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# Asymptotic multistability and local S-asymptotic $\omega$-periodicity for the nonautonomous fractional-order neural networks with impulses

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**Abstract** This paper focuses on the investigation of asymptotic multistability and on local S-asymptotic $\omega$-periodicity for nonautonomous fractional-order neural networks (FONNs) with impulses. Several criteria on the existence, uniqueness, and invariant sets of nonautonomous FONNs with impulses are derived by constructing convergent sequences and comparison principles, respectively. In addition, using the Lyapunov direct method, some novel conditions of boundedness and local asymptotic stability of the FONNs discussed are obtained. Also, the sufficient conditions for local S-asymptotic $\omega$-periodicities of the system are presented. Finally, a discussion using two examples verifies the validity of our findings, which imply that global asymptotic stability is a special case of asymptotic multistability.

**Keywords** fractional-order neural networks, local S-asymptotic $\omega$-periodicity, asymptotic multistability, impulse, nonautonomous

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## 1 Introduction

Many researchers have focused on neural networks (NNs) owing to their extensive applications, such as image processing, signal processing, pattern recognition, optimization control, argument estimation, and artificial intelligence. Such applications not only rely on the existence of equilibrium points, unique equilibrium point, and on the qualitative properties of stability, but also rely on dynamic behaviors, such as periodic oscillatory behavior, nearly periodic oscillatory properties, chaos, and bifurcation. Recently, on the basis of fractional-order calculus [1–3], a large variety of fractional-order neural networks (FONNs) have been established and investigated [4–7]. The FONNs have better infinity memory and heredity than integer-order neural networks. As is well known, the traditional concepts of integer-order periodic solutions cannot be applied to fractional order (FO) differential equations because of the absence of periodic solutions of FO differential equations [8]. In the existing literature, there are some results on lasymptotic $\omega$-periodicity, local S-asymptotic $\omega$-periodicity (SAP) and near-periodicity (see [9–11]). However, there are only a few reports on the SAP for FONNs. The global SAP was considered by Chen for nonautonomous FONNs in [12] and for the nonautonomous FONNs with time-varying delays in [13]. Wan and Wu [14] discussed local SAP for the FONNs. However, they did not consider impulsive effects.

In fact, impulsive perturbations exist widely in various fields, such as economics, electromagnetic wave radiation, electronics, and telecommunications (see [15–22]). Stamova and Henderson [23] studied the practical stability of the FONNs with impulses. Yang et al. [24] proposed Mittag-Leffler stability for those FONNs with delays and impulses. Wang et al. [25] explored global asymptotic stability for complex-valued FONNs with delays and impulses.

Some reports on the multistability of NNs have emerged in [26–28]. This multistability is important for a deeper understanding of NN dynamical systems. Wang and Chen [29] have proposed the $\mu$-stability

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for NNs with unbounded time-varying delays. Cheng et al. [30] explored multistability for NNs with delays via sequential contracting. Liu et al. presented multistability for nonmonotonic NNs with unbounded time-varying delays [31] and for nonmonotonic NNs with mixed time delays [32], respectively. However, there are no reports on the asymptotic multistability and local SAP of FONNs with impulses and nonautonomous parameters.

Motivated by the above discussion, we consider asymptotic multistability and local SAP for nonautonomous FONNs with impulses. Section 2 presents the FONNs, some definitions and lemmas. Invariant sets and boundness are explored in Section 3. Section 4 discusses asymptotic multistability and local SAP. Two examples are presented in Section 5. Section 6 completes this paper.

Notations: In this paper, a given vector $z=(z_1,z_2,\ldots,z_N)^{\rm T}$, in which the "T" represents the transpose, and $\|z\|=\sum_{i=1}^{N}|z_i|$. $C_b([\tau_0,\infty),\mathbb{R}^n)$ represents the space consisting of continuous and bounded functions $[\tau_0,\infty)\to\mathbb{R}^n$ with $\|\cdot\|_\infty$, which are uniformly convergent. Additionally, $C^r([\tau_0,\infty),\mathbb{R}^n)$ represents the space which contains $r$-order continuous differentiable functions $[\tau_0,\infty)\to\mathbb{R}^n$. Besides, $\mathbb{N}$ denotes the natural number. Let $\mathcal{F}=\{\tau_k:\tau_k\in[0,\infty),\tau_k<\tau_{k+1},k=0,1,2,\ldots,\lim_{k\to\infty}\tau_k=\infty\}$ denote the set of all strictly monotonically increasing and unbounded sequences. $\mathbb{PC}=\mathbb{PC}[\mathbb{R}^+,\mathbb{R}^N]=\{\vartheta:\mathbb{R}^+\to\mathbb{R}^n,\ \vartheta$ is continuous except for $\tau_k$, $\tau_k\in\mathcal{F}$, $\vartheta(\tau_k^-)$ and $\vartheta(\tau_k^+)$ exist, $\vartheta(\tau_k^-)=\vartheta(\tau_k)\}$. $\mathbb{PC}_b[\mathbb{R}^+,\mathbb{R}^N]$ denotes the set of all continuous functions which are piecewise bounded, which is a subspace of $\mathbb{PC}[\mathbb{R}^+,\mathbb{R}^N]$.

## 2 Preliminaries

We consider a nonautonomous FONN with impulses as follows:

$$\begin{cases} {}^C D^{\beta} z_i(\tau) = -a_i(\tau)z_i(\tau)+\sum\limits_{j=1}^{N} c_{ij}(\tau)h_j(z_j(\tau))+d_i(\tau), & \tau\neq\tau_k,\ \tau\geqslant 0,\\ \Delta z_i(\tau_k)=z_i(\tau_k^+)-z_i(\tau_k^-)=b_{ik}(z_i(\tau_k^-)), & k\in\mathbb{Z}_+,\ i=1,2,\ldots,N, \end{cases} \tag{1}$$

where $N$ stands for the number of neurons. $\mathbb{Z}_+$ stands for the set of positive integer. $z_i(\tau)$ represents the state variable of the neuron. $a_i(\tau)$ corresponds to the rate with which the $i$th unit resets its potential to other states in isolation when disconnected from network and external inputs. $c_{ij}(\tau)$ presents the strengths of the $j$th neuron in the $i$th neuron; $h_j(z_j(\tau))$ stands for the neuron activation function. $d_i(\tau)$ is the external input vector. The neuron states $z_i(\tau_k)=z_i(\tau_k^-)$ and $z_i(\tau_k^+)$ are the states of the $i$th neuron before and after an impulsive perturbation at time $\tau_k$, respectively. $b_{ik}$ is the abrupt change of the state $z_i(\tau)$ of the $i$th neuron at the impulsive moment $\tau_k$. The initial conditions satisfy that $z_i(\tau_0)=z_{i0}, i=1,2,\ldots,N$. Eq. (1) can be rewritten as the following matrix expression:

$$\begin{cases} {}^C D^{\beta} z(\tau)=-A(\tau)z(\tau)+C(\tau)H(z(\tau))+D(\tau)=G(\tau,z(\tau)), & \tau\neq\tau_k,\\ \Delta z(\tau_k)=z(\tau_k^+)-z(\tau_k^-)=B_k(z(\tau_k^-)), & k\in\mathbb{Z}_+, \end{cases} \tag{2}$$

where $z(\tau)=[z_1(\tau),\ldots,z_N(\tau)]^{\rm T}\in\mathbb{R}^N, H(z(\tau))=[h_1(z_1(\tau)),\ldots,h_N(z_N(\tau))]^{\rm T}, C(\tau)=(c_{ij}(\tau))_{N\times N}$, $A(\tau)={\rm diag}\,[a_1(\tau),\ldots,a_N(\tau)]\,, D(\tau)=[d_1(\tau),\ldots,d_N(\tau)]^{\rm T}, z_0=[z_{10},z_{20},\ldots,z_{N0}]^{\rm T}$, $B_k=[b_{1k},b_{2k},\ldots,b_{Nk}]^{\rm T}$.

**Definition 1** ([1])**.** For the function $f\in {\rm AC}^n([\tau_0,+\infty),\mathbb{R}^N)(n-1<\beta<n)$, the $\beta$ order Caputo derivative is defined as follows:

$${}^C_{\tau_0}D^{\beta}_{\tau}f(\tau)=\frac{1}{\Gamma(n-\beta)}\int_{\tau_0}^{\tau}\frac{f^{(n)}(s)}{(\tau-s)^{\beta-n+1}}{\rm d}s,\quad \tau\geqslant\tau_0,$$

where ${\rm AC}([\tau_0,+\infty),\mathbb{R}^N)$ denotes the absolutely continuous functions on $([\tau_0,+\infty),\mathbb{R}^N)$ and ${\rm AC}^n([\tau_0,+\infty),\mathbb{R}^N)$ is the function space of $f$, where $f\in C^{n-1}([\tau_0,+\infty),\mathbb{R}^N)$ and $f^{(n-1)}\in{\rm AC}([\tau_0,+\infty),\mathbb{R}^N)$.

**Lemma 1** ([2])**.** For the function $f(\tau)\in{\rm AC}^n([\tau_0,\infty),\mathbb{R}^N)$, we have

$${}^C_{\tau_0}D^{\beta}_{\tau}\left({}^C_{\tau_0}D^{-\beta}_{\tau}\right)f(\tau)=f(\tau)\ \text{ and }\ {}^C_{\tau_0}D^{-\beta}_{\tau}\left({}^C_{\tau_0}D^{\beta}_{\tau}\right)f(\tau)=f(\tau)-\sum_{i=0}^{n-1}\frac{(\tau-\tau_0)^i}{i!}f^{(i)}(\tau_0),$$

where $n-1<\beta<n$ and $\tau\geqslant\tau_0$.

