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Stability analysis for semi-Markovian switched stochastic systems with asynchronously impulsive jumps

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Abstract The almost surely (a.s.) exponential stability is studied for semi-Markovian switched stochastic systems with randomly impulsive jumps. We start from the case that switches and impulses occur synchronously, in which the impulsive switching signal is a semi-Markovian process. For the case that switches and impulses occur asynchronously, the impulsive arrival time sequence and the types of jump maps are driven by a renewal process and a Markov chain, respectively. By applying the multiple Lyapunov function approach, sufficient conditions of exponential stability a.s. are obtained based upon the ergodic property of semi-Markovian process. The validity of the proposed theoretical results is demonstrated by a numerical example.

Keywords stochastic systems, multiple Lyapunov function, almost surely exponential stability, impulsive jumps, semi-Markovian switching, renewal process

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1 Introduction

Stochastic hybrid systems represent a class of dynamical systems involving continuous evolution, instantaneous change and random effects. As two important types of stochastic hybrid systems, randomly switched systems and randomly impulsive systems have received increasing interests over the past decades; see [1–6] for randomly switched systems, [7, 8] for randomly impulsive systems. Randomly switched systems are composed of a family of subsystems and a random switching signal orchestrating the switching among the subsystems. Recently, randomly switched systems have received increasing research interests which involve stability analysis [1, 3–6], controller synthesis [2], fault detection [9], and filter design [10]. Randomly impulsive systems combine continuous-time dynamics with abrupt state changes occurring at random moments. In practice, randomly switched systems and randomly impulsive systems have a variety of applications in diverse areas such as networked control systems, biological systems, air traffic management systems and economic systems [11–13].

In the real world, dynamic systems may encounter sharp state jumps and abrupt dynamical changes, which cannot be well modeled as purely switched systems or purely impulsive systems. Thus, a more comprehensive model, namely, impulsive switched system [14–18], is proposed to characterize switched systems with impulsive jumps at switching instants. In this case, the switching signal and impulsive signal can be integrated as an impulsive switching signal [19]. For instance, the synchronization problem of nonlinear systems was discussed in [14] by utilizing hybrid impulsive and switching control. For randomly impulsive switching signals, the impulsive switching time sequence is a stochastic process. And Markovian impulsive switching signal is an important class of them, in which the impulsive switching signal is modeled by a Markovian process [16]. It is worth mentioning that the sojourn time for each subsystem is required to obey the exponential distribution in Markovian impulsive switched systems. To

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remove this restriction, we introduce the concept of semi-Markovian process (see [20,21]), which has been referred in switched systems [4,6,22–27], to describe a more general impulsive switching signal. For semi-Markovian impulsive switching signal, the sojourn times of subsystems do not necessarily obey exponential distribution and transition rates are time-varying. Up to now, researches about semi-Markovian impulsive switched systems are few.

On the other hand, switches and impulses can also occur asynchronously in numerous physical and man-made systems. For example, under switching communication topologies, networked systems with continuous-time and impulsive communications can be described as switched systems with asynchronous impulsive jumps [28]. It is known that the stability of switched stochastic systems with randomly impulsive jumps can be impacted by different types of subsystems and impulsive jumps in different manners [16, 18], that is, some subsystems and impulsive jumps can make positive effects on stability, and others can potentially destroy stability. The impulsive switching frequency in impulsive switched systems can be restricted by utilizing dwell-time conditions such as fixed dwell-time condition in [16] and average dwell-time condition in [18]. However, this approach cannot be applied to switched systems with asynchronous impulsive jumps directly, because there exist two kinds of discrete-time signals, namely, switching and impulses. To overcome this difficulty, Ref. [29] provided the stabilizing impulses condition for Makovian switched systems with asynchronous impulsive jumps. Unfortunately, this condition is not available for semi-Makovian switched systems in the presence of stabilizing and destabilizing impulses. Therefore, an effective method is needed to tackle and quantify the mixed effects of subsystem's stability, switches, impulses, and stochastic noises for the whole system's stability.

