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Tensor restricted isometry property analysis for a large class of random measurement ensembles

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Dear editor,

Low-rank tensor recovery (LRTR) [1] is a natural higher-order generalization of the compressed sensing (CS) [2] and the low-rank matrix recovery (LRMR) [3, 4]. It has been applied extensively in various fields of artificial intelligence, including computer vision, image processing and machine learning. Let $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{n_1\times n_2\times n_3}$ be a low-rank tensor (third-order), $\mathfrak{M}$ be a random map from $\mathbb{R}^{n_1\times n_2\times n_3}$ to $\mathbb{R}^m$ $(m \ll n_1n_2n_3)$ and $\boldsymbol{w} \in \mathbb{R}^m$ be a vector of measurement errors with a noise level $\|\boldsymbol{w}\|_2 \leqslant \epsilon$. Suppose that $\boldsymbol{y} = \mathfrak{M}(\boldsymbol{\mathcal{X}})+\boldsymbol{w}$ is a linear noise measurement. Then, LRTR aims at recovering $\boldsymbol{\mathcal{X}}$ from $\boldsymbol{y}$. It is often difficult to achieve this goal. On one hand, the naive approach of solving the nonconvex program

$$\min_{\boldsymbol{\mathcal{X}}\in\mathbb{R}^{n_1\times n_2\times n_3}} \operatorname{rank}(\boldsymbol{\mathcal{X}}) \quad \text{s.t.} \ \|\boldsymbol{y} - \mathfrak{M}(\boldsymbol{\mathcal{X}})\|_2 \leqslant \epsilon \tag{1}$$

is the NP-hard in general, where the operation $\operatorname{rank}(\boldsymbol{\mathcal{X}})$ acts as a sparsity regularization of tensor singular values of $\boldsymbol{\mathcal{X}}$. On the other hand, some existing tensor ranks do not work well, such as CP rank and Tucker rank. The reason for this is that the calculation of CP rank of a tensor is usually NP-hard and the convex surrogate of the Tucker rank, sum of nuclear norms (SNN), is not the tightest convex relaxation. To avoid these defects, Lu et al. [5] paid attention to the novel tensor tubal rank of $\boldsymbol{\mathcal{X}}$, denoted by $\operatorname{rank}_t(\boldsymbol{\mathcal{X}})$, induced by tensor-tensor product (t-product) and tensor singular value decomposition (t-SVD). More importantly, it has been proved that some commonly used original tensor data, including color image data, video data and face data from the Berkeley segmentation dataset, the UMist faces dataset and the YUV video sequences dataset, have a low-tubal-rank structure. So, some related problems such as image denoising, video foreground and background segmentation, face recognition, can be solved effectively by t-SVD and low-tubal-rank methods. Further, Lu et al. considered the following convex tensor nuclear norm minimization (TNNM) model:

$$\min_{\boldsymbol{\mathcal{X}}\in\mathbb{R}^{n_1\times n_2\times n_3}} \|\boldsymbol{\mathcal{X}}\|_{\circledast} \quad \text{s.t.} \ \|\boldsymbol{y} - \mathfrak{M}(\boldsymbol{\mathcal{X}})\|_2 \leqslant \epsilon, \tag{2}$$

where $\|\boldsymbol{\mathcal{X}}\|_{\circledast}$ is referred to as tensor nuclear norm (TNN) which has been proved to be the convex envelop of tensor average rank within the unit ball of the tensor spectral norm. To facilitate the design of algorithms and the needs of practical applications, in previous study [6], Zhang et al. presented a theoretical analysis for regularized tensor nuclear norm minimization (RTNNM) model, which takes the form

$$\min_{\boldsymbol{\mathcal{X}}\in\mathbb{R}^{n_1\times n_2\times n_3}} \|\boldsymbol{\mathcal{X}}\|_{\circledast} + \frac{1}{2\lambda}\|\boldsymbol{y} - \mathfrak{M}(\boldsymbol{\mathcal{X}})\|_2^2, \tag{3}$$

