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Stochastic maximum principle for optimal control problems of delayed systems

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Appendix A Proof of Proposition 1

Proposition 1. Eq. (2) and (3) are equivalent to the following

$$\begin{cases} -dP_1(t) = \Big\{E^t\big[e^{\lambda_1(t-T)}g_y^*(T)\big]\chi_{(T-\delta_1,T]}(t) + \mathcal{H}_x^*(t) + \mathbb{E}^t\Big[\int_t^{t+\delta_1} e^{\lambda_1(t-s)}\mathcal{H}_y^*(s)\chi_{[0,T]}(s)ds\Big] \\ \qquad\qquad +\mathbb{E}^t\big[\mathcal{H}_z^*(t+\delta_1)\big]\chi_{[0,T-\delta_1]}(t)\Big\}dt - Q_1(t)dW(t), \quad 0 \leqslant t \leqslant T, \\ P_1(T) = g_x^*(T), \end{cases} \tag{A1}$$

and

$$\begin{cases} -dP_2(t) = \big[-\lambda_1 P_2(t) + \mathcal{H}_y^*(t)\big]dt - Q_2(t)dW(t), \quad 0 \leqslant t \leqslant T, \\ P_2(T) = g_y^*(T). \end{cases} \tag{A2}$$

Proof. We only need to prove (A1). Applying Itô's formula to $e^{-\lambda_1 t}P_2(t)$ yields

$$P_2(t) = \mathbb{E}^t\left[e^{\lambda_1(t-T)}g_y^*(T) + \int_t^T e^{\lambda_1(t-s)}\mathcal{H}_y^*(s)ds\right].$$

Set $\Sigma(t) = P_2(t) - e^{-\lambda_1\delta_1}\mathbb{E}^t[P_2(t+\delta_1)]\chi_{[0,T-\delta_1]}(t)$. Then we have $\Sigma(t) = \Sigma_1(t) + \Sigma_2(t)$, where

$$\begin{cases} \Sigma_1(t) = E^t\big[e^{\lambda_1(t-T)}g_y^*(T)\big]\chi_{(T-\delta_1,T]}(t), \\ \Sigma_2(t) = \mathbb{E}^t\left[\int_t^{t+\delta_1} e^{\lambda_1(t-s)}\mathcal{H}_y^*(s)ds\right]\chi_{[0,T-\delta_1]}(t) + \mathbb{E}^t\left[\int_t^T e^{\lambda_1(t-s)}\mathcal{H}_y^*(s)ds\right]\chi_{(T-\delta_1,T]}(t). \end{cases}$$

It's easy to check that

$$\Sigma_2(t) = \mathbb{E}^t\left[\int_t^{t+\delta_1} e^{\lambda_1(t-s)}\mathcal{H}_y^*(s)\chi_{[0,T]}(s)ds\right].$$

This leads to (A1), in view of (2).

Appendix B Proof of Theorems 1 and 2

Theorem 1. Let Assumptions 1-5 hold for an admissible control $u^*(\cdot) \in \mathcal{U}$. Then $u^*(\cdot)$ is an optimal control of Problem (SOC) if it satisfies

$$\mathbb{E}\int_0^T \big[\langle u(t) - u^*(t), H_u^*(t)\rangle + \langle \nu(t) - \nu^*(t), H_\nu^*(t)\rangle + \langle \mu(t) - \mu^*(t), H_\mu^*(t)\rangle\big]dt \geqslant 0, \quad \forall u(\cdot) \in \mathcal{U}. \tag{B1}$$

Theorem 2. Let Assumptions 1 and 6 hold. If $u^*(\cdot) \in \mathcal{U}$ is an optimal control of Problem (SOC), then it satisfies (B1).