In fact, numerous practical systems are often perturbed by stochastic noises. For instance, air traffic management systems are often subject to mode-switching and various stochastic disturbances such as wind and air turbulence [30]; financial systems may undergo sudden changes of volatility rates [13]. Stochastic noise also can be used to stabilize an unstable system. In [31], the stochastic stabilization problem of continuous-time nonlinear systems was addressed by artificial multiplicative noise based on aperiodically sampled data. In general, stochastic noises can be modeled as the Brownian motion. Switched systems with the effects of stochastic noises are also known as switched stochastic systems [4, 32]. It is natural to consider a more general system model, i.e., switched stochastic systems with impulsive jumps [16, 29, 33]. Moreover, impulsive jumps do not always occur at fixed times, which means that the states of systems may change instantaneously at random times. In order to simulate this phenomenon, random impulsive systems, in which the impulsive time sequence is a stochastic process, were introduced in some literature (see [7,8]). However, there is no work on semi-Markovian switched stochastic systems with asynchronously randomly impulsive jumps.

Motivated by above discussion, we aim to investigate the exponential stability almost surely (a.s.) problem for semi-Markovian switched stochastic systems with synchronously and asynchronously randomly impulsive jumps, respectively. For the case that switches and impulses occur synchronously, the impulsive switching signal is modeled by a semi-Markovian process. For the case that switches and impulses occur asynchronously, the impulsive arrival time sequence and the types of jump maps are driven by a renewal process and a Markov chain, respectively. The contributions of this paper can be highlighted as follows: (1) The sufficient conditions of exponential stability a.s. are obtained for semi-Markovian impulsive switched stochastic systems. These results provide the unified framework to study exponential stability a.s. for impulsive switched stochastic systems driven by Markovian process or renewal process. (2) By using the limit properties of semi-Markovian process and renewal process, sufficient conditions of exponential stability a.s. are derived for semi-Markovian switched stochastic systems with asynchronously randomly impulsive jumps. This class of systems cover the semi-Markovian switched stochastic systems in [5, 6] and randomly impulsive systems as special cases. (3) An effective method is proposed to balance impulses, switches and stochastic noises in order to guarantee system stability.

Notation. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ stand for the n-dimensional Euclidean space and the set of $n \times m$ matrices, respectively. $\mathbb{N}^+$ represents the set of positive integers and $\mathbb{N} \triangleq \mathbb{N}^+ \cup \{0\}$. $\|\cdot\|$ represents the Euclidean vector norm. The superscript T denotes the transpose of matrix (or vector). $\mathbb{P}$ and E stand for the probability measure and the mathematical expectation, respectively. $\mathcal{C}^2$ denotes the set of all nonnegative functions $V(x, i) : \mathbb{R}^n \times \mathcal{S}_S \to [0, \infty)$, which are twice differentiable in x. The symbol tr$[\cdot]$ denotes trace operator. The notation $I(\cdot)$ stands for the indicator function.

2 Preliminaries

Consider the switched stochastic system with impulse effects at random times:

$$\begin{cases} \mathrm{d}x(t) = f(x(t),\sigma(t))\mathrm{d}t + g(x(t),\sigma(t))\mathrm{d}W(t), & t \neq t_k, \\ x(t) = h(x(t^-), r(k)), & t = t_k,\ k \in \mathbb{N}^+, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state. $W(t)$ is a d-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$. $\sigma(t) : [0,\infty) \to \mathcal{S}_S = \{1,2,\ldots,r\}$ is the switching signal, which is a piecewise constant function specifying the index of the active subsystem. A strictly increasing sequence $\mathcal{I}_S \triangleq \{T_1, T_2, \ldots\}$ is called switching time sequence satisfying $\lim_{k\to\infty} T_k = \infty$, in which the switching signal $\sigma(t)$ randomly chooses a value from index set $\mathcal{S}_S$. The functions $f : \mathbb{R}^n \times \mathcal{S}_S \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathcal{S}_S \to \mathbb{R}^{n\times d}$ are assumed to be continuous and uniformly locally Lipschitz with respect to x and $f(0,i) = g(0,i) = 0$, $i \in \mathcal{S}_S$. $\{r(k), k \in \mathbb{N}^+\}$ is a stochastic process taking values in the index set $\mathcal{S}_J$ and $r(k)$ represents the type of jump map at the kth impulse arrival time t_k. The strictly increasing sequence $\mathcal{I}_J \triangleq \{t_1, t_2, \ldots\}$ is called impulse arrival time sequence satisfying $\lim_{k\to\infty} t_k = \infty$, in which $r(k)$ randomly chooses a value from index set $\mathcal{S}_J$. The function $h : \mathbb{R}^n \times \mathcal{S}_J \to \mathbb{R}^n$ is assumed to be continuous and uniformly locally Lipschitz with respect to x and $h(0, r(k)) = 0$, $r(k) \in \mathcal{S}_J$. We assume that $r(k)$, t_k, $\sigma(t)$ are independent with $W(t)$ and there exists a unique solution of system (1) (see [16, 34, 35]).