**Definition 2.** The function $z(\tau) \in \mathbb{PC}_b([\tau_0,\infty),\mathbb{R}^N)\bigcap\bigcup_{k=0}^{\infty}\mathrm{AC}^1((\tau_k,\tau_{k+1}],\mathbb{R}^N)$ is a solution of (1), if $z(\tau)$ satisfies ${}^C D^\beta z(\tau) = G(\tau, z(\tau))$ for $\tau \in [\tau_0,\infty), \tau \neq \tau_k, k \in \mathbb{Z}_+$ and impulsive conditions $z(\tau_k^+) = z(\tau_k^-) + B_k(z(\tau_k^-)), k \in \mathbb{Z}_+$.

**Lemma 2** ([3]). For $\beta \in (0,1)$ and a continuous mapping $g : [\tau_0,\infty) \to \mathbb{R}$, the function $z(\tau)$ is the solution for the following equation:

$$z(\tau) = \begin{cases} z_0 + \frac{1}{\Gamma(\beta)}\int_{\tau_0}^{\tau}(\tau-s)^{\beta-1}g(s)\mathrm{d}s, & \tau \in [\tau_0,\tau_1], \\ z_0 + \sum_{i=1}^{k} B_i(z(\tau_i^-)) + \frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{\beta-1}g(s)\mathrm{d}s & \\ \quad +\frac{1}{\Gamma(\beta)}\int_{\tau_k}^{\tau}(\tau-s)^{\beta-1}g(s)\mathrm{d}s, & \tau \in (\tau_k,\tau_{k+1}],\ k \in \mathbb{Z}_+, \end{cases}$$

if and only if $z(\tau)$ is the solution of

$$\begin{cases} {}^C D^\beta z(\tau) = g(\tau),\ \tau \neq \tau_k,\ k \in \mathbb{Z}_+, \\ \Delta z(\tau_k) = z(\tau_k^+) - z(\tau_k^-) = B_i(z(\tau_k^-)),\ k \in \mathbb{Z}_+. \end{cases} \tag{3}$$

**Definition 3** ([10]). Function $h(\tau) \in \mathbb{PC}_b([\tau_0,\infty),\mathbb{R}^N)\bigcap\bigcup_{k=0}^{\infty}\mathrm{AC}^1((\tau_k,\tau_{k+1}],\mathbb{R}^N)$ is S-asymptotically $\omega$-periodically piecewise continuous in $[\tau_0,\infty)$, if it satisfies $\lim_{\tau\to\infty}|h(\tau+\omega)-h(\tau)| = 0$, where $\omega > 0$ is the asymptotical period of $h(\tau)$.

**Definition 4** ([14]). Suppose $\mathcal{D} \subset \mathbb{R}^N$ is a positive invariant set. If the solution of system (1) with $z(\tau_0) = z_0 \in \mathcal{D}$ converges to $z_\omega(\tau) \in \mathcal{D}$, system (1) is locally asymptotically $\omega$-periodic, and $z_\omega(\tau)$ is a $\omega$-periodic function.

**Definition 5** ([26]). If the solution $z(\tau)$ for (1) with $z(\tau_0) = z_0 \in \mathcal{D}$ satisfies $z(\tau) \in \mathcal{D}$ for $\tau > \tau_0$, then the set $\mathcal{D}$ is called an invariant set of (1).

**Definition 6** ( [33]). Let $\mathcal{D} \subset \mathbb{R}^N$ denote a positive invariant set and let $z^*(\tau) \in \mathcal{D}$ stand for the equilibrium solution of (1). System (1) is

(a) locally stable, if $(\forall\tau_0 \in \mathbb{R}^+)(\forall\epsilon > 0)(\exists\delta = \delta(\tau_0,\epsilon) > 0), (\forall\tau \geqslant \tau_0) : \|z(\tau;\tau_0,z_0) - z^*\| < \epsilon$;

(b) locally attractive, if $\lim_{\tau\to\infty} z(\tau;\tau_0,z_0) = z^*$;

(c) locally asymptotically stable, if and only if it is locally stable and locally attractive.

When $\mathcal{D} = \mathbb{R}^N$, system (1) is globally stable, globally attractive and globally asymptotically stable, respectively.

In this paper, the following assumptions are true.

(H1) Let $h_i(z)$ be continuous function and there exist two constants $p_i, q_i$ satisfying $p_i \leqslant h_i(z) \leqslant q_i$.

(H2) $c_{ij}(\tau), a_i(\tau)$ and $d_i(\tau)$ are all continuous and bounded on $\mathbb{R}^+$ and $\sup_{m\to\infty}\sum_{j=1}^{m}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-1)^{\beta-1}\|A(s)\|\mathrm{d}s \leqslant \sigma < \infty, \sup_{m\to\infty}\sum_{j=1}^{m}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-1)^{\beta-1}\|C(s)\|\mathrm{d}s \leqslant \sigma < \infty, \sup_{m\to\infty}\sum_{j=1}^{m}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-1)^{\beta-1}\|D(s)\|\mathrm{d}s \leqslant \sigma < \infty, \check{c}_{ij} = \min_{\tau_0\leqslant\tau<\infty} c_{ij}(\tau), \hat{c}_{ij} = \max_{\tau_0\leqslant\tau<\infty} c_{ij}(\tau), \check{d}_i = \min_{\tau_0\leqslant\tau<\infty} d_i(\tau), \hat{d}_i = \max_{\tau_0\leqslant\tau<\infty} d_i(\tau)$.

(H3) There exist constants $-\infty \leqslant s_i^{(0)} < r_i^{(0)} < s_i^{(1)} < r_i^{(1)} < \cdots < s_i^{(D_i-1)} < r_i^{(D_i-1)} < s_i^{(D_i)} < r_i^{(D_i)} \leqslant \infty$, and $\check{\lambda}_i^j, \hat{\lambda}_i^j, \check{\nu}_i^k, \hat{\nu}_i^k$ which satisfy $\check{\lambda}_i^j \leqslant \frac{h_i(x)-h_i(y)}{x-y} \leqslant \hat{\lambda}_i^j$, for $x, y \in (s_i^j, r_i^j), j = 0,1,\ldots,D_i \check{\nu}_i^k \leqslant \frac{h_i(x)-h_i(y)}{x-y} \leqslant \hat{\nu}_i^k$, for $x, y \in [r_i^{k-1}, s_i^k], k = 1,\ldots,D_i$.

(H4) There exist constants $\gamma_{ik}$ such that $b_{ik}(z_i(\tau_k)) = \gamma_{ik}z_i(\tau_k)$.

**Lemma 3** ([34]). Function $g(\tau)$ is differential on $[\tau_0,+\infty)$ with $\beta \in (0,1)$. For $\tau_0 \leqslant \tau < \bar{\tau}$, $g(\tau) < 0$ and $g(\bar{\tau}) = 0$, we have ${}_{\tau_0}^{C}D_\tau^\beta g(\tau) > 0$, for $\tau = \bar{\tau}$.

**Lemma 4** ([35]). $f(\tau)$ stands for a successive function on $[\tau_0,\infty)$. If there is a constant $\beta > 0$ satisfying

$$\begin{cases} {}_{\tau_0}^{C}D_\tau^\alpha f(\tau) \leqslant -\beta f(\tau), & \alpha \in (0,1), \\ f(\tau_0) = f_0, \end{cases}$$

then $f(\tau) \leqslant f_0 E_\alpha(-\beta(\tau-\tau_0)^\alpha),\ \tau \geqslant \tau_0$.

**Lemma 5.** If the following conditions are true, then the solution of (1) with $z(\tau_0) = z_0$ is unique.

(B1) A positive constant $M$ satisfies $\lim_{k\to\infty}\|\sum_{j=0}^{k} B_j(z(\tau_j))\| = \lim_{k\to\infty}\|\sum_{j=0}^{k}\hat{\gamma}_j z(\tau_j)\| \leqslant M < \infty$, where $z(\cdot)$ is continuous functions of bounded piecewise.

(B2) $\lim_{k\to\infty}\sum_{j=1}^{k}\hat{\gamma}_j + \frac{\sigma+\sigma L}{\Gamma(\beta)} < 1$, where $L = \max_{i=1,2,\ldots,N,j=0,1,\ldots,D_i}\{|\check{\lambda}_i^j|, |\hat{\lambda}_i^j|\}, \hat{\gamma}_j = \max_{1\leqslant i\leqslant N}\{\gamma_{ij}\}$.

*Proof.* The proof is given by two steps.

