot{e}(s)R\dot{e}(s)\mathrm{d}s. \tag{8}$$

The Lipschitz property of f implies the following inequality:

$$0 \leqslant (\alpha_1\tilde{\theta}(t) + \alpha_2\theta(t))\xi^{\mathrm{T}}(t)(e^{\mathrm{T}}(t)G^{\mathrm{T}}Ge(t) - F^{\mathrm{T}}(t)F(t))\xi(t), \tag{9}$$

where $\theta(t) = \psi(t)/h$, and $\tilde{\theta}(t) = 1 - \theta(t)$.

Using (8) and (9), the upper right-hand Dini derivative of $V(t)$ along the trajectory of (4) is bounded by

$$D^+V(t) \leqslant -2\alpha V(t) + \xi^{\mathrm{T}}(t)\Xi(t)\xi(t) + \gamma\|\omega_\varepsilon(t)\|^2, \tag{10}$$

where $\Xi(t) = \tilde{\theta}(t)\tilde{\Xi}_1 + \theta(t)(\Xi_2 + h\mathcal{A}^{\mathrm{T}}R\mathcal{A})$, wherein $\mathcal{A} = [\mathcal{A}_0\ \ I\ \ I]$ with $\mathcal{A}_0 = A_0\mathcal{I}_1 - LC_0\mathcal{I}_3$, and $\tilde{\Xi}_1$ is derived from Σ_1 with $\Upsilon_1 + h\mathrm{e}^{2\alpha h}MR^{-1}M^{\mathrm{T}}$ instead of Υ_1. Using Schur complement, it is easy to verify that condition (7) implies $\Xi(t) < 0$. It follows from (10) that $D^+V(t) \leqslant -2\alpha V(t) + \gamma\|\omega_\varepsilon(t)\|^2$, which yields $V(t) \leqslant \mathrm{e}^{-2\alpha(t-t_0)}V(t_0) + \int_{t_0}^{t}\gamma\mathrm{e}^{-2\alpha(t-s)}\|\omega(s,\varepsilon)\|^2\mathrm{d}s$.

Note that condition (6) ensures the existence of a scalar λ_1 such that $V(t) \geqslant \lambda_1\|e(t)\|^2$. In view of (5), we obtain

$$\|e(t)\| \leqslant c_0\mathrm{e}^{-\alpha(t-t_0)}\|e(t_0)\| + \bar{E}_\varepsilon(t),$$

where $c_0 = \sqrt{\lambda_2/\lambda_1}$, $\bar{E}_\varepsilon(t) = \sqrt{(\gamma/\lambda_1)E_\varepsilon(t)}$, wherein $\lambda_2 = \lambda_{\max}(P)$. From the definition of $E_\varepsilon(t)$, we have $\lim_{t\to\infty}\bar{E}_\varepsilon(t) = O(\varepsilon)$. This completes the proof.

Numerical example. Consider system (1) with the following data:

$$A_{11} = \begin{bmatrix} -1 & 1 & 1 \\ -\dfrac{100}{7} & 0 & 0 \\ 9 & 0 & \dfrac{18}{7} \end{bmatrix},\ A_{12} = \begin{bmatrix} 0.01 & 0 \\ 0.01 & 0.01 \\ 0 & 0.02 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 0.1 & 0.2 & 0.1 \\ 0 & 0.3 & 0.3 \end{bmatrix},\ A_{22} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix},$$

$$f(x) = \frac{1}{2}(|x_3+1| - |x_3-1|)\,[0\ \ 0\ \ 3/7]^{\mathrm{T}},$$

$$C = [0\ \ 0\ \ 1],\ D = [0.6\ \ 0.4],\ B = 0.$$

By applying Theorem 1, we obtain the maximum value of h as $h = 0.054$. The corresponding observer gain $L^{\mathrm{T}} = [9.5542\ \ 13.8793\ \ 7.1078]$. With regard to the different values of ε, the impact of fast dynamics on the convergence properties of observer (2) is illustrated in Figure 1. It can be seen that for a small enough $\varepsilon > 0$, the observation error approximates zero as $t \to \infty$.

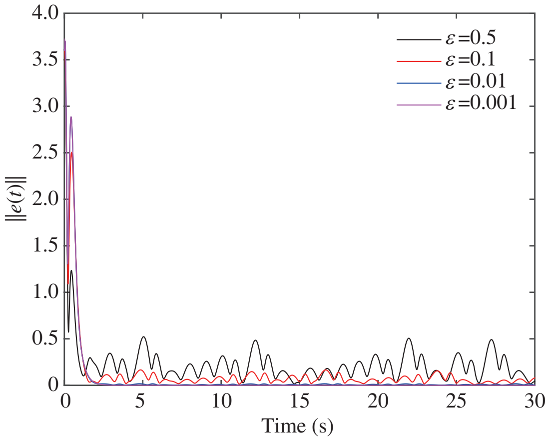

Figure 1 (Color online) Observation errors for different values of ε.

Conclusion. Within a singular perturbation framework, a continuous-discrete time reduced-order observer was designed for a class of SPSs with sampled measurements, assuming that the fast dynamics are asymptotically stable. It was proved that if a set of ε-independent linear matrix inequalities is feasible, then the observation error decays exponentially to $O(\varepsilon)$. In other words, if the stable fast dynamics are very fast and the sampling period is small enough, the proposed observer is able to provide an accurate estimate of the slow state.

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