It is noted that there exist two kinds of discrete-time random signals in system (1), namely, switching and impulses, which make effects on the stability. Next, some notations are introduced for such two kinds of discrete-time random signals. For switching signal $\sigma(t)$, let $\mathcal{N}_S^i(t)$ be the activated number of the ith subsystem in the interval $(0,t]$, $\mathcal{N}_{ij}(t)$ be the number of switching from the ith subsystem to the jth subsystem in the interval $(0,t]$ and $\mathcal{T}_i(t)$ be the total time for system (1) active on the ith subsystem in the interval $(0,t]$, where $i \in \mathcal{S}_S$. For the impulse arrival time sequence $\{t_k, k \in \mathbb{N}^+\}$, let $\mathcal{N}_J(t)$ be the total number of impulse occurring in the interval $(0,t]$ with $\mathcal{N}_J(0) = 0$. Let $\xi(k) = t_k - t_{k-1}$ denote the impulsive interval between the $(k-1)$th impulse arrival time and kth impulse arrival time.

Definition 1 ([20, 21])**.** The switching signal $\sigma(t)$ is said to be semi-Markovian switching, if let $\sigma(t) = \sigma(T_k) = \sigma_k$ for $t \in [T_k, T_{k+1}), k \in \mathbb{N}$, (i) the discrete-time process $\{\sigma_k, k \in \mathbb{N}\}$ is a Markov chain with $P = [p_{ij}]_{r\times r}, i,j \in \mathcal{S}_S$, where $p_{ij} = \mathbb{P}\{\sigma(T_{k+1}) = j \mid \sigma(T_k) = i\}$ stands for the transition probability of the subsystem transiting from i to j at time T_{k+1}; and (ii) the distribution function of $\tau(k+1) = T(k+1) - T(k)$ is given by

$$F_{ij}(s) = \mathbb{P}\big\{\tau(k+1) \leqslant s \mid \sigma_k = i, \sigma_{k+1} = j\big\}, \quad i,j \in \mathcal{S}_S, s \geqslant 0,$$

which has the continuous differentiable density $f_{ij}(t)$ and does not depend on k.

From Definition 1, let τ_i be the sojourn time on the ith subsystem, $i \in \mathcal{S}_S$, and then its distribution function can be defined by

$$F_i(s) = \mathbb{P}\big\{\tau_i \leqslant s\big\} = \mathbb{P}\big\{T(k+1) - T(k) \leqslant s \mid \sigma_k = i\big\} = \sum_{j\in\mathcal{S}_S} F_{ij}(s)p_{ij}, \quad i \in \mathcal{S}_S,\ s \geqslant 0,\ \forall k \in \mathbb{N}.$$

Here, we assume that $p_{ii} = 0$ and the embedded Markov chain $\{\sigma_k, k \in \mathbb{N}\}$ is irreducible. Then, $\{\sigma_k, k \in \mathbb{N}\}$ has a stationary distribution $\bar{\pi}$ satisfying $\bar{\pi}P = \bar{\pi}$ and $\sum_{i\in\mathcal{S}_S} \bar{\pi}_i = 1$.

Definition 2. The system (1) is said to be exponentially stable a.s., if

$$\limsup_{t\to\infty} \frac{1}{t} \ln \|x(t; x_0, \sigma_0)\| < 0, \ \text{a.s.}$$

holds for any initial conditions $x_0 \in \mathbb{R}^n$ and $\sigma_0 \in \mathcal{S}_S$.