Step 1. To verify the existence of the solution of (1). From Lemma 2, we have

$$z(\tau)=\begin{cases} z_0+\frac{1}{\Gamma(\beta)}\int_{\tau_0}^{\tau}(\tau-s)^{\beta-1}G(s,z(s))\mathrm{d}s, & \tau\in[\tau_0,\tau_1],\\ z_0+\sum_{j=1}^{k}B_j(z(\tau_j^-))+\frac{1}{\Gamma(\beta)}\sum_{j=1}^{k}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{\beta-1}G(s,z(s))\mathrm{d}s & \\ \quad+\frac{1}{\Gamma(\beta)}\int_{\tau_k}^{\tau}(\tau-s)^{\beta-1}G(s,z(s))\mathrm{d}s, & \tau\in(\tau_k,\tau_{k+1}],\ k\in\mathbb{Z}_+. \end{cases}$$

In the following, we assume that $\tau>\tau_1$. We construct a sequence $\{z_i^m(\tau,\tau_0,z_0)\}, z_i^0(\tau,\tau_0,z_0)=z_i^0$, for $\tau\in(\tau_k,\tau_{k+1}],k\in\mathbb{Z}_+$, satisfying

$$\begin{aligned} z_i^{m+1}(\tau)=z_i^0&+\sum_{j=1}^{k}b_{ij}(z_i^m(\tau_j^-))+\frac{1}{\Gamma(\beta)}\sum_{j=1}^{k}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{\beta-1}h_i(s,z^m(s))\mathrm{d}s\\ &+\frac{1}{\Gamma(\beta)}\int_{\tau_k}^{\tau}(\tau-s)^{\beta-1}h_i(s,z^m(s))\mathrm{d}s,\quad i=1,2,\ldots,n, \end{aligned}$$

where $h_i(s,z^m(s))=-a_i(s)z_i^m(s)+\sum_{j=1}^{N}c_{ij}(s)h_j(z_j^m(s))+d_i(s)$. Therefore, $\|z^{m+1}(\tau)-z^m(\tau)\|\leqslant\bar{d}_1+\frac{1}{\Gamma(\beta)}\left(\bar{d}_2+\bar{d}_3\right)$, where $\bar{d}_2=\sum_{i=1}^{N}\sum_{j=1}^{k}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{\beta-1}\left|h_i(s,z^m(s))-h_i(s,z^{m-1}(s))\right|\mathrm{d}s$, $\bar{d}_1=\sum_{i=1}^{N}\sum_{j=1}^{k}\left|b_{ij}(z_i^m(\tau_j^-))-b_{ij}(z_i^{m-1}(\tau_j^-))\right|$, $\bar{d}_3=\sum_{i=1}^{N}\int_{\tau_k}^{\tau}(\tau-s)^{\beta-1}\left|h_i(s,z^m(s))-h_i(s,z^{m-1}(s))\right|\mathrm{d}s$.

In view of assumption (H4) and condition (B1), we get

$$\bar{d}_1\leqslant\sum_{i=1}^{N}\sum_{j=1}^{k}\hat{\gamma}_j\left|z_i^m(\tau_j^-)-z_i^{m-1}(\tau_j^-)\right|\leqslant\sum_{j=1}^{k}\hat{\gamma}_j\|z^m(\tau)-z^{m-1}(\tau)\|_\infty,\tag{4}$$

where $\hat{\gamma}_j=\max_{1\leqslant i\leqslant n}\{|\gamma_{ij}|\}$. Considering assumptions (H1)–(H3), one obtains

$$\begin{aligned} \bar{d}_2&\leqslant\sum_{j=1}^{k}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{\beta-1}\sum_{i=1}^{N}\left|h_i(s,z^m(s))-h_i(s,z^{m-1}(s))\right|\mathrm{d}s\\ &\leqslant\sum_{j=1}^{k}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{\beta-1}\sum_{i=1}^{N}\left|-a_i(s)\left(z_i^m(s)-z_i^{m-1}(s)\right)+\sum_{j=1}^{N}c_{ij}(s)\left(h_j(z_j^m(s))-h_j(z_j^{m-1}(s))\right)\right|\mathrm{d}s\\ &\leqslant\sum_{j=1}^{k}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{\beta-1}\|a(s)\|\|z^m(s)-z^{m-1}(s)\|+\|C(s)\|L\|z^m(s)-z^{m-1}(s)\|\mathrm{d}s\\ &\leqslant(\sigma+\sigma L)\|z^m(s)-z^{m-1}(s)\|_\infty. \end{aligned}$$

From condition (H1), it follows that

$$\begin{aligned} \bar{d}_3&=\sum_{i=1}^{N}\int_{\tau_k}^{\tau}(\tau-s)^{\beta-1}\left|h_i(s,z^m(s))-h_i(s,z^{m-1}(s))\right|\mathrm{d}s\\ &\leqslant\int_{\tau_k}^{\tau}(\tau-s)^{\beta-1}\sum_{i=1}^{N}\left|-a_i(s)\left(z_i^m(s)-z_i^{m-1}(s)\right)+\sum_{j=1}^{N}c_{ij}(s)\left(h_j(z_j^m(s))-h_j(z_j^{m-1}(s))\right)\right|\mathrm{d}s\\ &\leqslant\int_{\tau_k}^{\tau}(\tau-s)^{\beta-1}\left(\|A(s)\|\|z^m(s)-z^{m-1}(s)\|+\|C(s)\|L\|z^m(s)-z^{m-1}(s)\|\right)\mathrm{d}s\\ &\leqslant\int_{\tau_k}^{\tau}(\tau-s)^{\beta-1}\left(\|A(s)\|+\|C(s)\|L\right)\mathrm{d}s\|z^m(s)-z^{m-1}(s)\|_\infty\\ &\leqslant(\epsilon+\epsilon L)\|z^m(s)-z^{m-1}(s)\|_\infty. \end{aligned}$$

Considering $\|z^1(\tau)-z^0(\tau)\|\leqslant M+\frac{\sigma}{\Gamma(\beta)}\left(\|z^0\|_\infty+L\|z^0\|_\infty+\|G(0)\|\right),\tau\in(\tau_k,\tau_{k+1}],k\in\mathbb{Z}_+$, we get

$$\|z^{m+1}(\tau)-z^m(\tau)\|\leqslant\left(\sum_{j=1}^{k}\hat{\gamma}_j+\frac{\sigma+\sigma L+\epsilon+\epsilon L}{\Gamma(\beta)}\right)\|z^m(\tau)-z^{m-1}(\tau)\|_\infty,\quad\tau\in(\tau_k,\tau_{k+1}],\ k\in\mathbb{Z}_+,$$

which means that

$$\|z^{m+1}(\tau)-z^{m}(\tau)\| \leqslant \left(\sum_{j=1}^{k}\hat{\gamma}_j+\frac{\sigma+\sigma L}{\Gamma(\beta)}\right)\|z^{m+1}(\tau)-z^{m}(\tau)\|_\infty$$
$$\leqslant \left(\sum_{j=1}^{k}\hat{\gamma}_j+\frac{\sigma+\sigma L}{\Gamma(\beta)}\right)^{m}\left(M+\frac{\sigma}{\Gamma(\beta)}\left(\|z^0\|_\infty+L\|z^0\|_\infty+\|G(0)\|\right)\right).$$

As $(\sum_{j=1}^{k}\hat{\gamma}_j+\frac{\sigma+\sigma L}{\Gamma(\beta)})<1$, then $z(\tau,\tau_0,z_0)$ is a solution of (1).

Step 2. Now we shall testify the uniqueness of the solution of (1). Suppose $z(\tau)=[z_1(\tau),\ldots,z_N(\tau)]^{\mathrm{T}}$ and $y(\tau)=[y_1(\tau),\ldots,y_N(\tau)]^{\mathrm{T}}$ are two solutions for the same initial conditions, such that

$$z_i(\tau)=\begin{cases} z_{i0}+\frac{1}{\Gamma(\beta)}\int_{t_0}^{\tau_1}(\tau-s)^{\beta-1}h_i(s,z(s))\mathrm{d}s, & \tau\in[\tau_0,\tau_1],\\ z_{i0}+\sum_{j=1}^{k}b_{ij}(z_i(\tau_j^-))+\frac{1}{\Gamma(\beta)}\sum_{j=1}^{k}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{\beta-1}h_i(s,z(s))\mathrm{d}s & \\ \quad+\frac{1}{\Gamma(\beta)}\int_{\tau_k}^{\tau}(\tau-s)^{\beta-1}h_i(s,z(s))\mathrm{d}s, & \tau\in(\tau_k,\tau_{k+1}],\ k\in\mathbb{Z}_+,\end{cases}$$

and

$$y_i(\tau)=\begin{cases} z_{i0}+\frac{1}{\Gamma(\beta)}\int_{\tau_0}^{\tau_1}(\tau-s)^{\beta-1}h_i(s,y(s))\mathrm{d}s, & \tau\in[\tau_0,\tau_1],\\ z_{i0}+\sum_{j=1}^{k}b_{ij}(y_i(t_j^-))+\frac{1}{\Gamma(\beta)}\sum_{j=1}^{k}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{\beta-1}h_i(s,y(s))\mathrm{d}s & \\ \quad+\frac{1}{\Gamma(\beta)}\int_{\tau_k}^{\tau}(\tau-s)^{\beta-1}h_i(s,y(s))\mathrm{d}s, & \tau\in(\tau_k,\tau_{k+1}],\ k\in\mathbb{Z}_+,\end{cases}$$

where $h_i(s,x(s))=-a_i(s)x_i(s)+\sum_{j=1}^{N}c_{ij}(s)h_j(x_j(s))+d_i(s)$. Then, when $\tau\in(\tau_k,\tau_{k+1}],k\in\mathbb{Z}_+$, one has

$$\|z(\tau)-y(\tau)\| \leqslant \sum_{i=1}^{N}\sum_{j=1}^{k}|\gamma_{ij}||z_i(\tau_j^-)-y_i(\tau_j^-)|+\frac{1}{\Gamma(\beta)}\sum_{i=1}^{N}\sum_{j=1}^{k}\int_{\tau_{j-1}}^{\tau_j}(\tau_j-s)^{\beta-1}(h_i(s,x(s))$$
$$-h_i(s,y(s)))\mathrm{d}s+\frac{1}{\Gamma(\beta)}\sum_{i=1}^{N}\int_{\tau_k}^{\tau}(\tau-s)^{\beta-1}(h_i(s,z(s))-h_i(s,y(s)))\mathrm{d}s$$
$$\leqslant \left(\sum_{j=1}^{k}|\gamma_{ij}|+\frac{1}{\Gamma(\beta)}(\sigma+\sigma L)\right)\|z-y\|_\infty+\frac{\epsilon+\epsilon L}{\Gamma(\beta)}\|z-y\|_\infty.$$