Proof. Let $u(\cdot) \in \mathcal{U}$ be any admissible control. It's easy to check that

$$dY^u(t) = \big[X^u(t) - \lambda_1 Y^u(t) - e^{-\lambda_1\delta_1}Z^u(t)\big]dt, \quad 0 \leqslant t \leqslant T.$$

Then applying Itô's formula to $\langle \hat{X}(t), P_1(t)\rangle + \langle \hat{Y}(t), P_2(t)\rangle$ and using a change of variables leads to

$$J(u(\cdot)) - J(u^*(\cdot)) = \mathbb{E}\Big\{\big[\hat{g}(T) - \langle \hat{X}(T), g_x^*(T)\rangle - \langle \hat{Y}(T), g_y^*(T)\rangle\big] + \int_0^T \big[\hat{H}(t) - \langle \hat{X}(t), H_x^*(t)\rangle - \langle \hat{Y}(t), H_y^*(t)\rangle - \langle \hat{Z}(t), H_z^*(t)\rangle\big]dt\Big\}.$$

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Here we use the following notations: $\hat{X}(t) = X^u(t) - X^*(t)$, $\hat{Y}(t) = Y^u(t) - Y^*(t)$, $\hat{Z}(t) = Z^u(t) - Z^*(t)$, $\hat{g}(T) = g(X^u(T), Y^u(T)) - g^*(T)$, and $\hat{H}(t) = H(t, \Pi^u(t), P(t), Q(t)) - H^*(t)$. Then the two conclusions could be obtained in the standard way together with the change of variables.

Appendix C Open-loop solvability of LQ problem

In this section, we consider the well-known LQ case when the system is described by a linear delayed SDE and the cost functional takes a quadratic form.

To get point-wise results, we only study the case when the average delays of the controls vanish. Take $U = \mathbb{R}^m$. Let the coefficients of the controlled system and the cost functional take the following forms:

$$\begin{cases} b(t,x,y,z,u,\mu) = A_1(t)x + A_2(t)y + A_3(t)z + C_1(t)u + C_2(t)\mu, \\ \sigma(t,x,y,z,u,\mu) = a_1(t)x + a_2(t)y + a_3(t)z + c_1(t)u + c_2(t)\mu, \\ g(x,y) = \frac{1}{2}\big[\langle G_1 x, x\rangle + \langle G_2 y, y\rangle\big], \\ f(t,x,y,z,u,\mu) = \frac{1}{2}\big[\langle H_1(t)x,x\rangle + \langle H_2(t)y,y\rangle + \langle H_3(t)z,z\rangle + \langle R_1(t)u,u\rangle + \langle R_2(t)\mu,\mu\rangle\big], \end{cases}$$

where $A_i(\cdot), a_i(\cdot), H_i(\cdot), C_j(\cdot), c_j(\cdot), R_j(\cdot)$, $i = 1,2,3$, $j = 1,2$ are bounded progressively measurable processes, and G_1, G_2 are bounded $\mathcal{F}_T$-measurable random variables. The coefficients are of appropriate dimensions. In this case, the assumptions 1 and 6 hold true.

Let $X^*(\cdot)$ be the trajectory corresponding to an admissible control $u^*(\cdot) \in \mathcal{U}$. The two adjoint equations are as follows:

$$\begin{cases} -dP_1(t) = \Big\{A_1^\top(t)P_1(t) + a_1^\top(t)Q_1(t) + P_2(t) + H_1(t)X^*(t) + \mathbb{E}^t\big[A_3^\top(t+\delta_1)P_1(t+\delta_1) + a_3^\top(t+\delta_1)Q_1(t+\delta_1) \\ \qquad\qquad -e^{-\lambda_1\delta_1}P_2(t+\delta_1) + H_3(t+\delta_1)Z^*(t+\delta_1)\big]\chi_{[0,T-\delta_1]}(t)\Big\}dt - Q_1(t)dW(t), \quad 0 \leqslant t \leqslant T, \\ P_1(T) = G_1 X^*(T), \end{cases} \tag{C1}$$

$$\begin{cases} -dP_2(t) = \Big\{A_2^\top(t)P_1(t) + a_2^\top(t)Q_1(t) - \lambda_1 P_2(t) + H_2(t)Y^*(t)\Big\}dt - Q_2(t)dW(t), \quad 0 \leqslant t \leqslant T, \\ P_2(T) = G_2 Y^*(T). \end{cases} \tag{C2}$$

The maximum principle condition is as follows:

$$\begin{aligned} &\big\{R_1(t) + \mathbb{E}^t[R_2(t+\delta_2)]\chi_{[0,T-\delta_2]}(t)\big\}u^*(t) + C_1^\top(t)P_1(t) + c_1^\top(t)Q_1(t) \\ &+ \mathbb{E}^t\big[C_2^\top(t+\delta_2)P_1(t+\delta_2) + c_2^\top(t+\delta_2)Q_1(t+\delta_2)\big]\chi_{[0,T-\delta_2]}(t) = 0, \quad a.s., a.e. \end{aligned} \tag{C3}$$