with a positive parameter λ. However, the RTNNM model (3) is more applicable than the constrained-TNNM model (2) when the noise level is not given or cannot be accurately estimated. The tensor restricted isometry property (t-RIP) was first defined based on t-SVD in [6] as an analysis framework for LRTR via (3). For an integer r, the r-tensor restricted isometry constants of a linear map $\mathfrak{M} : \mathbb{R}^{n_1\times n_2\times n_3} \to \mathbb{R}^m$ are defined as the smallest constants satisfying

$$(1 - \delta_r)\|\boldsymbol{\mathcal{X}}\|_F^2 \leqslant \|\mathfrak{M}(\boldsymbol{\mathcal{X}})\|_2^2 \leqslant (1 + \delta_r)\|\boldsymbol{\mathcal{X}}\|_F^2,$$

for all tensor $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{n_1\times n_2\times n_3}$ whose tubal rank is at most r. Moreover, Theorem 4.1 in [6] shows that if $\mathfrak{M}$ satisfies the t-RIP with $\delta_{\mathrm{tr}} < \sqrt{(t-1)/(n_3^2+t-1)}$, for certain $t > 1$, the solution to (3) can robustly recover the low-tubal-rank tensor $\boldsymbol{\mathcal{X}}$.

Note that Zhang et al. [6] derived a deterministic condition of robust recovery for the RTNNM model (3) based on the t-RIP. Unfortunately, a way to construct the linear map $\mathfrak{M}$ that satisfies t-RIP is yet to be known. The aim of this study is to show the existence of these satisfactory

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linear maps under suitable conditions on the number of measurements in terms of the tubal rank r and the dimensions n_1, n_2 and n_3 of the tensor using probabilistic arguments. We consider the sub-Gaussian measurement ensemble such that all elements are drawn independently according to a sub-Gaussian distribution. This includes zero-mean Gaussian distributions, symmetric Bernoulli distributions, and all zero-mean bounded distributions. For such linear maps, the t-RIP holds with high probability in the stated parameter regime.

In 2018, Lu et al. [1] provided an exact recovery result based on the Gaussian width for TNNM model (2). Specifically, they pointed out that the unknown tensor of size $n_1 \times n_2 \times n_3$ with tubal rank r can be exactly recovered with high probability by solving (2) when the given number of Gaussian measurements is of the order $O(r(n_1+n_2-r)n_3)$. In 2019, Wang et al. [7] presented a generalized tensor Dantzig selector for a low-tubal-rank tensor recovery problem with noisy measurements $\boldsymbol{y} = \mathfrak{M}(\boldsymbol{\mathcal{X}}) + \boldsymbol{w}$, where $\boldsymbol{w}$ is the noise term. They showed that when the sample size $m = \Omega(r(n_1 + n_2 - r)n_3)$, the solution $\hat{\boldsymbol{\mathcal{X}}}$ of generalized tensor Dantzig selector satisfies $\|\hat{\boldsymbol{\mathcal{X}}} - \boldsymbol{\mathcal{X}}\|_F^2 \leqslant O(r(n_1 + n_2 - r)n_3 m^{-1})$ with high probability. In the noiseless setting (i.e., $\boldsymbol{w} = \boldsymbol{0}$), their results degenerate to Lu's case. All recovery results mentioned are probabilistic. Some deterministic results involving tensor RIP have emerged in LRTR. In 2013, the first tensor deterministic condition—tensor RIP based on Tucker decomposition which can guarantee that a given linear map $\mathfrak{M}$ can be utilized for LRTR was proposed by Shi et al. [8]. They showed that a tensor $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{n_1\times n_2\times n_3}$ with Tucker rank-(r_1, r_2, r_3) can be exactly recovered in the noiseless case if the linear map $\mathfrak{M}$ satisfies the tensor RIP with the constant $\delta_\Lambda < 0.4931$ for $\Lambda \in \{(2r_1, n_2, n_3), (n_1, 2r_2, n_3), (n_1, n_2, 2r_3)\}$. Such tensor RIP is hardly practical because it depends on a rank tuple that differs greatly from the definition of a familiar matrix rank, which will cause that some existing analysis tools and techniques cannot be used for tensor cases. Moreover, it is still an open problem for them which linear maps satisfy such tensor RIP.