Therefore, $\|z-y\|_\infty\leqslant\|z(\tau)-y(\tau)\|\leqslant\left(\sum_{j=1}^{k}|\gamma_{ij}|+\frac{1}{\Gamma(\beta)}(\sigma+\sigma L)\right)\|z-y\|_\infty$, which is because that $\epsilon$ is small enough. From $\sum_{j=1}^{k}|\gamma_{ij}|+\frac{1}{\Gamma(\beta)}(\sigma+\sigma L)<1$, it results in the contraction. Hence, solution $z(\tau)$ of (1) is unique.

## 3 The invariant sets and boundedness

The invariant sets and boundedness for system (1) are discussed in this section. To facilitate the below discussion, several mathematical notations are introduced. Let

$$\bar{\Re}_j=\left\{[r_j^{(0)},s_j^{(1)}],[r_j^{(1)},s_j^{(2)}],\ldots,[r_j^{(D_j-1)},s_j^{(D_j)}]\right\},\quad \tilde{\Re}_j=\left\{(s_j^{(0)},r_j^{(0)}),(s_j^{(1)},r_j^{(1)}),\ldots,(s_j^{(D_j)},r_j^{(D_j)})\right\};$$
$$\Theta=\left\{\prod_{j=1}^{N}U_j,U_j\in\bar{\Re}_j\bigcup\tilde{\Re}_j\right\},\qquad \tilde{\Theta}=\left\{\prod_{j=1}^{N}U_j,U_j\in\tilde{\Re}_j\right\}.$$

There are $\prod_{i=1}^{N}(2D_i+1)$ elements and $\prod_{i=1}^{N}(D_i+1)$ elements in $\Theta$ and $\tilde{\Theta}$, respectively. Let $\check{s}_i=\sum_{j=1,j\neq i}^{N}\min(\check{c}_{ij}p_j,\hat{c}_{ij}p_j,\check{c}_{ij}q_j,\hat{c}_{ij}q_j)+\check{d}_i$, $\hat{s}_i=\sum_{j=1,j\neq i}^{N}\max(\check{c}_{ij}p_j,\hat{c}_{ij}p_j,\check{c}_{ij}q_j,\hat{c}_{ij}q_j)+\hat{d}_i$. Set $\check{\aleph}_i(\tau,z_i(\tau))=-a_i(\tau)z_i(\tau)+c_{ii}(\tau)h_i(z_i(\tau))+\check{s}_i$, $\hat{\aleph}_i(\tau,z_i(\tau))=-a_i(\tau)z_i(\tau)+c_{ii}(\tau)h_i(z_i(\tau))+\hat{s}_i$, $\aleph_i(\tau,z_i(\tau))=-a_i(\tau)z_i(\tau)+\sum_{j=1}^{N}c_{ij}(\tau)h_j(z_j(\tau))+d_i(\tau)$. From these above definitions, $\check{\aleph}_i(\tau,z_i(\tau))\leqslant{}^{C}D^{\beta}z_i(\tau)=\aleph_i(\tau,z_i(\tau))\leqslant\hat{\aleph}_i(\tau,z_i(\tau))$, $\tau\neq\tau_k,k\in\mathbb{Z}_+$.

**Lemma 6.** Suppose (B1) and (B2) are true. Furthermore, if the the following condition holds:

$$\check{\aleph}_i(\tau, s_i^j) > 0, \quad \hat{\aleph}_i(\tau, r_i^j) < 0, \quad \tau \geqslant \tau_0,\ \tau \neq \tau_k,$$

then we can find the positive invariant set $\prod_{j=1}^N (D_j + 1)$ for system (1) in $\tilde{\Theta}$.
*Proof.* $\check{z}(\tau), \bar{z}(\tau)$ and $\hat{z}(\tau)$ show solutions for the following equantions, respectively:

$${}^C D^\beta z_i(\tau) = \check{\aleph}_i(\tau, z_i(\tau)), \quad {}^C D^\beta z_i(\tau) = \aleph_i(\tau, z_i(\tau)), \quad {}^C D^\beta z_i(\tau) = \hat{\aleph}_i(\tau, z_i(\tau)),$$

with the same initial value $(\tau_0, z_0), \tau_0 \in \tilde{\Theta}$, respectively. From Lemma 2, when $\tau \in (\tau_k, \tau_{k+1}]$, we obtain

$$\begin{aligned}
\check{z}_i(\tau) =& z_{i0} + \sum_{j=1}^k \left( b_{ij}(\check{z}_i(\tau_j^-)) + \frac{1}{\Gamma(\beta)} \int_{\tau_{j-1}}^{\tau_j} (\tau_j - 1)^{\beta-1} \check{\aleph}_i(s, \check{z}_i(s)) \mathrm{d}s \right) \\
&+ \frac{1}{\Gamma(\beta)} \int_{\tau_k}^{\tau} (\tau - 1)^{\beta-1} \check{\aleph}_i(s, \check{z}_i(s)) \mathrm{d}s, \\
\hat{z}_i(\tau) =& z_{i0} + \sum_{j=1}^k \left( b_{ij}(\hat{z}_i(\tau_j^-)) + \frac{1}{\Gamma(\beta)} \int_{\tau_{j-1}}^{\tau_j} (\tau_j - 1)^{\beta-1} \hat{\aleph}_i(s, \hat{z}_i(s)) \mathrm{d}s \right) \\
&+ \frac{1}{\Gamma(\beta)} \int_{\tau_k}^{\tau} (\tau - 1)^{\beta-1} \hat{\aleph}_i(s, \hat{z}_i(s)) \mathrm{d}s, \\
\bar{z}_i(\tau) =& z_{i0} + \sum_{j=1}^k \left( b_{ij}(\bar{z}_i(\tau_j^-)) + \frac{1}{\Gamma(\beta)} \int_{\tau_{j-1}}^{\tau_j} (\tau_j - 1)^{\beta-1} \bar{\aleph}_i(s, \bar{z}_i(s)) \mathrm{d}s \right) \\
&+ \frac{1}{\Gamma(\beta)} \int_{\tau_k}^{\tau} (\tau - 1)^{\beta-1} \bar{\aleph}_i(s, \bar{z}_i(s)) \mathrm{d}s.
\end{aligned}$$

Hence,

$$\check{z}_i(\tau) \leqslant \bar{z}_i(\tau) \leqslant \hat{z}_i(\tau). \tag{5}$$

Next, we will prove $\hat{z}_i(\tau) < r_i^{(j)}, \tau \neq \tau_k, \tau \geqslant \tau_0$. Otherwise, we can find $\hat{\tau} > \tau_0, \hat{\tau} \neq \tau_k$ which satisfies $\hat{z}_i(\hat{\tau}) = r_i^{(j)}$. Set $\hat{\omega}_i(\tau) = \hat{z}_i(\tau) - r_i^{(j)}$, and then

$$\begin{cases} \hat{\omega}_i(\tau) = 0, & \tau = \hat{\tau}, \\ \hat{\omega}_i(\tau) < 0, & \tau_0 \leqslant \tau < \hat{\tau},\ \hat{\tau} \neq \tau_k. \end{cases}$$

Considering Lemma 3, it follows ${}^C D^\beta \hat{\omega}_i(\hat{\tau}) = {}^C D^\beta \hat{z}_i(\hat{\tau}) = \hat{\aleph}_i(\hat{\tau}, r_i^{(j)}) > 0, \hat{\tau} \in (\tau_0, \tau_1) \bigcup_{k \in \mathbb{Z}_+} (\tau_k, \tau_{k+1})$. This is inconsistent with $\hat{\aleph}_i(\tau, r_i^j) < 0, \tau \geqslant \tau_0$. Therefore

$$\hat{z}_i(\tau) < r_i^j, \quad \tau \geqslant \tau_0. \tag{6}$$

Suppose $\check{\tau} > \tau_0, \check{\tau} \neq \tau_k$ and $\check{z}_i(\check{\tau}) = s_i^j$. Letting $\check{\omega}_i(\tau) = s_i^{(j)} - \check{z}_i(\tau)$, hence

$$\begin{cases} \check{\omega}_i(\tau) = 0, & \tau = \check{\tau}, \\ \check{\omega}_i(\tau) < 0, & \tau_0 \leqslant \tau < \check{\tau},\ \check{\tau} \neq \tau_k. \end{cases}$$

Considering Lemma 3, then ${}^C D^\beta \check{\omega}_i(\check{\tau}) = -{}^C D^\beta \check{z}_i(\check{\tau}) = -\check{\aleph}_i(\check{\tau}, s_i^{(j)}) > 0,\ \check{\tau} \in (\tau_0, \tau_1) \bigcup_{k \in \mathbb{Z}_+} (\tau_k, \tau_{k+1})$. This contradicts with $\check{\aleph}_i(s_i^j) > 0, \tau \geqslant \tau_0$. Hence

$$\check{z}_i(\tau) > s_i^{(j)}, \quad \tau \geqslant \tau_0. \tag{7}$$

From (5)–(7), we have $s_i^j \leqslant \bar{z}_i(\tau) \leqslant r_i^j,\ \tau \geqslant \tau_0$. Because the number of elements in $\tilde{\Theta}$ is $\prod_{j=1}^N (D_j + 1)$, there are $\prod_{j=1}^N (D_j + 1)$ positive invariants sets for (1).