Let $\mathcal{U}_1$ be the subset of $\mathcal{U}$ whose element vanishes for $t \in [-\delta_2, 0)$. For $u_1(\cdot) \in \mathcal{U}_1$, let $X^\varepsilon(\cdot)$ be the trajectory corresponding to $u^*(\cdot) + \varepsilon u_1(\cdot) \in \mathcal{U}$. Set $X_1(\cdot) = [X^\varepsilon(\cdot) - X^*(\cdot)]/\varepsilon$. Then $X_1(\cdot)$ satisfies

$$\begin{cases} dX_1(t) = \big[A_1(t)X_1(t) + A_2(t)Y_1(t) + A_3(t)Z_1(t) + C_1(t)u_1(t) + C_2(t)\mu_1(t)\big]dt \\ \qquad\qquad +\big[a_1(t)X_1(t) + a_2(t)Y_1(t) + a_3(t)Z_1(t) + c_1(t)u_1(t) + c_2(t)\mu_1(t)\big]dW(t), \quad 0 \leqslant t \leqslant T, \\ X_1(t) = 0, \ -\delta_1 \leqslant t \leqslant 0. \end{cases} \tag{C4}$$

Besides, it's easy to check that

$$J(u^*(\cdot) + \varepsilon u_1(\cdot)) - J(u^*(\cdot)) = \varepsilon\Lambda_1 + \frac{\varepsilon^2}{2}\Lambda_2, \tag{C5}$$

where

$$\begin{aligned} \Lambda_1 =& \mathbb{E}\bigg\{\big\langle G_1X^*(T), X_1(T)\big\rangle + \big\langle G_2Y^*(T), Y_1(T)\big\rangle + \int_0^T \big[\big\langle H_1(t)X^*(t), X_1(t)\big\rangle \\ &+ \big\langle H_2(t)Y^*(t), Y_1(t)\big\rangle + \big\langle H_3(t)Z^*(t), Z_1(t)\big\rangle + \big\langle R_1(t)u^*(t), u_1(t)\big\rangle + \big\langle R_2(t)\mu^*(t), \mu_1(t)\big\rangle\big]dt\bigg\}, \end{aligned}$$

and

$$\begin{aligned} \Lambda_2 =& \mathbb{E}\bigg\{\big\langle G_1X_1(T), X_1(T)\big\rangle + \big\langle G_2Y_1(T), Y_1(T)\big\rangle + \int_0^T \big[\big\langle H_1(t)X_1(t), X_1(t)\big\rangle \\ &+ \big\langle H_2(t)Y_1(t), Y_1(t)\big\rangle + \big\langle H_3(t)Z_1(t), Z_1(t)\big\rangle + \big\langle R_1(t)u_1(t), u_1(t)\big\rangle + \big\langle R_2(t)\mu_1(t), \mu_1(t)\big\rangle\big]dt\bigg\}. \end{aligned} \tag{C6}$$

Note that $u^*(\cdot) \in \mathcal{U}$ is an open-loop optimal control if and only if it holds that

$$J(u^*(\cdot) + \varepsilon u_1(\cdot)) \geqslant J(u^*(\cdot)), \quad \forall \varepsilon \in \mathbb{R}, \ \forall u_1(\cdot) \in \mathcal{U}_1,$$

which, in view of (C5), is equivalent to

$$\Lambda_1 = 0 \text{ and } \Lambda_2 \geqslant 0, \quad \forall u_1(\cdot) \in \mathcal{U}_1.$$

Note that $dY_1(t) = \big[X_1(t) - \lambda_1 Y_1(t) - e^{-\lambda_1\delta_1}Z_1(t)\big]dt$. Applying Itô's formula to $\langle X_1(t), P_1(t)\rangle + \langle Y_1(t), P_2(t)\rangle$ and using a change of variables leads to

$$\begin{aligned} \Lambda_1 =& \mathbb{E}\bigg\{\int_0^T \Big\langle u_1(t), C_1^\top(t)P_1(t) + c_1^\top(t)Q_1(t) + R_1(t)u^*(t) \\ &+ \mathbb{E}^t\big[C_2^\top(t+\delta_2)P_1(t+\delta_2) + c_2^\top(t+\delta_2)Q_1(t+\delta_2) + R_2(t+\delta_2)u^*(t)\big]\chi_{[0,T-\delta_2]}(t)\Big\rangle dt\bigg\}. \end{aligned}$$

Thus, $\Lambda_1 = 0$ holds for all $u_1(\cdot) \in \mathcal{U}_1$ if and only if the maximum principle condition (C3) holds true.