Zhang et al. [6] used the t-RIP to answer the question regarding the conditions under which the robust solution to model (3) can be obtained. In this study, we continue the work and answer a quintessential and an all-important question: which linear map $\mathfrak{M}$ satisfies the t-RIP? The practical significance of this topic is to provide theoretical support for the robust recovery of low-tubal-rank tensor data from a small number of linear measurements in some real problems, such as magnetic resonance imaging (MRI), hyper-spectral imaging, and video security monitoring. For illustration, we consider an MRI. Imaging speed is important in many MRI applications. However, the speed depends largely on the amount of data collected in MRI. If the reconstructed image with high resolution can be obtained by using a small amount of data, then we can reduce the scanning time, sampling cost and pain of patients. So, how to design the sampling operator and how many samples are needed to ensure the accurate estimation of the image on the hardware side, are problems to be solved. Thus, our research results will provide some theoretical guarantees to similar application environments.

The following two results are the main contributions of this study.

Theorem 1. Fix $\delta, \varepsilon \in (0, 1)$ and let $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{n_1\times n_2\times n_3}$ be any given third-order tensor whose tubal rank is at most r, and then a random draw of a sub-Gaussian measurement ensemble $\mathfrak{M} : \mathbb{R}^{n_1\times n_2\times n_3} \to \mathbb{R}^m$ satisfies $\delta_r \leqslant \delta$, with probability at least $1 - \varepsilon$, in the condition that

$$m \geqslant C\delta^{-2} \max\left\{r(n_1 + n_2 + 1)n_3, \log(\varepsilon^{-1})\right\},$$

where the constant $C > 0$ only depends on the sub-Gaussian parameter.

Corollary 1 is trivial but an important special case of Theorem 1.

Corollary 1. Let $\mathfrak{M} : \mathbb{R}^{n_1\times n_2\times n_3} \to \mathbb{R}^m$ be a zero-mean Gaussian or symmetric Bernoulli measurement ensemble. Then there exists a universal constant $C > 0$, such that the tensor restricted isometry constant of $\mathfrak{M}$ satisfies $\delta_r \leqslant \delta$, with probability at least $1 - \varepsilon$, in the condition that

$$m \geqslant C\delta^{-2} \max\left\{r(n_1 + n_2 + 1)n_3, \log(\varepsilon^{-1})\right\}.$$

Discussion. We know that sub-Gaussian distributions belong to a larger class of random distributions, including zero-mean Gaussian distributions, symmetric Bernoulli distributions and all zero-mean bounded distributions. Thus, in some sense, Theorem 1 completely characterizes the behavior of numerous random measurement ensembles in term of the t-RIP. In CS and LRMR, Gaussian random matrix or Bernoulli random matrix is often used as a universal measurement matrix (ensemble) because it satisfies vector RIP with high probability. Accordingly, Corollary 1 guarantees that the zero-mean Gaussian or symmetric Bernoulli measurement ensemble can also be used for LRTR.