**Remark 1.** Different from systems in [14, 34], the system discussed in this paper involves impulses, in which the state functions $z_i(\tau)$ are discontinuous in $[\tau_0, \infty)$. Therefore, the method used for the positive invariant sets in [14] cannot be used up directly to this paper. The conditions $\check{\aleph}_i(\tau, s_i^{(j)}) > 0, \hat{\aleph}_i(\tau, r_i^{(j)}) < 0, \tau \geqslant \tau_0$ of invariant sets for the FONNs with impulses and nonautonomous parameters are dual functions, which are more complicated than those with autonomous parameters, such as in [14].

**Lemma 7** ([12]). If $f(\tau) \in C^1([\tau_0, \infty), \mathbb{R}^N)$, for $\beta \in (0,1)$, then

$$ {}_{\tau_0}^{C}D_\tau^\beta |f(\tau)| \leqslant \mathrm{sgn}(f(\tau))\, {}_{\tau_0}^{C}D_\tau^\beta f(\tau), \quad t \in [\tau_0, \infty), $$

where sgn$(\cdot)$ denotes sign$(\cdot)$.

**Theorem 1.** Suppose (B1) and (B2) are true. If the following conditions are also true, then the solution of (1) is bounded.

(B3) There is some constant $\varpi > 1$ such that $\varpi \hat{\gamma} E_\beta(-\bar{c}T^\beta) < 1$, where $\hat{\gamma} = \max_{1 \leqslant i \leqslant N, k \in \mathbb{Z}_+}(|1+\gamma_{ik}|)$.

(B4) The inequality $\sum_{i=1}^N |z_{i0}| - \frac{N\bar{\psi}}{\bar{a}} > 0$ is fulfilled, where $\check{a}_i = \min_{\tau_0 \leqslant \tau \leqslant \infty}(|a_i(\tau)|), \psi_i = \sum_{j=1}^N \max(|\check{c}_{ij}|, |\hat{c}_{ij}|) \max(|p_i|, |q_i|) + \max(|\check{d}_i|, |\hat{d}_i|)$, $\bar{a} = \min_{1 \leqslant i \leqslant N}(\check{a}_i)$ and $\bar{\psi} = \max_{1 \leqslant i \leqslant N}(\psi_i)$.

*Proof.* $z(\tau)$ denotes the solution of (1). Considering Lemma 7, when $\tau \neq \tau_k$, one has

$$ {}^C D^\beta |z_i(\tau)| \leqslant \mathrm{sgn}(z_i(\tau))\, {}^C D^\beta z_i(\tau) \leqslant \mathrm{sgn}(z_i(\tau)) \aleph_i(\tau) \leqslant -\check{a}_i |z_i(\tau)| + \psi_i, $$

where $\check{a}_i = \min_{\tau_0 \leqslant \tau \leqslant \infty}(|a_i(\tau)|), \psi_i = \sum_{j=1}^N \max(|\check{c}_{ij}|, |\hat{c}_{ij}|) \max(|p_i|, |q_i|) + \max(|\check{d}_i|, |\hat{d}_i|)$. Then

$$ {}^C D^\beta \left( \sum_{i=1}^N |z_i(\tau)| \right) \leqslant \sum_{i=1}^N \mathrm{sgn}(z_i(\tau))\, {}^C D^\beta z_i(\tau) \leqslant \sum_{i=1}^N \mathrm{sgn}(z_i(\tau)) \aleph_i(\tau) \leqslant -\bar{a} \sum_{i=1}^N |z_i(\tau)| + N\bar{\psi}, $$

where $\bar{a} = \min_{1 \leqslant i \leqslant N}(\check{a}_i)$ and $\bar{\psi} = \max_{1 \leqslant i \leqslant N}(\psi_i)$. Set $V(\tau) = \sum_{i=1}^N |z_i(\tau)| - \frac{N\bar{\psi}}{\bar{a}}$. Hence

$$ {}^C D^\beta V(\tau) \leqslant -\bar{a} V(\tau), \quad \tau \neq \tau_k. $$

From Lemma 4, we get

$$ V(\tau) \leqslant V(\tau_k) E_\alpha(-\bar{a}(\tau - \tau_k)^\alpha), \quad \tau \in (\tau_k, \tau_{k+1}]. $$

While $\tau = \tau_k$, then

$$ \sum_{i=1}^N |z_i(\tau_k^+)| - \frac{N\bar{\psi}}{\bar{a}} = \sum_{i=1}^N |1 + \gamma_{ik}||z_i(\tau_k^-)| - \frac{N\bar{\psi}}{\bar{a}} \leqslant \hat{\gamma} \left( \sum_{i=1}^N |z_i(\tau_k^-)| - \frac{N\bar{\psi}}{\bar{a}} \right) \leqslant \hat{\gamma} V(\tau_k), \tag{8} $$

where $\hat{\gamma} < 1$. Considering assumption (H4), condition (B3), and (8), we have

$$ \begin{aligned} V(\tau) &\leqslant V(\tau_0) E_\beta(-\bar{a}T^\beta) \hat{\gamma} E_\beta(-\bar{a}T^\beta) \cdots \hat{\gamma} E_\beta(-\bar{a}(\tau - \tau_k)^\beta) \\ &\leqslant V(\tau_0) E_\beta(-\bar{a}T^\beta) \hat{\gamma} E_\beta(-\bar{a}T^\beta) \cdots \hat{\gamma} E_\beta(-\bar{a}(T)^\beta) \\ &\leqslant V(\tau_0) E_\beta(-\bar{a}T^\beta) \frac{1}{\varpi^k}, \quad \tau \in (\tau_k, \tau_{k+1}], \end{aligned} $$

and therefore

$$ \sum_{i=1}^N |z_i(\tau)| \leqslant \left( \sum_{i=1}^N |z_{i0}| - \frac{N\bar{\psi}}{\bar{a}} \right) E_\beta(-\bar{a}T^\beta) \frac{1}{\varpi^k} + \frac{N\bar{\psi}}{\bar{a}}, \quad \tau \in (\tau_k, \tau_{k+1}]. $$

Therefore, all solutions with the initial value $(\tau_0, z_0)$ of (1) are bounded on $[\tau_0, \infty)$.

## 4 Asymptotic multistability and local S-asymptotic $\omega$-periodicity

In this section, asymptotic multistability and local SAP of (1) are discussed.

**Theorem 2.** Suppose assumptions (B1)–(B4) are true. Besides, if the following condition is fulfilled, and let $z^*(\tau)$ be the solution of system (1), then system (1) is of asymptotic multistability in $\tilde{\Theta}$.

(B5) $\check{a}_i - \sum_{j=1}^{N}|c_{ji}(\tau)|L_i > 0$, in which $C_{ii} = \max_{0\leqslant j\leqslant D_i}(\hat{c}_{ii}\check{\lambda}_i^j, \hat{c}_{ii}\hat{\lambda}_i^j), L_i = \max_{0\leqslant j\leqslant D_i}(|\check{\lambda}_i^j|, |\hat{\lambda}_i^j|)$.