To sum up, we get the following open-loop solvability result for this LQ problem.

Proposition 2. An admissible control $u^*(\cdot) \in \mathcal{U}$ is an open-loop optimal control of the LQ problem if and only if (i) the maximum principle condition (C3) holds true, and (ii) $\Lambda_2 \geqslant 0$ holds for all $u_1(\cdot) \in \mathcal{U}_1$, where Λ_2 is defined by (C6) and $X_1(\cdot)$ is the unique solution of (C4).

By a change of variables we get

$$\begin{aligned}\Lambda_2 =&\mathbb{E}\Big\{\big\langle G_1 X_1(T), X_1(T)\big\rangle + \big\langle G_2 Y_1(T), Y_1(T)\big\rangle + \int_0^T \Big\{\big\langle\big(H_1(t) + \mathbb{E}^t[H_3(t+\delta_1)]\chi_{[0,T-\delta_1]}(t)\big)X_1(t), X_1(t)\big\rangle \\ &+ \big\langle H_2(t)Y_1(t), Y_1(t)\big\rangle + \big\langle\big[R_1(t) + \mathbb{E}^t[R_2(t+\delta_2)]\chi_{[0,T-\delta_2]}(t)\big]u_1(t), u_1(t)\big\rangle\Big\}dt\Big\}.\end{aligned}$$

Thus, by Proposition 2 we have the following corollary.

Corollary 1. Assume that G_1, G_2, $H_1(t) + \mathbb{E}^t[H_3(t+\delta_1)]\chi_{[0,T-\delta_1]}(t)$, $H_2(t)$ and $R_1(t) + \mathbb{E}^t[R_2(t+\delta_2)]\chi_{[0,T-\delta_2]}(t)$ are all positive semi-definite, a.s, a.e. Then an admissible control $u^*(\cdot) \in \mathcal{U}$ is an open-loop optimal control if and only if it satisfies (C3).

Furthermore, we have the unique solvability result.

Corollary 2. In addition to the assumptions in Corollary 1, it's also assumed that $\tilde{R}(t) := R_1(t)+\mathbb{E}^t[R_2(t+\delta_2)]\chi_{[0,T-\delta_2]}(t)$ is uniformly positive definite, a.s., a.e. Then this LQ problem has a unique open-loop optimal control $u^*(\cdot)$ given by

$$u^*(t) = -\tilde{R}^{-1}(t)\Big\{C_1^\top(t)P_1(t) + c_1^\top(t)Q_1(t) + \mathbb{E}^t\big[C_2^\top(t+\delta_2)P_1(t+\delta_2) + c_2^\top(t+\delta_2)Q_1(t+\delta_2)\big]\chi_{[0,T-\delta_2]}\Big\}, \quad 0 \leqslant t \leqslant T.$$

Remark 1. One LQ problem with delays and mean-field terms is studied in [1] and the open-loop optimal control is obtained. Compared with this literature, the terminal cost functional g in our model also contains y and the positive conditions of the coefficients are weaker. One LQ problem with point-wise delays is considered in [2]; in the indefinite framework, the open-loop optimal control is obtained with a relaxed compensator method. Compared with this literature, the present model involves average delays of the state.

Appendix D Application to a kind of optimal consumption problem

Let us consider a kind of optimal consumption problem, which is modified from [3].

