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A local observability analysis method for a time-varying nonlinear system and its application in the continuous self-calibration system

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Appendix A The local observability matrix of the constant nonlinear system

A type of discrete-time constant nonlinear system is represented by following state space equations:

$$\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{f}(\boldsymbol{x}(k)) \\ \boldsymbol{z}(k) = \boldsymbol{C}\boldsymbol{x}(k) \end{cases} \tag{A1}$$

where, $\boldsymbol{x}(k) \in \mathbf{R}^n$ is an n-dimensional vector; $\boldsymbol{z}(k) \in \mathbf{R}^m$ is an m-dimensional vector; $\boldsymbol{A} \in \mathbf{R}^{n\times n}$ and $\boldsymbol{C} \in \mathbf{R}^{m\times n}$ are $n \times n$ and $m \times n$ constant matrix respectively; $\boldsymbol{f} : \mathbf{R}^n \to \mathbf{R}^n$ is a nonlinear bounded mapping.

For equation (A1), there are

$$\begin{cases} \boldsymbol{z}(0) = \boldsymbol{C}\boldsymbol{x}(0) \\ \boldsymbol{x}(1) = \boldsymbol{A}\boldsymbol{x}(0) + \boldsymbol{f}(\boldsymbol{x}(0)) \\ \boldsymbol{z}(1) = \boldsymbol{C}\boldsymbol{x}(1) = \boldsymbol{C}\boldsymbol{A}\boldsymbol{x}(0) + \boldsymbol{C}\boldsymbol{f}(\boldsymbol{x}(0)) \\ \boldsymbol{x}(2) = \boldsymbol{A}\boldsymbol{x}(1) + \boldsymbol{f}(\boldsymbol{x}(1)) = \boldsymbol{A}^2\boldsymbol{x}(0) + \boldsymbol{A}\boldsymbol{f}(\boldsymbol{x}(0)) + \boldsymbol{f}(\boldsymbol{x}(1)) \\ \boldsymbol{z}(2) = \boldsymbol{C}\boldsymbol{x}(2) = \boldsymbol{C}\boldsymbol{A}^2\boldsymbol{x}(0) + \boldsymbol{C}\boldsymbol{A}\boldsymbol{f}(\boldsymbol{x}(0)) + \boldsymbol{C}\boldsymbol{f}(\boldsymbol{x}(1)) \\ \quad\vdots \\ \boldsymbol{x}(k) = \boldsymbol{A}\boldsymbol{x}(k-1) + \boldsymbol{f}(\boldsymbol{x}(k-1)) = \cdots = \boldsymbol{A}^k\boldsymbol{x}(0) + \sum\limits_{i=0}^{k-1} \boldsymbol{A}^{k-i-1}\boldsymbol{f}(\boldsymbol{x}(i)) \\ \boldsymbol{z}(k) = \boldsymbol{C}\boldsymbol{x}(k-1) = \boldsymbol{C}\boldsymbol{A}^k\boldsymbol{x}(0) + \sum\limits_{i=0}^{k-1} \boldsymbol{C}\boldsymbol{A}^{k-i-1}\boldsymbol{f}(\boldsymbol{x}(i)) \end{cases} \tag{A2}$$

The local observability is defined in [1]. More specifically, a system represented by (A2) is said to be "locally observable" if the system state $\boldsymbol{x}(k)$ can be determined from the knowledge of $\boldsymbol{z}(j)$ for j=k to j=k+n-1. For $j = [k, k+n-1]$, there are

$$\begin{cases} \boldsymbol{z}(k) = \boldsymbol{C}\boldsymbol{x}(k) \\ \boldsymbol{z}(k+1) = \boldsymbol{C}\boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{C}\boldsymbol{f}(\boldsymbol{x}(k)) \\ \boldsymbol{z}(k+2) = \boldsymbol{C}\boldsymbol{A}^2\boldsymbol{x}(k) + \boldsymbol{C}\boldsymbol{A}\boldsymbol{f}(\boldsymbol{x}(k)) + \boldsymbol{C}\boldsymbol{f}(\boldsymbol{x}(k+1)) \\ \qquad\vdots \\ \boldsymbol{z}(k+n-1) = \boldsymbol{C}\boldsymbol{A}^{n-1}\boldsymbol{x}(k) + \sum\limits_{i=0}^{n-2} \boldsymbol{C}\boldsymbol{A}^{n-i-2}\boldsymbol{f}(\boldsymbol{x}(i)) \end{cases} \tag{A3}$$

