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Stochastic maximum principle for optimal control problems involving delayed systems

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Dear editor,

The main objective of this study is to investigate one type of stochastic optimal control problem for a delayed system using the maximum principle method. The existing research can be categorized into two categories. In the first category, the adjoint equation comprises two backward stochastic differential equations (BSDEs) and one backward ordinary differential equation (BODE), which is assumed to have a zero solution; see e.g., [1–3]. However, there is still scope for improvement in this research direction. Here, a special setting is considered for the system parameters; see e.g., (3.9) and (3.12) in [2]. The BODE is assumed to result in a zero solution when the necessary maximum principle is strictly applied; see e.g., Theorem 5.1 in [3]. In the second category, a type of anticipated/time-advanced BSDE is introduced as the adjoint equation; see e.g., [1, 4–7].

In this study, (i) the averaged and point-wise time delays of the state and control processes are observed to be involved with the system equation and the cost function, ensuring that the problem that is being studied is in accordance with the general framework. (ii) Further, the necessary and sufficient maximum principles are clearly established, proving that the sufficient maximum principle conforms to some slightly relaxed conditions. (iii) Next, two types of adjoint equations are introduced, among which the first type contains an anticipated BSDE (see (3)) and a BSDE (see (4)), which can be expressed using one Hamiltonian H. The other type of adjoint equation comprises one time-advanced BSDE (see (5)) and one BSDE (see (6)), both of which can be expressed using another Hamiltonian $\mathcal{H}$. BODE is not needed as part of the adjoint equation. Furthermore, to the best of the author's knowledge, the two aforementioned types of adjoint equations are proved to be equivalent for the very first time. Thus, a unified adjoint equation can be obtained (see (2)).

Formulation. Here, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, where the expectation is denoted by $\mathrm{E}[\cdot]$. Let $W(t), t \geqslant 0$ be a one-dimensional standard Brownian motion, the augmented filtration of which is $\mathbb{F} = \{\mathcal{F}_t, t \geqslant 0\}$. Let $\mathrm{E}^t[\cdot] = \mathrm{E}[\cdot|\mathcal{F}_t]$ for each value of t. Further, assume that T is a finite time horizon, λ_1, λ_2 are constants and $\delta_1, \delta_2, \delta_3$ are positive constants, such that $\max\{\delta_1, \delta_2, \delta_3\} < T$. Let $L^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$ denote the class of $\mathbb{R}^n$-valued progressively measurable processes that are square integrable and $S^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$ denote the subspace of $L^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$, where each element $\psi(\cdot)$ is continuous and satisfies $\mathrm{E}\big[\sup_{0\leqslant t\leqslant T} |\psi(t)|^2\big] < \infty$. Let $x_0(\cdot) : [-\delta_1, 0] \to \mathbb{R}^n$ and $u_0(\cdot) : [-\delta_2, 0] \to \mathbb{R}^m$ be two functions that satisfy $\int_{-\delta_1}^0 |x_0(t)|^2 \mathrm{d}t < \infty$ and $\int_{-\delta_2}^0 |u_0(t)|^2 \mathrm{d}t < \infty$. Let U be a nonempty convex subset of $\mathbb{R}^m$. Let $\mathcal{U}$ denote the class of functions $u(\cdot) : \Omega \times [-\delta_2, T] \to \mathbb{R}^m$ which satisfies that $u(t) = u_0(t)$ for $t \in [-\delta_2, 0)$. $u(\cdot)|_{[0,T]}$ takes values in U and $u(\cdot)|_{[0,T]} \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^m)$. The element of $\mathcal{U}$ is called an admissible control. Here, each $u(\cdot) \in \mathcal{U}$ is accompanied by a moving-average delayed control $\nu(\cdot)$ and a point-wise delayed control $\mu(\cdot)$ which are defined as $\nu(t) = \int_{-\delta_2}^0 \mathrm{e}^{\lambda_2 s} u(t+s)\mathrm{d}s$ and $\mu(t) = u(t-\delta_2)$, respectively. By changing the variables, a constant c_1 can be obtained that depends on (λ_2, δ_2) and satisfies

$$\mathrm{E}\left[\sup_{0\leqslant t\leqslant T} |\nu(t)|^2 + \int_0^T |\mu(t)|^2 \mathrm{d}t\right] \leqslant c_1 \mathrm{E}\int_{-\delta_2}^T |u(t)|^2 \mathrm{d}t.$$