*Proof.* $z(\tau) = [z_1(\tau), \ldots, z_N(\tau)]^{\mathrm{T}}$ denotes the solution of (1) with $z(\tau_0) = z_0 \in \tilde{\Theta}$, which is different from $z^*(\tau)$. Set $\bar{V}(\tau) = \sum_{i=1}^{N}|z_i(\tau) - z_i^*(\tau)|$. From Lemma 6, when $\tau \neq \tau_k$, we have

$$\begin{aligned}
{}^{C}D^{\beta}\bar{V}(\tau) &\leqslant \sum_{i=1}^{N} {}^{C}D^{\beta}|z_i(\tau) - z_i^*(\tau)| \\
&\leqslant \sum_{i=1}^{N} \operatorname{sgn}(z_i(\tau) - z_i^*(\tau)) \left(-a_i(\tau)\left(z_i(\tau) - z_i^*(\tau)\right) + \sum_{j=1}^{N} c_{ij}(\tau)\left(h_j(z_j(\tau)) - h_j(z_j^*(\tau))\right)\right) \\
&\leqslant \sum_{i=1}^{N}\left(-\check{a}_i|z_i(\tau) - z_i^*(\tau)| + \sum_{j=1}^{N}|c_{ij}(\tau)|L_j|(z_j(\tau)) - z_j^*(\tau)|\right) \\
&\leqslant \sum_{i=1}^{N}\left(-\lambda_i|z_i(\tau) - z_i^*(\tau)|\right) \leqslant -\lambda\bar{V}(\tau),
\end{aligned}$$

where $\check{a}_i = \min_{\tau_0\leqslant\tau<\infty}(a_i(\tau)), \lambda_i = \check{a}_i - \sum_{j=1}^{N}|c_{ji}(\tau)|L_i, \lambda = \max_{i=1,2,\ldots,N}(\lambda_i)$. From Lemma 4, we get $\bar{V}(\tau) \leqslant \bar{V}(\tau_k)E_{\beta}(-\lambda(\tau-\tau_k)^{\beta}),\ \tau \in (\tau_k, \tau_{k+1}]$. Then

$$\begin{aligned}
\bar{V}(\tau) &\leqslant \hat{\gamma}\bar{V}(\tau_k^-)E_{\beta}(-\lambda(\tau-\tau_k)^{\beta}) \\
&\leqslant \bar{V}(\tau_0)E_{\beta}(-\lambda T^{\beta})\hat{\gamma}E_{\beta}(-\lambda T^{\beta})\cdots\hat{\gamma}E_{\beta}(-\lambda(\tau-\tau_k)^{\beta}) \\
&\leqslant \bar{V}(\tau_0)E_{\beta}(-\lambda T^{\beta})\frac{1}{\varpi^{k-1}}\hat{\gamma}E_{\beta}(-\lambda(\tau-\tau_k)^{\beta}), \quad \tau \in (\tau_k, \tau_{k+1}],
\end{aligned}$$

where $\hat{\gamma} = \max_{1\leqslant i\leqslant N, k\in\mathbb{Z}_+}(|1+\gamma_{ik}|)$. When $\tau = \tau_k$, one has $\bar{V}(\tau_k^-) \leqslant \bar{V}(\tau_0)E_{\beta}(-\lambda T^{\beta})\frac{1}{\varpi^{k-1}}$. Then we obtain $\bar{V}(\tau_k^+) = |z_i(\tau_k^+) - z_i^*(\tau_k^+)| \leqslant \hat{\gamma}|z_i(\tau_k^-) - z_i^*(\tau_k^-)| \leqslant \hat{\gamma}\bar{V}(\tau_0)E_{\beta}(-\lambda T^{\beta})\frac{1}{\varpi^{k-1}}$, and

$$\bar{V}(\tau) \leqslant \bar{V}(\tau_0)E_{\beta}(-\lambda T^{\beta})\frac{1}{\varpi^{k}}, \quad \tau \in [\tau_k, \tau_{k+1}].$$

Therefore, we have $|z(\tau) - z^*(\tau)| \leqslant \frac{1}{\varpi^{k-1}}|z_0 - z_0^*|E_{\beta}(-\lambda T^{\beta}),\ \tau \in [\tau_k, \tau_{k+1}]$, which shows solution for (1) is asymptotically stable. From Lemma 6, there are $\prod_{j=1}^{N}(D_j+1)$ positive invariant sets in $\tilde{\Theta}$. Therefore, system (1) is asymptotically multistable.

**Remark 2.** Compared with the results on Mittag-Leffler multistability of autonomous FONNs [14], our asymptotic stability for the nonautonomous FONNs with impulses is more realistic. Besides, because of the discontinuity of $z_i(\tau)$, the method used in [14] cannot be directly utilized in this model. Therefore, our results are new.

**Theorem 3.** Let (B1)–(B5) hold. Furthermore, suppose that the following condition is fulfilled, then the solutions of piecewise succession for system (1) are locally S-asymptotically $\omega$-periodic in $\tilde{\Theta}$. And all solutions of (1) in $\tilde{\Theta}$ tend asymptotically to the $\omega$-periodic nonconstant solution.

(B6) $\sum_{i=1}^{N}|z_{i0} - z_i(\tau_0+\omega)| - \frac{\hat{\upsilon}}{\lambda} > 0$, where $\lambda = \sum_{i=1}^{N}\check{a}_i - \max_{\tau_0\leqslant\tau<\infty}(\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}|c_{ij}(\tau)|L_j), \hat{\upsilon} = \max_{\tau_0\leqslant\tau<\infty}\sum_{i=1}^{N}(d_i(\tau) - d_i(\tau+\omega)) + \frac{\sum_{i=1}^{N}\bar{\aleph}_i}{\Gamma(1+\beta)}\theta$.

*Proof.* The proof process is given by two steps.

Step 1. We prove the solutions in $\tilde{\Theta}$ are all locally S-asymptotically $\omega$-periodic. $z(\tau) = [z_1(\tau), \ldots, z_N(\tau)]^{\mathrm{T}}$ stands for a solution of (1) in $\tilde{\Theta}$ with $z(\tau_0) = z_0 = [z_1^0, \ldots, z_N^0] \in \tilde{\Theta}$. From Lemma 2, when $\tau \in (\tau_k, \tau_{k+1}]$, we have

$$\begin{aligned}
z_i(\tau) = {} & z_{i0} + \sum_{j=1}^{k}\left(b_{ij}(z_i(\tau_j^-)) + \int_{\tau_{j-1}}^{\tau_j}(t_j - s)^{\beta-1}\aleph_i(s, z_i(s))\mathrm{d}s\right) \\
&+ \frac{1}{\Gamma(\beta)}\int_{\tau_k}^{\tau+\omega}(\tau+\omega-s)^{\beta-1}\aleph_i(s, z_i(s))\mathrm{d}s,
\end{aligned}$$

$$\begin{aligned} z_i(\tau+\omega) =&\ z_{i0} + \sum_{j=1}^{[\frac{\tau+\omega}{T}]} \left( b_{ij}(z_i(\tau_j^-)) + \frac{1}{\Gamma(\beta)} \int_{\tau_{j-1}}^{\tau_j} (\tau_j - s)^{\beta-1} \aleph_i(s, z_i(s)) \mathrm{d}s \right) \\ &+ \frac{1}{\Gamma(\beta)} \int_{\tau_{[\frac{\tau+\omega}{T}]}}^{\tau+\omega} (\tau+\omega-s)^{\beta-1} \aleph_i(s, z_i(s)) \mathrm{d}s \\ =&\ z_{i0} + \sum_{j=1}^{[\frac{\tau+\omega}{T}]} \left( b_{ij}(z_i(\tau_j^-)) + \frac{1}{\Gamma(\beta)} \int_{\tau_{j-1}}^{\tau_j} (\tau_j - s)^{\beta-1} \aleph_i(s, z_i(s)) \mathrm{d}s \right) \\ &+ \frac{1}{\Gamma(\beta)} \int_{\tau_{[\frac{\tau+\omega}{T}]-\omega}}^{\tau} (\tau-s)^{\beta-1} \aleph_i(s+\omega, z_i(s+\omega)) \mathrm{d}s. \end{aligned}$$

Hence,

$$^{C}D^{\beta}(z_i(\tau) - z_i(\tau+\omega)) = -a_i(\tau)\left(z_i(\tau) - z_i(\tau+\omega)\right) + \sum_{j=1}^{N} c_{ij}(\tau)\left(h_j(z_i(\tau)) - h_j(z_j(\tau+\omega))\right) + {}^{C}D^{\beta}\bar{d}_i(\tau),$$

where

$$\bar{d}_i(\tau) = \sum_{j=k+1}^{[\frac{\tau+\omega}{T}]} \left( b_{ij}(z_i(\tau_j^-)) + \int_{\tau_{j-1}}^{\tau_j} (\tau_j - s)^{\beta-1} \aleph_i(s) \mathrm{d}s \right) + \int_{\tau_{[\frac{\tau+\omega}{T}]-\omega}}^{\tau_k} (\tau-s)^{\beta-1} \aleph_i(s+\omega, z_i(s+\omega)) \mathrm{d}s.$$

From Lemma 7, it follows

$$^{C}D^{\beta}|z_i(\tau) - z_i(\tau+\omega)| \leqslant \phi_i(\tau) + {}^{C}D^{\beta}|\bar{d}_i(\tau)|, \tag{9}$$

where $\phi_i(\tau) = -\check{a}_i|z_i(\tau) - z_i(\tau+\omega)| + \sum_{j=1}^{N}|c_{ij}(\tau)|L_j|z_i(\tau) - z_i(\tau+\omega)| + |d_i(\tau) - d_i(\tau+\omega)|$. By computation, we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}\tau}\bar{d}_i(\tau)\right| \leqslant \frac{1-\beta}{\Gamma(\beta)} \int_{\tau_{[\frac{\tau+\omega}{T}]-\omega}}^{\tau_k} (\tau-s)^{\beta-2} |\aleph_i(s+\omega, z_i(s+\omega))| \mathrm{d}s.$$