Let $\mathbf{Q} = \begin{bmatrix} \boldsymbol{C}^{\mathrm{T}} & (\boldsymbol{C}\boldsymbol{A})^{\mathrm{T}} & (\boldsymbol{C}\boldsymbol{A}^2)^{\mathrm{T}} & \cdots & (\boldsymbol{C}\boldsymbol{A}^{n-1})^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, $\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{z}(k)^{\mathrm{T}} & \boldsymbol{z}(k+1)^{\mathrm{T}} & \cdots & \boldsymbol{z}(k+n-1)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$,
$\boldsymbol{F}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{0}_{3\times 1}^{\mathrm{T}} & (\boldsymbol{C}\boldsymbol{f}(\boldsymbol{x}(k)))^{\mathrm{T}} & \cdots & (\sum\limits_{i=0}^{n-2} \boldsymbol{C}\boldsymbol{A}^{n-i-2}\boldsymbol{f}(\boldsymbol{x}(i)))^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$.

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Lemma 1. If $\mathrm{rank}(\boldsymbol{Q}) = n$, nonlinear equation set

$$\boldsymbol{Z} = \boldsymbol{Q}\boldsymbol{x}(k) + \boldsymbol{F}(\boldsymbol{x}) \tag{A4}$$

has a solution in $\mathbf{R}^n$.

Proof. Equation (A4) is left by $\boldsymbol{Q}^{\mathrm{T}}$ and there is

$$\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Z} = \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x}(k) + \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{F}(\boldsymbol{x}) \tag{A5}$$

There is $\mathrm{rank}(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q}) = \mathrm{rank}(\boldsymbol{Q}) = n$, so $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q}$ is reversible and equation (A5) can be rewritten as

$$\boldsymbol{x}(k) = (\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q})^{-1}(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Z} - \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{F}(\boldsymbol{x})) \tag{A6}$$

Because $\boldsymbol{f}$ is a nonlinear bounded mapping, $\boldsymbol{F}(\boldsymbol{x})$ is also bounded. In other words, there is a real number F_0 which lets $|\boldsymbol{F}(\boldsymbol{x})| \leqslant F_0$. $\boldsymbol{Z}$ is known, so there is a real number Z_0 which lets $|\boldsymbol{Z}| \leqslant Z_0$. From equation (A6), there is

$$\begin{aligned} \|\boldsymbol{x}(k)\| &= \left\|(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q})^{-1}(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Z} - \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{F}(\boldsymbol{x}))\right\| \\ &\leqslant \left\|(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q})^{-1}\right\| (\left\|\boldsymbol{Q}^{\mathrm{T}}\right\| \|\boldsymbol{Z}\| + \left\|\boldsymbol{Q}^{\mathrm{T}}\right\| \|\boldsymbol{F}(\boldsymbol{x})\|) \\ &\leqslant \left\|(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q})^{-1}\right\| (\left\|\boldsymbol{Q}^{\mathrm{T}}\right\| Z_0 + \left\|\boldsymbol{Q}^{\mathrm{T}}\right\| F_0) \end{aligned} \tag{A7}$$

Let $D := \{\boldsymbol{x}(k) \in \mathbf{R}^n : \|\boldsymbol{x}(k)\| \leqslant \left\|(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q})^{-1}\right\| (\|\boldsymbol{Q}^{\mathrm{T}}\| Z_0 + \|\boldsymbol{Q}^{\mathrm{T}}\| F_0)\} \subseteq \mathbf{R}^n$, and mapping P is defined as $P : D \to D, P\boldsymbol{x}(k) = (\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q})^{-1}(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Z} - \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{F}(\boldsymbol{x}))$. It is obviously that mapping P is continuous. Based on the Brouwer fixed point theorem [2], we can know that P has a fixed point in D. Therefore, equation (A4) has a solution in $\mathbf{R}^n$.

Theorem 1. If $\mathrm{rank}(\boldsymbol{Q}) = n$, the discrete-time constant nonlinear system of equation (A1) is locally observable. The matrix $\boldsymbol{Q}$ is called as the local observability matrix of system (A1).

Proof. From **Lemma 1**, we can know that equation (A4) has a solution in $\mathbf{R}^n$ if $\mathrm{rank}(\boldsymbol{Q}) = n$. Therefore, based on the definition of local observability, it is obviously that the discrete-time constant nonlinear system of equation (A1) is locally observable.

References

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