Let $\pi = (x, y, z, u, \nu, \mu) \in \mathbb{R}^{3n+3m}$. Let $b(\cdot, \pi)$ and $\sigma(\cdot, \pi)$ be progressively measurable $\mathbb{R}^n$-valued functions for any π.

Assumption 1. $b(t, \pi)$ and $\sigma(t, \pi)$ can be differentiated with respect to π, having continuous and bounded partial derivatives. $b(\cdot, 0)$ and $\sigma(\cdot, 0)$ are square integrable.

The controlled system evolves as

$$\begin{cases} \mathrm{d}X^u(t) = b(t, \Pi^u(t))\mathrm{d}t + \sigma(t, \Pi^u(t))\mathrm{d}W(t), & 0 \leqslant t \leqslant T, \\ X^u(t) = x_0(t), & -\delta_1 \leqslant t \leqslant 0, \end{cases}$$

where $\Pi^u(t) = (X^u(t), Y^u(t), Z^u(t), u(t), \nu(t), \mu(t))$, $Y^u(t) = \int_{-\delta_1}^0 \mathrm{e}^{\lambda_1 s} X^u(t+s)\mathrm{d}s$, $Z^u(t) = X^u(t-\delta_1)$. $Y^u(\cdot)$ and $Z^u(\cdot)$ are considered to be the moving-average delayed state and point-wise delayed state of $X^u(\cdot)$, respectively. Under Assumption 1, the system equation results in a unique solution $X^u(\cdot) \in S^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$ for any $u(\cdot) \in \mathcal{U}$. There also exists a constant c_2, which depends on (λ_1, δ_1) and satisfies

$$\mathrm{E}\left[\sup_{0\leqslant t\leqslant T} |Y^u(t)|^2 + \int_0^T |Z^u(t)|^2 \mathrm{d}t\right] \leqslant c_2 \mathrm{E}\int_{-\delta_1}^T |X^u(t)|^2 \mathrm{d}t.$$

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Let us introduce the cost function J as

$$J(u(\cdot)) = \mathrm{E}\left[g(X^u(T), Y^u(T)) + \int_0^T f(t, \Pi^u(t))\mathrm{d}t\right],$$

where $g(x,y)$ is $\mathcal{F}_T$-measurable for any (x,y) and $f(\cdot,\pi)$ is progressively measurable for any π.

Assumption 2. In case of $u(\cdot) \in \mathcal{U}$,

$$\mathrm{E}\left[|g(X^u(T), Y^u(T))| + \int_0^T |f(t, \Pi^u(t))|\mathrm{d}t\right] < \infty.$$

The stochastic optimal control problem under consideration is formulated as follows: to find $u^*(\cdot) \in \mathcal{U}$ such that

$$J(u^*(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}} J(u(\cdot)). \tag{1}$$

In this case, the admissible control $u^*(\cdot)$ that satisfies (1) is referred to as an optimal control. The corresponding average and point-wise delayed controls are denoted by $\nu^*(\cdot)$ and $\mu^*(\cdot)$, respectively. Set $l^*(t) = l^{u^*}(t)$ where l denotes X, Y, Z, $h^*(t) = h(t, \Pi^{u^*}(t))$ where h denotes b, σ, f and their partial derivatives, and $\iota^*(T) = \iota(X^*(T), Y^*(T))$ where ι denotes g along with its partial derivatives.