Based on Lemma 6, we can find a constant $\bar{\aleph}_i > 0$ which satisfies $\sup_{\tau_0 \leqslant \tau < \infty} |\aleph_i(\tau, z_i(\tau))| \leqslant \bar{\aleph}_i$. From the above inequality, for $\tau \in (\tau_k, \tau_{k+1}]$, one can obtain

$$\begin{aligned} ^{C}D^{\beta}|\bar{d}_i(\tau)| &\leqslant \frac{1}{\Gamma(1-\beta)} \int_{\tau_k}^{\tau} (\tau-v)^{-\beta} \left|\frac{\mathrm{d}}{\mathrm{d}v} d_i(v)\right| \mathrm{d}v \\ &\leqslant \frac{1}{\Gamma(1-\beta)} \int_{\tau_k}^{\tau} (\tau-v)^{-\beta} \frac{1-\beta}{\Gamma(\beta)} \int_{\tau_{[\frac{\tau+\omega}{T}]-\omega}}^{\tau_k} (v-s)^{\beta-2} |\aleph_i(s+\omega, z_i(s+\omega))| \mathrm{d}s\mathrm{d}v \\ &\leqslant \frac{\bar{\aleph}_i}{\Gamma(1-\beta)\Gamma(\beta)} \int_{\tau_k}^{\tau} (\tau-v)^{-\beta} \left( (v-t_k)^{\beta-1} - (v+\omega-\tau_{[\frac{t+\omega}{T}]})^{\beta-1} \right) \mathrm{d}v \\ &\leqslant \frac{\bar{\aleph}_i}{\Gamma(1+\beta)} {}^{C}D^{\beta} \left( (\tau-\tau_k)^{\beta} - (\tau+\omega-\tau_{[\frac{\tau+\omega}{T}]})^{\beta} \right). \end{aligned} \tag{10}$$

From (9) and (10), we get

$$^{C}D^{\beta}|z_i(\tau) - z_i(\tau+\omega)| \leqslant \phi_i(\tau) + \frac{\bar{\aleph}_i}{\Gamma(1+\beta)} {}^{C}D^{\beta} \left( (\tau-\tau_k)^{\beta} - (\tau+\omega-\tau_{[\frac{\tau+\omega}{T}]})^{\beta} \right).$$

Setting $\nu = \tau - \tau_k, \tau \in (\tau_k, \tau_{k+1}]$, one can get

$${}_{\tau_k}^{C}D_{\tau}^{\beta} \left( (\tau-\tau_k)^{\beta} - (\tau+\omega-\tau_{[\frac{\tau+\omega}{T}]})^{\beta} \right) = {}_{0}^{C}D_{\nu}^{\beta} \left( \nu^{\beta} - (\nu+\omega-\tau_{[\frac{\nu+\omega}{T}]})^{\beta} \right), \quad \nu \in (0, T],$$

which implies that there is a positive constant $\theta$ satisfying ${}_{\tau_k}^{C}D_{\tau}^{\beta}((\tau-\tau_k)^{\beta}-(\tau+\omega-\tau_{[\frac{\tau+\omega}{T}]})^{\beta})\leqslant\theta$. Let $\tilde{V}(\tau)=\sum_{i=1}^{N}|z_i(\tau)-z_i(\tau+\omega)|$. Hence,

$$\begin{aligned}
{}^{C}D^{\beta}\tilde{V}(\tau)&=\sum_{i=1}^{N}{}^{C}D^{\beta}|z_i(\tau)-z_i(\tau+\omega)|\\
&\leqslant\sum_{i=1}^{N}\left(\phi_i(\tau)+\frac{\bar{\aleph}_i}{\Gamma(1+\beta)}{}^{C}D^{\beta}\left((\tau-\tau_k)^{\beta}-(\tau+\omega-\tau_{[\frac{\tau+\omega}{T}]})^{\beta}\right)\right)\\
&\leqslant-\sum_{i=1}^{N}\left(\breve{a}_i-\sum_{j=1,j\neq i}^{N}|c_{ij}(\tau)|L_j\right)\bar{V}(\tau)+\sum_{i=1}^{N}\left((d_i(\tau)-d_i(\tau+\omega))+\frac{\bar{\aleph}_i}{\Gamma(1+\beta)}\theta\right)\\
&\leqslant-\lambda\tilde{V}(\tau)+\hat{\upsilon},
\end{aligned}$$

where $\lambda=\sum_{i=1}^{N}\breve{a}_i-\max_{\tau_0\leqslant\tau<\infty}(\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}|c_{ij}(\tau)|L_j)$, $\hat{\upsilon}=\max_{\tau_0\leqslant\tau<\infty}\sum_{i=1}^{N}(d_i(\tau)-d_i(\tau+\omega))+\frac{\sum_{i=1}^{N}\bar{\aleph}_i}{\Gamma(1+\beta)}\theta$. Let $W(\tau)=\tilde{V}(\tau)-\frac{\hat{\upsilon}}{\lambda}$, and then

$$ {}^{C}D^{\beta}W(\tau)\leqslant-\lambda W(\tau). \tag{11}$$

When $\tau=\tau_k$,

$$W(\tau_k^{+})=\tilde{V}(\tau_k^{+})-\frac{\hat{\upsilon}}{\lambda}\leqslant\hat{\gamma}\tilde{V}(\tau_k^{-})-\frac{\hat{\upsilon}}{\lambda}\leqslant\hat{\gamma}W(\tau_k^{-}), \tag{12}$$

where $\hat{\gamma}=\max_{1\leqslant i\leqslant N,k\in\mathbb{Z}_+}(\gamma_{ik})<1$. By Lemma 4 and (11), we get $W(\tau)\leqslant W(\tau_k)E_{\beta}(-\lambda(\tau-\tau_k)^{\beta}),\tau\in(\tau_k,\tau_{k+1}]$, which means that

$$\begin{aligned}
W(\tau)&\leqslant W(\tau_0)E_{\beta}(-\lambda T^{\beta})\hat{\gamma}E_{\beta}(-\lambda T^{\beta})\cdots\hat{\gamma}E_{\beta}(-\lambda(\tau-\tau_k)^{\beta})\\
&\leqslant W(\tau_0)E_{\beta}(-\lambda T^{\beta})\frac{1}{\varpi^{k-1}}\hat{\gamma}E_{\beta}(-\lambda(\tau-\tau_k)^{\beta}).
\end{aligned}\tag{13}$$

Therefore, from (12) and (13), we obtain $W(\tau)\leqslant W(\tau_0)E_{\beta}(-\lambda T^{\beta})\frac{1}{\varpi^{k}}$, $\tau\in[\tau_k,\tau_{k+1}]$. Therefore, all the solutions of piecewise succession for (1) are locally S-asymptotically $\omega$-periodic in $\tilde{\Theta}$.

Step 2. Now we will explore the asymptotical periodicity of (1). Let $z(\tau)$ denote the solution of (1) in $\tilde{\Theta}$. Then $\{z(\tau+k\omega)\}_{k\in\mathbb{N}}$ is uniformly bounded and equi-continuous. From the diagonal selection principle and Arzela-Ascoli theorem, there is a sub-sequence $\{k_1\omega\}_{k_1\in\mathbb{N}}$ of $\{k\omega\}_{k\in\mathbb{N}}$, which is uniformly convergent and the limit $z^*(\tau)$ is $\omega$-periodically non-constant. Hence, the solution $y(\tau)$ for (1) with the same initial conditions in $\tilde{\Theta}$ satisfies

$$\|y(\tau)-z^*(\tau)\|\leqslant\|y(\tau)-z(\tau)\|+\|z(\tau)-z(\tau+k_1\omega)\|+\|z(\tau+k_1\omega)-z^*(\tau)\|.$$

From the asymptotic stability, SAP of piecewise succession and the definition of $z^*(\tau)$, we can get that $\|y(\tau)-z(\tau)\|$, $\|z(\tau)-z(\tau+k_1\omega)\|$, and $\|z(\tau+k_1\omega)-z^*(\tau)\|$ all tend to zero, that is $\lim_{\tau\to\infty}\|y(\tau)-z^*(\tau)\|=0$. Namely, $-\kappa<z_i^*(\tau)-y_i(\tau)<\kappa$, then $s_i-\kappa<y_i(\tau)-\kappa<z_i^*(\tau)<y_i(\tau)+\kappa<r_i+\kappa$, where $\kappa$ is a sufficiently small positive constant with $\lim_{\tau\to\infty}\kappa=0$. Thus $s_i\leqslant z_i^*(\tau)\leqslant r_i$, which implies $z_i^*(\tau)\in\tilde{\Theta}$. Considering the periodicity of $z_i^*(\tau)$, we have that $z_i^*(\tau)$ is locally asymptotically $\omega$-periodic. Therefore, system (1) is locally asymptotical $\omega$-periodic.

**Remark 3.** We can easily find some functions that satisfy the condition (B6), for instance, $I_i(\tau)=\sin(2\pi\tau),\lim_{\tau\to\infty}|\sin(2\pi\tau)-\sin(2\pi(\tau+1))|=0$.

**Remark 4.** We investigate local SAP for (1), by blocking the state space, determining the positive invariant sets and adopting the Lyapunov direct method. Our methods are standard with some novelty. Especially, when $D_i=0$, we can only find a positively invariant set in $\tilde{\Theta}$ and the global asymptotic stability is a special case of our results.