The degrees of freedom of $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{n_1\times n_2\times n_3}$ and $\bar{\boldsymbol{\mathcal{X}}}$ (the result of DFT on $\boldsymbol{\mathcal{X}}$ along the third dimension) are the same because the DFT is the invertible. Suppose $\text{rank}_t(\boldsymbol{\mathcal{X}}) = r$, and then we have $\text{rank}(\bar{\boldsymbol{X}}^{(i)}) \leqslant r$, $i = 1, \ldots, n_3$, where $\bar{\boldsymbol{X}}^{(i)}$ is the i-th frontal slice of $\bar{\boldsymbol{\mathcal{X}}}$. Then $\bar{\boldsymbol{X}}^{(i)}$ has at most $r(n_1 + n_2 - r)$ degrees of freedom, and thus $\boldsymbol{\mathcal{X}}$ has at most $r(n_1 + n_2 - r)n_3$ degrees of freedom. So, the required number of measurements is very reasonable when compared with the degrees of freedom. This reason is consistent with the viewpoint of Recht et al. on Theorem 4.2 in [4]. Now, we further ascertain the strength of the bound. Observe that $O(r(n_1 + n_2 + 1)n_3)$ can be rewritten as $O(n_{(1)}n_3 r)$ where $n_{(1)} = \max(n_1, n_2)$. Thus, the bound indicates that one only needs a constant number of measurements per degree of freedom of the underlying rank-r tensor in order to obtain the t-RIP at rank r. The above explanation coincides with the statement on Theorem 2.3 in [3] by Candès et al. If $n_3 = 1$, the third-order tensor $\boldsymbol{\mathcal{X}}$ will reduce to a second-order tensor, i.e., a matrix. Accordingly, the tensor tubal rank will reduce to the matrix rank, and t-RIP will reduce to the Definition 2.1 in [3]. Thus, the required number of measurements for random sub-Gaussian measurement ensembles in Theorem 1 includes the results of Theorem 2.3 in [3] for LRMR.

It is worth mentioning that there exists a good study on the low-tubal-rank tensor recovery from Gaussian measurements by Lu et al. [1]. Based on the Gaussian width rather than RIP, they showed that the sampling number $O(r(n_1 + n_2 - r)n_3)$ (or $O(rn_{(1)}n_3)$) is sufficient and order optimal compared to the degrees of freedom of a tensor with tubal rank r. As mentioned previously, the recovery results provided by Lu et al. are probabilistic: they are valid for a random measurement ensemble $\mathfrak{M}$ and with high probability. One may wonder if there exists a deterministic condition which can guarantee that a given operator $\mathfrak{M}$ can be used for tensor recovery. Consequently, Theorem 1 and Corollary 1 in this study were just inspired by this problem.

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Supporting information Notations, definitions, and probabilistic tools are introduced in Appendixes A and B. Appendixes C and D present the proof of Theorem 1 and some numerical experiments. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

References

1 Lu C, Feng J, Lin Z, et al. Exact low tubal rank tensor recovery from Gaussian measurements. In: Proceedings of the 27th International Joint Conference on Artificial Intelligence, Stockholm, 2018. 2504–2510

2 Wang J, Zhang F, Huang J, et al. A nonconvex penalty function with integral convolution approximation for compressed sensing. Signal Process, 2019, 158: 116–128

3 Candès E J, Plan Y. Tight oracle inequalities for low-rank matrix recovery from a minimal number of noisy random measurements. IEEE Trans Inform Theor, 2011, 57: 2342–2359

4 Recht B, Fazel M, Parrilo P A. Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization. SIAM Rev, 2010, 52: 471–501

5 Lu C, Feng J, Chen Y, et al. Tensor robust principal component analysis with a new tensor nuclear norm. IEEE Trans Pattern Anal Mach Intell, 2019. doi: 10.1109/TPAMI.2019.2891760

6 Zhang F, Wang W, Huang J, et al. RIP-based performance guarantee for low-tubal-rank tensor recovery. 2019. ArXiv: 1906.01774

7 Wang A, Song X, Wu X, et al. Generalized Dantzig selector for low-tubal-rank tensor recovery. In: Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing, Brighton, 2019. 3427–3431

8 Shi Z, Han J, Zheng T, et al. Guarantees of augmented trace norm models in tensor recovery. In: Proceedings of the 23rd International Joint Conference on Artificial Intelligence, Beijing, 2013. 1670–1676