Definition 3 ([3])**.** For any $\mathcal{C}^2$ function $V(x,i), i \in \mathcal{S}_S$, the two operators $\mathcal{L}$ and $\mathcal{H}$, which are associated with the system (1), are defined as

$$\mathcal{L}V(x,i) \triangleq \frac{\partial V(x,i)}{\partial x} f(x,i) + \frac{1}{2}\mathrm{tr}\left[g^{\mathrm{T}}(x,i)\frac{\partial^2 V(x,i)}{\partial x^2} g(x,i)\right], \quad \mathcal{H}V(x,i) \triangleq \frac{\partial V(x,i)}{\partial x} g(x,i).$$

By means of operators $\mathcal{L}$, $\mathcal{H}$ and Itô formula, we can obtain that for each $i \in \mathcal{S}_S$,

$$\mathrm{d}V(x(t),i) = \mathcal{L}V(x(t),i)\mathrm{d}t + \mathcal{H}V(x(t),i)\mathrm{d}W(t). \tag{2}$$

3 Exponential stability a.s. of semi-Markovian switched stochastic systems with synchronous impulsive jumps

In this section, we consider the first situation that semi-Markovian switching and impulsive jumps occur synchronously and the types of impulsive jumps are determined by the modes of subsystems. In this case, system (1) becomes to be a semi-Markovian impulsive switched system. Next, we will study the exponential stability a.s. for system (1). Moreover, some corollaries will be given to show that our results can be considered as a further development of some existing results.

Assumption 1. The semi-Markovian switching and impulsive jumps occur synchronously and the types of impulsive jumps are determined by the modes of subsystems, i.e., $T_k = t_k,\ \mathcal{S}_J = \mathcal{S}_S \times \mathcal{S}_S$ and $r(k) = \sigma_{k-1}, \sigma_k$.

Lemma 1 ([6, 21])**.** Let $\{\sigma(t), t \geqslant 0\}$ be a semi-Markovian process and $\{\sigma_k, k \in \mathbb{N}\}$ is its associated embedded Markov chain; then

$$\lim_{t\to\infty} \frac{\mathcal{T}_i(t)}{t} = \pi_i, \quad \text{a.s.}, \forall i \in \mathcal{S}_S, \tag{3}$$

$$\lim_{t\to\infty} \frac{\mathcal{N}_S^i(t)}{t} = \frac{\pi_i}{m_i}, \quad \text{a.s.}, \forall i \in \mathcal{S}_S, \tag{4}$$

where $m_i = \mathrm{E}[\tau_i]$ and $\pi = (\pi_1, \pi_2, \ldots, \pi_r)$ is the stationary distribution of $\sigma(t)$, which can be given as $\pi_i = \frac{\bar{\pi}_i m_i}{\sum_{j\in\mathcal{S}_S} \bar{\pi}_j m_j}$.

Theorem 1. Under Assumption 1, if there exist a function $V \in \mathcal{C}^2$, positive numbers c, p, μ_{ij}, and constants $\beta_i \geqslant 0, \lambda_i \in \mathbb{R}$ such that

$$c\|x(t)\|^p \leqslant V(x(t), i), \tag{5}$$

$$\mathcal{L}V(x(t), i) \leqslant \lambda_i V(x(t), i), \tag{6}$$

$$|\mathcal{H}V(x(t), i)|^2 \geqslant \beta_i V^2(x(t), i), \tag{7}$$

$$V(h(x(t^-), i, j), j) \leqslant \mu_{ij} V(x(t^-), i), \tag{8}$$

$$\sum_{i\in\mathcal{S}_S} \pi_i \left(\lambda_i + \sum_{j\in\mathcal{S}_S} \frac{p_{ij}}{m_i} \ln \mu_{ij} - 0.5\beta_i \right) < 0, \tag{9}$$

hold for all $i, j \in \mathcal{S}_S$, then semi-Markovian switched stochastic system (1) with synchronous impulsive jumps is exponentially stable a.s..

Proof. It follows from Itô formula [35] and (2) that

$$\mathrm{d}[\ln V(x(s), i)] = \left[\frac{\mathcal{L}V(x(s), i)}{V(x(s), i)} - \frac{1}{2} \frac{|\mathcal{H}V(x(s), i)|^2}{V^2(x(s), i)} \right] \mathrm{d}s + \frac{\mathcal{H}V(x(s), i)}{V(x(s), i)} \mathrm{d}W(s),\ i \in \mathcal{S}_S. \tag{10}$$