Adjoint equations. Let us introduce the following adjoint equation:

$$\begin{cases} -\mathrm{d}P(t) = \big\{A_1(t)P(t) + B_1(t)Q(t) + \tilde{b}(t) \\ \qquad + \mathrm{E}^t\big[A_2(t+\delta_1)P(t+\delta_1) \\ \qquad + B_2(t+\delta_1)Q(t+\delta_1)\big]\chi_{[0,T-\delta_1]}(t)\big\}\mathrm{d}t \\ \qquad - Q(t)\mathrm{d}W(t), \quad 0 \leqslant t \leqslant T, \\ P(T) = \zeta, \end{cases} \tag{2}$$

where $A_1(t) = \begin{pmatrix} b_x^*(t)^{\mathrm{T}} & I \\ b_y^*(t)^{\mathrm{T}} & -\lambda_1 I \end{pmatrix}$, $B_1(t) = \begin{pmatrix} \sigma_x^*(t)^{\mathrm{T}} & 0 \\ \sigma_y^*(t)^{\mathrm{T}} & 0 \end{pmatrix}$, $A_2(t) = \begin{pmatrix} b_z^*(t)^{\mathrm{T}} & -\mathrm{e}^{-\lambda_1\delta_1} I \\ 0 & 0 \end{pmatrix}$, $B_2(t) = \begin{pmatrix} \sigma_z^*(t)^{\mathrm{T}} & 0 \\ 0 & 0 \end{pmatrix}$, $\zeta = \begin{pmatrix} g_x^*(T) \\ g_y^*(T) \end{pmatrix}$, and $\tilde{b}(t) = \begin{pmatrix} f_x^*(t) + \mathrm{E}^t[f_z^*(t+\delta_1)]\chi_{[0,T-\delta_1]}(t) \\ f_y^*(t) \end{pmatrix}$. I denotes the $n\times n$ identity matrix, whereas χ denotes the indicator function. Eq. (2) is a kind of anticipated BSDE that was first introduced in [8].

Assumption 3. When $\xi = x, y, z$,

$$\mathrm{E}\left[|g_\xi^*(T)|^2 + \int_0^T |f_\xi^*(t)|^2\mathrm{d}t\right] < \infty.$$

Under Assumptions 1 and 3, Theorem 4.2 in [8] shows that Eq. (2) has a unique solution $(P(\cdot), Q(\cdot)) \in S^2_{\mathbb{F}}(0,T;\mathbb{R}^{2n}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^{2n})$. Let $P(\cdot) = (P_1(\cdot)^{\mathrm{T}}, P_2(\cdot)^{\mathrm{T}})^{\mathrm{T}}$ and $Q(\cdot) = (Q_1(\cdot)^{\mathrm{T}}, Q_2(\cdot)^{\mathrm{T}})^{\mathrm{T}}$. Next, define a Hamiltonian function H as

$$\begin{aligned} H(t,\pi,p,q) &= \langle b(t,\pi), p_1\rangle + \langle \sigma(t,\pi), q_1\rangle \\ &\quad + \langle x - \lambda_1 y - \mathrm{e}^{-\lambda_1\delta_1} z, p_2\rangle + f(t,\pi), \end{aligned}$$

where $p = (p_1, p_2)$, $q = (q_1, q_2)$. By setting $H^*(t) = H(t, \Pi^*(t), P(t), Q(t))$, we obtain

$$\begin{cases} -\mathrm{d}P_1(t) = \big\{\mathrm{E}^t[H_z^*(t+\delta_1)]\chi_{[0,T-\delta_1]}(t) \\ \qquad + H_x^*(t)\big\}\mathrm{d}t - Q_1(t)\mathrm{d}W(t), \\ P_1(T) = g_x^*(T), \end{cases} \tag{3}$$

and

$$\begin{cases} -\mathrm{d}P_2(t) = H_y^*(t)\mathrm{d}t - Q_2(t)\mathrm{d}W(t), \\ P_2(T) = g_y^*(T). \end{cases} \tag{4}$$

The system of adjoint equations used in [1–3] is a special case of (3) and (4) where it is assumed that $H_z^*(t) \equiv 0$.