Suppose that the wealth evolves by

$$\begin{cases} dX(t) = \big[A_1(t)X(t) + A_2(t)Y(t) + A_3(t)Z(t) - C_1(t)u(t) - C_2(t)\mu(t)\big]dt \\ \qquad\qquad +\big[a_1(t)X(t) + a_2(t)Y(t) + a_3(t)Z(t) + \sigma(t, u(t), \mu(t))\big]dW(t), \quad 0 \leqslant t \leqslant T, \\ X(t) = x_0(t), \quad -\delta_1 \leqslant t \leqslant 0, \end{cases} \tag{D1}$$

where $A_i(t), a_i(t), C_j(t),\ i = 1, 2, 3,\ j = 1, 2$ are bounded stochastic processes, $\sigma(t, u, \mu)$ is a random function satisfying Assumption 1, and $Y(t), Z(t), \mu(t)$ are defined as before. The control process $u(\cdot)$ is interpreted as the consumption rate. The objective is to find an appropriate $u^*(\cdot)$ which minimizes

$$J(u(\cdot)) = \mathbb{E}\Big\{ - [X(T) + \alpha Y(T)] - \int_0^T e^{-\rho t}\frac{u^\gamma(t)}{\gamma}dt\Big\},$$

where $\alpha \in \mathbb{R}$, $\rho > 0$ and $\gamma \in (0, 1)$ are three constants. Note that the objective consists of two parts. One part is to maximize the combination of the wealth at the terminal time and an average wealth in a terminal segment; the other is a maximization of a consumption utility. Note that [3] considers the case when the coefficients are deterministic and the delayed control $\mu(\cdot)$ vanishes. Take the control domain U by $U = [a, +\infty)$, where a is fixed positive constant. Note that we are considering a problem with control constraint and the control domain is not an open set. This is one feature of this example. All the assumptions of Theorems 1 and 2 hold true.

Let $(P(\cdot), Q(\cdot)) = \left(\begin{pmatrix} P_1(\cdot) \\ P_2(\cdot)\end{pmatrix}, \begin{pmatrix} Q_1(\cdot) \\ Q_2(\cdot)\end{pmatrix}\right)$ be the unique square-integrable solution of the following adjoint equation:

$$\begin{cases} -dP(t) = \Big\{\mathbb{E}^t\big[B(t+\delta_1)P(t+\delta_1) + \tilde{B}(t+\delta_1)Q(t+\delta_1)\big]\chi_{[0,T-\delta_1]}(t) \\ \qquad\qquad +A(t)P(t) + \tilde{A}(t)Q(t)\Big\}dt - Q(t)dW(t), \quad 0 \leqslant t \leqslant T, \\ P(T) = \zeta, \end{cases} \tag{D2}$$

where $A(t) = \begin{pmatrix} A_1(t) & 1 \\ A_2(t) & -\lambda_1 \end{pmatrix}$, $\tilde{A}(t) = \begin{pmatrix} a_1(t) & 0 \\ a_2(t) & 0 \end{pmatrix}$, $B(t) = \begin{pmatrix} A_3(t) & -e^{-\lambda_1\delta_1} \\ 0 & 0 \end{pmatrix}$, $\tilde{B}(t) = \begin{pmatrix} a_3(t) & 0 \\ 0 & 0 \end{pmatrix}$, and $\zeta = \begin{pmatrix} -1 \\ -\alpha \end{pmatrix}$. By Theorem 1, an admissible control $u^*(\cdot) \in \mathcal{U}$ is an optimal control if it satisfies

$$\begin{aligned}&(u - u^*(t))\Big\{ - C_1(t)P_1(t) + \sigma_u^*(t)Q_1(t) - e^{-\rho t}(u^*(t))^{\gamma-1} \\ &+ \mathbb{E}^t[-C_2(t+\delta_2)P_1(t+\delta_2) + \sigma_\mu^*(t+\delta_2)Q_1(t+\delta_2)]\chi_{[0,T-\delta_2]}(t)\Big\} \geqslant 0, \quad a.e., a.s., \quad \forall u \geqslant a,\end{aligned} \tag{D3}$$

We solve (D3) to get

$$u^*(t) = \begin{cases} \omega(t), & \text{if } \omega(t) \geqslant a, \\ a, & \text{if } \omega(t) < a, \end{cases} \tag{D4}$$