Hence, we obtain

$$\ln V(x(t), \sigma(t)) = \ln V(x(T_k), \sigma_k) + \int_{T_k}^t \left[\Phi(s) - \frac{1}{2}\Psi(s) \right] \mathrm{d}s + \int_{T_k}^t \Upsilon(s) \mathrm{d}W(s),\ t \in [T_k, T_{k+1}),\ k \in \mathbb{N}, \tag{11}$$

and

$$\ln V(x(T_k^-), \sigma_{k-1}) = \ln V(x(T_{k-1}), \sigma_{k-1}) + \int_{T_{k-1}}^{T_k} \left[\Phi(s) - \frac{1}{2}\Psi(s) \right] \mathrm{d}s + \int_{T_{k-1}}^{T_k} \Upsilon(s) \mathrm{d}W(s),\ k \in \mathbb{N}^+, \tag{12}$$

where $\Phi(s) \triangleq \frac{\mathcal{L}V(x(s),\sigma(s))}{V(x(s),\sigma(s))}$, $\Psi(s) \triangleq \frac{|\mathcal{H}V(x(s),\sigma(s))|^2}{V^2(x(s),\sigma(s))}$ and $\Upsilon(s) \triangleq \frac{\mathcal{H}V(x(s),\sigma(s))}{V(x(s),\sigma(s))}$. By (8), we have

$$\ln V(x(T_k), \sigma_k) = \ln V(h(x(T_k^-), \sigma_{k-1}, \sigma_k), \sigma_k) \leqslant \ln \mu_{\sigma_{k-1},\sigma_k} + \ln V(x(T_k^-), \sigma_{k-1}),\ k \in \mathbb{N}^+. \tag{13}$$

Combining (11), (12) with (13) yields that for any $t \in [T_k, T_{k+1}), k \in \mathbb{N}^+$,

$$\ln V(x(t), \sigma(t)) \leqslant \ln \mu_{\sigma_{k-1},\sigma_k} + \ln V(x(T_{k-1}), \sigma_{k-1}) + \int_{T_{k-1}}^t \left[\Phi(s) - \frac{1}{2}\Psi(s) \right] \mathrm{d}s + \int_{T_{k-1}}^t \Upsilon(s) \mathrm{d}W(s).$$

Repeating the above procedure, we have for all $t \in [T_k, T_{k+1}), k \in \mathbb{N}^+$

$$\ln V(x(t), \sigma_k) \leqslant \ln V_0 + \sum_{l=1}^{k} \ln \mu_{\sigma_{l-1},\sigma_l} + \int_0^t \left[\Phi(s) - \frac{1}{2}\Psi(s)\right] \mathrm{d}s + M(t), \tag{14}$$

where $M(t) \triangleq \int_0^t \Upsilon(s)\mathrm{d}W(s)$ is a continuous local martingale with $M(0) = 0$ and $V_0 = V(x_0, \sigma_0)$. Thus, for any $t \geqslant 0$,

$$\ln V(x(t), \sigma(t)) \leqslant \ln V_0 + \sum_{i,j\in\mathcal{S}_S} \mathcal{N}_{ij}(t) \ln \mu_{ij} + \int_0^t \left[\Phi(s) - \frac{1}{2}\Psi(s)\right] \mathrm{d}s + M(t). \tag{15}$$

Notice that the quadratic variation of $M(t)$ is $\langle M(t), M(t)\rangle = \int_0^t \Psi(s)\mathrm{d}s$. Setting $\varepsilon \in (0, 1)$ and using the similar analysis used in Theorem 3.1 of [5] or Theorem 2.2 of [36], we can obtain that there exists an integer $N_0 = N_0(\omega)$, such that

$$M(t) \leqslant \frac{2}{\varepsilon} \ln N + \int_0^t \frac{\varepsilon}{2}\Psi(s)\mathrm{d}s, \ \forall N > N_0, \ \text{a.s.}, \tag{16}$$

holds for $0 \leqslant t \leqslant N$. Substituting (16) into (15), we obtain

$$\ln V(x(t), \sigma(t)) \leqslant \ln V_0 + \sum_{i,j\in\mathcal{S}_S} \mathcal{N}_{ij}(t) \ln \mu_{ij} + \frac{2}{\varepsilon} \ln N + \int_0^t \left[\Phi(s) - \frac{1-\varepsilon}{2}\Psi(s)\right] \mathrm{d}s, \ 0 \leqslant t \leqslant N. \tag{17}$$

Combining (17) with (6) and (7) yields that for any $0 \leqslant t \leqslant N, N \geqslant N_0$,

$$\begin{aligned} \ln V(x(t), \sigma(t)) &\leqslant \ln V_0 + \sum_{i,j\in\mathcal{S}_S} \mathcal{N}_{ij}(t) \ln \mu_{ij} + \frac{2}{\varepsilon} \ln N + \int_0^t \left(\lambda_{\sigma(s)} - \frac{1-\varepsilon}{2}\beta_{\sigma(s)}\right) \mathrm{d}s \\ &= \ln V_0 + \sum_{i,j\in\mathcal{S}_S} \mathcal{N}_{ij}(t) \ln \mu_{ij} + \frac{2}{\varepsilon} \ln N + \sum_{i\in\mathcal{S}_S} \left(\lambda_i - \frac{1-\varepsilon}{2}\beta_i\right) \mathcal{T}_i(t), \ \text{a.s.}. \end{aligned} \tag{18}$$