Next, a new Hamiltonian $\mathcal{H}$ is defined as follows:

$$\mathcal{H}(t,\pi,p_1,q_1) = \langle b(t,\pi), p_1\rangle + \langle \sigma(t,\pi), q_1\rangle + f(t,\pi).$$

Proposition 1. Eqs. (3) and (4) are equivalent to

$$\begin{cases} -\mathrm{d}P_1(t) = \{\mathrm{E}^t\big[\mathcal{H}_z^*(t+\delta_1)\big]\chi_{[0,T-\delta_1]}(t) \\ \quad + E^t\big[\mathrm{e}^{\lambda_1(t-T)}g_y^*(T)\big]\chi_{(T-\delta_1,T]}(t) \\ \quad + \mathrm{E}^t\big[\int_t^{t+\delta_1} \mathrm{e}^{\lambda_1(t-s)}\mathcal{H}_y^*(s)\chi_{[0,T]}(s)\mathrm{d}s\big] \\ \quad + \mathcal{H}_x^*(t)\}\mathrm{d}t - Q_1(t)\mathrm{d}W(t), \quad 0 \leqslant t \leqslant T, \\ P_1(T) = g_x^*(T), \end{cases} \tag{5}$$

and

$$\begin{cases} -\mathrm{d}P_2(t) = [-\lambda_1 P_2(t) + \mathcal{H}_y^*(t)]\mathrm{d}t \\ \qquad\qquad - Q_2(t)\mathrm{d}W(t), \quad 0 \leqslant t \leqslant T, \\ P_2(T) = g_y^*(T), \end{cases} \tag{6}$$

where $\mathcal{H}^*(t) = \mathcal{H}(t, \Pi^{u^*}(t), P_1(t), Q_1(t))$. Eq. (5) is just one type of time-advanced BSDE, coupled to (6). The result shows that the system presented in (5) and (6) is equivalent to that presented in (3) and (4) and that both of these systems originate from (2). In [1,7], the time-advanced adjoint equation does not comprise any delayed controls. In [6], the time-advanced adjoint equation does not comprise the average delayed control and the terminal cost function g is independent of y.

The maximum principle.

Assumption 4. The partial derivatives of f with respect to (u, ν, μ) exist.

Assumption 5. $H(t, \pi, P(t), Q(t))$ is convex in π and $g(x,y)$ is convex in (x,y).

Let us first establish the sufficient maximum principle.

Theorem 1. Let $u^*(\cdot) \in \mathcal{U}$ be an admissible control satisfying Assumptions 1–5. Then $u^*(\cdot)$ is an optimal control if the following maximum principle condition is satisfied:

$$\begin{aligned} &\mathrm{E}\int_0^T \big[\langle u(t) - u^*(t), H_u^*(t)\rangle + \langle \nu(t) - \nu^*(t), H_\nu^*(t)\rangle \\ &\quad + \langle \mu(t) - \mu^*(t), H_\mu^*(t)\rangle\big]\mathrm{d}t \geqslant 0, \quad \forall u(\cdot) \in \mathcal{U}. \end{aligned} \tag{7}$$

The continuity and square integrability of $f_u^*(\cdot)$, $f_\nu^*(\cdot)$, $f_\mu^*(\cdot)$ are not required to achieve the sufficient maximum principle.

Assumption 6. The partial derivatives of g with respect to (x,y) as well as those of f with respect to π are continuous with linear growth. Besides,

$$\mathrm{E}\left[|g(0)| + \int_0^T |f(t,0)|\mathrm{d}t\right] < \infty.$$

The next result is the necessary maximum principle.

Theorem 2. Let Assumptions 1 and 6 hold. If $u^*(\cdot) \in \mathcal{U}$ is an optimal control, then it satisfies (7).

When H is independent of ν, it can be easily verified that the condition (7) is equivalent to the following point-wise inequality:

$$\big\langle H_u^*(t) + \mathrm{E}^t[H_\mu^*(t+\delta_2)]\chi_{[0,T-\delta_2]}(t),\\ v - u^*(t)\big\rangle \geqslant 0, \quad \text{a.s., a.e.,} \quad \forall v \in U.$$

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Supporting information Appendixes A–D. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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