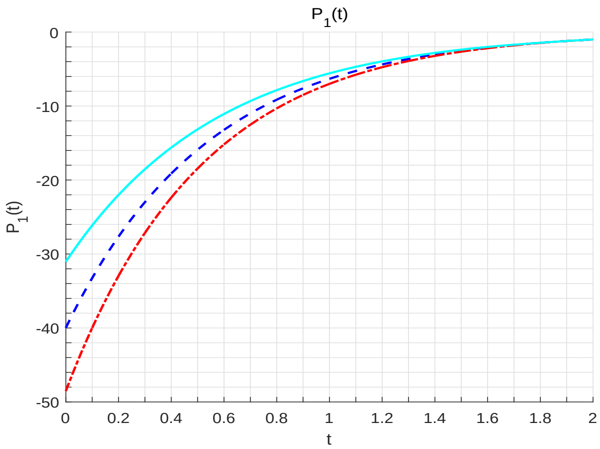

Figure D1 The function $P_1(t)$.

where

$$\omega(t)=\big(-e^{\rho t}\omega_1(t)\big)^{\frac{1}{\gamma-1}} \tag{D5}$$

with

$$\omega_1(t)=C_1(t)P_1(t)+\mathbb{E}^t[C_2(t+\delta_2)]P_1(t+\delta_2)\chi_{[0,T-\delta_2]}(t)-\sigma_u^*(t)Q_1(t)-\mathbb{E}^t[\sigma_\mu^*(t+\delta_2)Q_1(t+\delta_2)]\chi_{[0,T-\delta_2]}(t). \tag{D6}$$

Proposition 3. The optimal consumption strategy $u^*(\cdot)$ of this optimal consumption problem is given by (D4), where $\omega(\cdot)$ is defined by (D5) with $\omega_1(\cdot)$ being given by (D6) and $(P(\cdot),Q(\cdot))=\left(\begin{pmatrix}P_1(\cdot)\\P_2(\cdot)\end{pmatrix},\begin{pmatrix}Q_1(\cdot)\\Q_2(\cdot)\end{pmatrix}\right)$ being the unique square-integrable solution of the adjoint equation (D2).

In the remaining part of this section, in order to get an explicit form of $u^*(\cdot)$, we consider the special case when the coefficients $A_i(t),a_i(t)$, $i=1,2,3$ are deterministic, and $A_1(t),A_2(t)$ are time-invariant. In this case, it holds that $Q(t)\equiv 0$, while $P(t)$ is a deterministic function satisfying the following two-dimensional ODE:

$$\begin{cases}-\dot P(t)=AP(t)+B(t+\delta_1)P(t+\delta_1)\chi_{[0,T-\delta_1]}(t), \quad 0\leqslant t\leqslant T,\\ P(T)=\zeta,\end{cases} \tag{D7}$$

where $A=\begin{pmatrix}A_1 & 1\\ A_2 & -\lambda_1\end{pmatrix}$, $B(t)=\begin{pmatrix}A_3(t) & -e^{-\lambda_1\delta_1}\\ 0 & 0\end{pmatrix}$, and $\zeta=\begin{pmatrix}-1\\-\alpha\end{pmatrix}$. That is, $P_1(t)$ satisfies

$$\begin{cases}-\dot P_1(t)=A_1P_1(t)+A_3(t+\delta_1)P_1(t+\delta_1)\chi_{[0,T-\delta_1]}(t)+P_2(t)-e^{-\lambda_1\delta_1}P_2(t+\delta_1)\chi_{[0,T-\delta_1]}(t), \quad 0\leqslant t\leqslant T,\\ P_1(T)=-1,\end{cases} \tag{D8}$$

and $P_2(t)$ satisfies

$$\begin{cases}-\dot P_2(t)=-\lambda_1P_2(t)+A_2P_1(t), \quad 0\leqslant t\leqslant T,\\ P_2(T)=-\alpha.\end{cases} \tag{D9}$$

Besides, we have $\omega_1(t)=C_1(t)P_1(t)+\mathbb{E}^t[C_2(t+\delta_2)]P_1(t+\delta_2)\chi_{[0,T-\delta_2]}(t)$. Note that the functions $C_1(t)$, $C_2(t)$ and $\sigma(t,u,\mu)$ can still be random. In order to get the optimal control $u^*(\cdot)$, we only need to solve $P_1(\cdot)$. One can check that it's difficult to solve $P_1(\cdot)$ by focusing on the two coupled ODEs (D8) and (D9). The method we will use to get $P_1(\cdot)$ is to solve the two-dimensional ODE (D7) to obtain its solution $P(\cdot)$ and then derive $P_1(t)=(1\ \ 0)P(t)$. This is another feature of this example.