Hence, if $N - 1 \leqslant t \leqslant N$ and $N \geqslant N_0$, we can get that

$$\frac{1}{t} \ln V(x(t), \sigma(t)) \leqslant \frac{1}{N-1} \left(\ln V_0 + \frac{2}{\varepsilon} \ln N\right) + \sum_{i,j\in\mathcal{S}_S} \frac{\mathcal{N}_{ij}(t)}{t} \ln \mu_{ij} + \sum_{i\in\mathcal{S}_S} \left(\lambda_i - \frac{1-\varepsilon}{2}\beta_i\right) \frac{\mathcal{T}_i(t)}{t}, \quad \text{a.s.}. \tag{19}$$

Let $\breve{\mathcal{N}}_S^i(t) = \mathcal{N}_S^i(t) - I(\sigma(t) = i)$ be the total number of the events "deactivating the ith subsystem". It can be got from (4) that

$$\lim_{t\to\infty} \frac{\mathcal{N}_{ij}(t)}{t} = \lim_{t\to\infty} p_{ij} \frac{\breve{\mathcal{N}}_S^i(t)}{t} = \lim_{t\to\infty} p_{ij} \frac{\mathcal{N}_S^i(t) - I(\sigma(t) = i)}{t} = \pi_i \frac{p_{ij}}{m_i}, \ i, j \in \mathcal{S}_S, \quad \text{a.s.}. \tag{20}$$

Substituting (20) and (3) into (19), we can get that

$$\limsup_{t\to\infty} \frac{1}{t} \ln V(x(t), \sigma(t)) \leqslant \sum_{i\in\mathcal{S}_S} \pi_i \left(\lambda_i + \sum_{j\in\mathcal{S}_S} \frac{p_{ij}}{m_i} \ln \mu_{ij} - \frac{1-\varepsilon}{2}\beta_i\right), \ \text{a.s.}.$$

Letting $\varepsilon \to 0$, according to conditions (5) and (9), we obtain that

$$\limsup_{t\to\infty} \frac{1}{t} \ln \|x(t; x_0, \sigma_0)\| < 0, \ \text{a.s.}.$$

Thus, semi-Markovian switched stochastic system (1) with synchronous impulsive jumps is exponentially stable a.s..

Remark 1. Conditions (6) and (7) provide a quantitative estimate of the stability degree of each subsystem of system (1). For each subsystem $i \in \mathcal{S}_S$, the negative value of $\lambda_i - 0.5\beta_i$ corresponds to the case that the ith subsystem is exponentially stable a.s.. Compared with the switched system without impulsive effects, in which μ_{ij} is often assumed to be independent with j, the different types of impulses can affect the impulsive switched systems in different manners; that is, some impulses (called stabilizing impulses) can contribute towards stability ($0 < \mu_{ij} < 1$), and others (called destabilizing impulses) can potentially destroy stability ($\mu_{ij} > 1$). Furthermore, according to condition (9), we can observe that if the positive effect of stabilizing impulses is big enough, system (1) will be exponentially stable a.s., even all systems are unstable (i.e., $\lambda_i - 0.5\beta_i > 0, \forall i \in \mathcal{S}_S$).

If the distribution function $F_{ij}(s)$ is only dependent on the current subsystem mode $i \in \mathcal{S}_S$ and obeys an exponential distribution with parameter q_i, in other words,

$$F_{ij}(s) = \mathbb{P}\big\{\tau(k+1) \leqslant s \mid \sigma_k = i, \sigma_{k+1} = j\big\} = 1 - \mathrm{e}^{-q_i s}, \quad s \geqslant 0, \tag{21}$$

then the corresponding semi-Markovian impulsive switching signal reduces to Markovian impulsive switching signal. The generator of Markovian impulsive switching signal $\sigma(t)$ can be given as $Q = [q_{ij}]_{r\times r}$ with

$$\mathbb{P}\{\sigma(t+\Delta) = j|\sigma(t) = i\} = \begin{cases} q_{ij}\Delta + o(\Delta), & i \neq j, \\ 1 + q_{ii}\Delta + o(\Delta), & i = j, \end{cases}$$

where $i, j \in \mathcal{S}_S, \Delta > 0$, $q_{ij} \geqslant 0$ is the transition rate from i to j if $i \neq j$ and $q_{ii} = -\sum_{j\neq i} q_{ij} = -q_i$. In this case, the stationary distribution π of $\sigma(t)$ can be given by solving $\pi Q = 0$, and $\sum_{i\in\mathcal{S}_S} \pi_i = 1$.