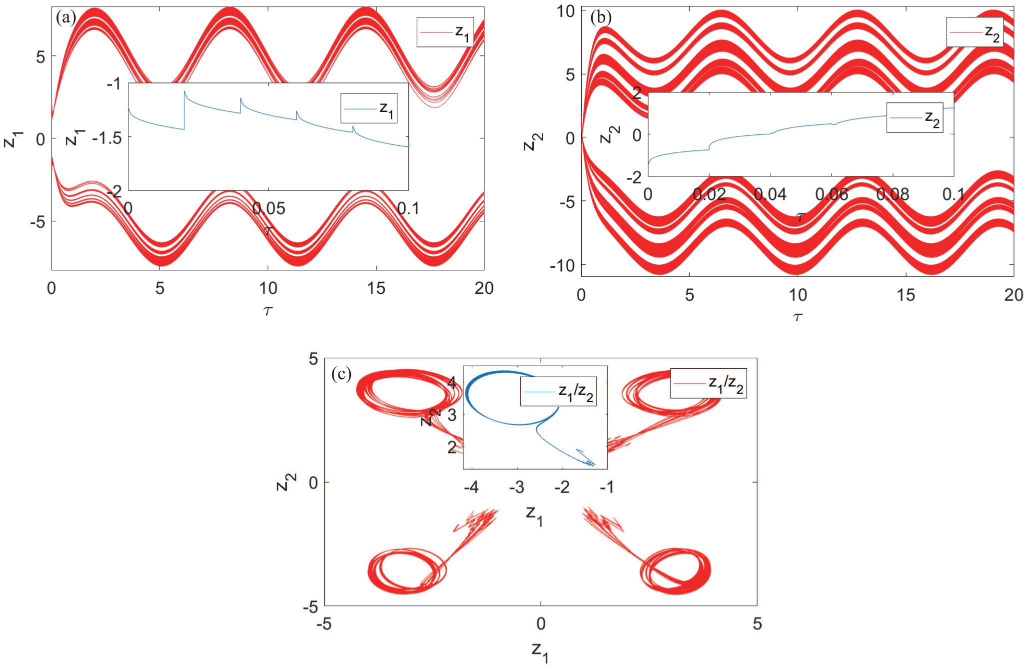


**Figure 1** (Color online) The transient of the states $z_i(\tau)$ $(i=1,2)$ of Example 1 for $\beta=0.5$.

## 5 Simulations

**Example 1.** A two-neuron FONN with impulses and nonautonomous arguments is presented as

$$\begin{cases} {}^C D^{\beta} z_i(\tau) = -a_i(\tau)z_i(\tau) + \sum_{j=1}^{2} c_{ij}(\tau)h_j(z_j(\tau)) + d_i(\tau), \quad i=1,2, \ \beta\in(0,1), \\ \Delta z_i(\tau_k^+) = z_i(\tau_k^+) - z_i(\tau_k^-) = b_{ik}(z_i(\tau_k^-)), \quad k\in\mathbb{Z}_+, \end{cases} \tag{14}$$

where $d_1(\tau)=0.5\sin(\tau), d_2(\tau)=-0.48\sin(\tau), a_i(\tau)=0.95+0.01\sin(\tau), h_i(z_i(\tau))=\tanh(z_i(\tau)), i=1,2, b_{ik}(z_i(\tau_k^-))=-\frac{1}{(k+1)^2}z_i(\tau_k^-), i=1,2, k\in\mathbb{Z}_+$, and $c_{11}=2.5+0.01\sin(\tau), c_{12}=0.08\sin(\tau), c_{21}=0.06\sin(\tau), c_{22}=2.8+0.01\sin(\tau)$. It is obvious that $p_i=-1, q_i=1, D_i=1, \check{a}_i=0.0.94, \hat{a}_i=0.96, i=1,2, \check{d}_1=-0.5, \hat{d}_1=0.5, \check{d}_2=-0.48, \hat{d}_2=0.48$ and $\hat{c}_{11}=2.51, \check{c}_{11}=2.49, \hat{c}_{12}=0.08, \check{c}_{12}=-0.08, \hat{c}_{21}=0.06, \check{c}_{21}=-0.06, \hat{c}_{22}=2.81, \check{c}_{22}=2.79$. By computation, for $i=1,2$, we derive $\bar{\Re}_i=\{[-1,1]\}, \tilde{\Re}_i=\{(-\infty,-1),(1,\infty)\}$. From Lemma 6, there exist $\sum_{i=1}^{2}(1+D_1)=4$ elements in $\tilde{\Theta}$, which are invariant sets of the solution of system (14). Besides, $\check{\lambda}_i^j=0, \hat{\lambda}_i^j=\max_{\tau\in(s_i^j,r_i^j)}\frac{\mathrm{d}(\tanh(\tau))}{\mathrm{d}\tau}=0.42$, where $(s_i^0,r_i^0)=(-\infty,-1),(s_i^1,r_i^1)=(1,\infty), i=1,2, j=0,1$, which means that the conditions of (B5) and (B6) are satisfied. According to Theorems 2 and 3, system (14) is of asymptotic multistability and local SAP. Figure 1 shows that our results are effective.

In the following, we present a globally asymptotically stable example.

**Example 2.** Consider a two-neuron FONN with impulses and nonautonomous parameters:

$$\begin{cases} {}^C D^{\beta} z_i(\tau) = -a_i(\tau)z_i(\tau) + \sum_{j=1}^{2} c_{ij}(\tau)h_j(z_j(\tau)) + d_i(\tau), \quad i=1,2, \ \tau\neq\tau_k, \ k\in\mathbb{Z}_+, \\ \Delta z_i(\tau_k^+) = z_i(\tau_k^+) - z_i(\tau_k^-) = b_{ik}(z_i(\tau_k^-)), \quad k\in\mathbb{Z}_+, \end{cases} \tag{15}$$

where $d_1=0.52\sin(\tau), d_2=0.5\sin(\tau), a_i(\tau)=1.1+0.01\sin(\tau), h_i(z_i(\tau))=\tanh(z_i(\tau)), b_{ik}(z_i(\tau_k^-))=-\frac{1}{k^2}z_i(\tau_k^-), i=1,2, k\in\mathbb{Z}_+$, and $c_{11}=-2.4+0.01\sin(\tau), c_{12}=0.1\sin(\tau), c_{21}=0.09\sin(\tau), c_{22}=-2.5+0.01\sin(\tau)$. By Theorems 2 and 3, Eq. (15) is globally asymptotically stable and globally asymptotically periodic. Figure 2 with 40 random initial values shows the transient of the states $z_i, i=1,2$ of system (15). The result shows that global asymptotic stability is a special case of asymptotic multistability.

**Remark 5.** Comparing with [14], our model with nonautonomous arguments and impulses is more complicated, and the previous results do not work in this case. Therefore, our results are new.

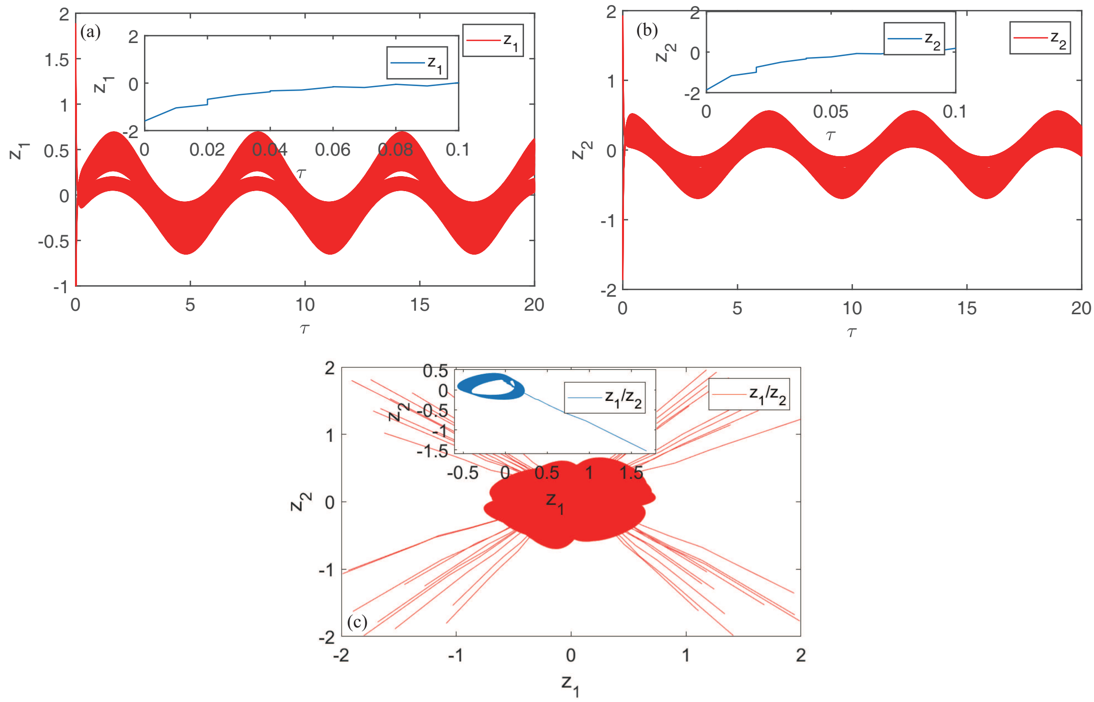


**Figure 2** (Color online) The transient of the states $z_i(\tau)$ $(i = 1, 2)$ of Example 2 for $\beta = 0.5$.

## 6 Conclusion

In this paper, asymptotic multistability and SAP of the nonautonomous FONNs with impulses are discussed. By constructing convergent series, the existence and uniqueness of the nonautonomous FONNs with impulses are obtained. In addition, the boundness and asymptotic multistability of this kind of systems are investigated by Lyapunov direct method. Some new sufficient conditions on the local SAP are derived. We will extend our results to fractional-order coupled system on a network with/without strong connectedness [36] in future.

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