Since (D7) is a piece-wise backward ODE, we solve it backwardly by dividing the time intervals. Let us set $j=[T/\delta_1]$ and define $P(t)=p_i(t)$ when $t\in[T-i\delta_1,T-(i-1)\delta_1]$ for $i=1,2,\cdots,j$, and $P(t)=p_{j+1}(t)$ for $t\in[0,T-j\delta_1]$ if T/δ_1 is not an integer. On $[T-\delta_1,T]$, we have

$$\begin{cases}-\dot p_1(t)=Ap_1(t), \quad t\in[T-\delta_1,T],\\ p_1(T)=\zeta.\end{cases}$$

Solving it yields

$$p_1(t)=e^{A(T-t)}\zeta, \quad t\in[T-\delta_1,T].$$

For $i=1,\cdots,j-1$, we have

$$\begin{cases}-\dot p_{i+1}(t)=Ap_{i+1}(t)+B(t+\delta_1)p_i(t+\delta_1), \quad t\in[T-(i+1)\delta_1,T-i\delta_1],\\ p_{i+1}(T-i\delta_1)=p_i(T-i\delta_1),\end{cases}$$

which is solved to get

$$p_{i+1}(t)=e^{A(T-i\delta_1-t)}p_i(T-i\delta_1)+\int_t^{T-i\delta_1}e^{A(s-t)}B(s+\delta_1)p_i(s+\delta_1)ds, \quad t\in[T-(i+1)\delta_1,T-i\delta_1].$$

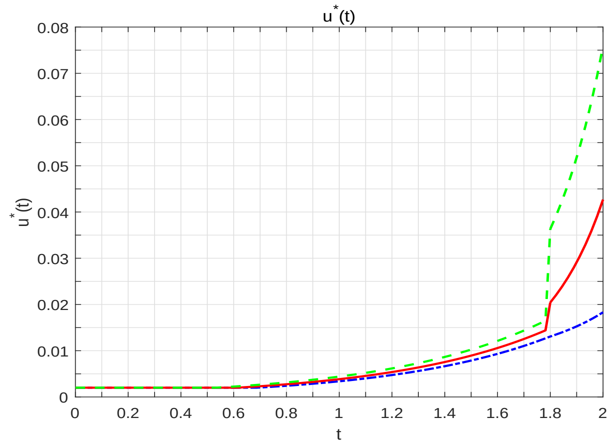

Figure D2 The optimal consumption strategy $u^*(t)$.

If T/δ_1 is not an integer, we have

$$\begin{cases} -\dot{p}_{j+1}(t) = Ap_{j+1}(t) + B(t+\delta_1)p_j(t+\delta_1), & t \in [0, T - j\delta_1], \\ p_{j+1}(T - j\delta_1) = p_j(T - j\delta_1), \end{cases}$$

that is,

$$p_{j+1}(t) = e^{A(T-j\delta_1-t)}p_j(T-j\delta_1) + \int_t^{T-j\delta_1} e^{A(s-t)}B(s+\delta_1)p_j(s+\delta_1)ds, \quad t \in [0, T-j\delta_1].$$

In comparison with [3, Example 1], one can see our improvement even in the case when $C_2(t) \equiv 0$.

Finally let us assign some concrete numbers to the coefficients and give numerical results. Let $A_1 = A_2 = A_3 = 1$, $\lambda_1 = 1$, $T = 2$, $\alpha = 1$, $\rho = 1$, $\gamma = 0.5$ and $a = 0.002$. Figure D1 shows the graphs of $P_1(t)$, $t \in [0, 2]$ in the cases when 0.4 (solid), 0.2 (dashed) and $\delta_1 = 0.1$ (dash-dot), respectively. With $\delta_1 = \delta_2 = 0.2$, Figure D2 shows the graphs of $u^*(t)$, $t \in [0, 2]$ in the cases when $(C_1(t), C_2(t)) \equiv (0.6, 0.4)$ (dashed), $(0.8, 0.2)$ (solid) and $(1, 0)$ (dash-dot), respectively.

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