Corollary 1 (Markovian process case). Under Assumption 1 and condition (21), if there exist a function $V \in \mathcal{C}^2$, positive numbers c, p, μ_{ij}, and constants $\beta_i \geqslant 0, \lambda_i \in \mathbb{R}$ such that (5)–(8) and

$$\sum_{i\in\mathcal{S}_S} \pi_i \left(\lambda_i + \sum_{j\in\mathcal{S}_S, j\neq i} q_{ij} \ln \mu_{ij} - 0.5\beta_i\right) < 0 \tag{22}$$

hold for all $i, j \in \mathcal{S}_S$, then Markovian switched stochastic system (1) with synchronous impulsive jumps is exponentially stable a.s..

Proof. From (21), we have $\mathrm{E}[\tau_i] = m_i = 1/q_i$. Next, we show that

$$\pi_i = \frac{\bar{\pi}_i/q_i}{\sum_{j\in\mathcal{S}_S} \bar{\pi}_j/q_j}, \quad \sum_{i\in\mathcal{S}_S} \pi_i = 1 \tag{23}$$

are equivalent to $\pi Q = 0$ and $\sum_{i\in\mathcal{S}_S} \pi_i = 1$.

Noting that $\bar{\pi}P = \bar{\pi}$ and $\sum_{i\in\mathcal{S}_S} \bar{\pi}_i = 1$, we obtain that $\bar{\pi}_i = \sum_{j\in\mathcal{S}_S} \bar{\pi}_j p_{ji}$. It follows that

$$q_i\pi_i = \frac{\bar{\pi}_i}{\sum_{j\in\mathcal{S}_S} \bar{\pi}_j/q_j} = \frac{\sum_{j\in\mathcal{S}_S} \bar{\pi}_j p_{ji}}{\sum_{j\in\mathcal{S}_S} \bar{\pi}_j/q_j} = \sum_{j\in\mathcal{S}_S} q_j\pi_j p_{ji}, \quad \sum_{i\in\mathcal{S}_S} \pi_i = 1. \tag{24}$$

By using $q_{ji} = q_j p_{ji}$, we have

$$q_i\pi_i = \sum_{j\neq i} \pi_j q_{ji}, \quad \sum_{i\in\mathcal{S}_S} \pi_i = 1, \tag{25}$$

which means that $\pi Q = 0$ and $\sum_{i\in\mathcal{S}_S} \pi_i = 1$.

Thus, condition (9) can be written as condition (22).

If the distribution function of sojourn time $\tau(k+1)$ is independent on the subsystem mode $i \in \mathcal{S}_S$, i.e.,

$$F_{ij}(s) = F_{ik}(s) = F_{lj}(s) = F(s), \ i, j, k, l \in \mathcal{S}_S, \ s \geqslant 0, \tag{26}$$

then the total number of switching $\mathcal{N}_S(t)$ is a renewal process and the corresponding semi-Markovian impulsive switching signal reduces to renewal process impulsive switching signal.

Corollary 2 (Renewal process case). Under Assumption 1 and condition (26), if there exist a function $V \in \mathcal{C}^2$, positive numbers c, p, μ_{ij}, and constants $\beta_i \geqslant 0, \lambda_i \in \mathbb{R}$ such that (5)–(8) and

$$\sum_{i\in\mathcal{S}_S} \bar{\pi}_i \left(\lambda_i + \sum_{j\in\mathcal{S}_S} \frac{p_{ij}}{m} \ln \mu_{ij} - 0.5\beta_i\right) < 0 \tag{27}$$

hold for all $i, j \in \mathcal{S}_S$, where $\mathrm{E}[\tau(1)] = m$, then stochastic system (1) switched by renewal process with synchronous impulsive jumps is exponentially stable a.s..

Proof. From condition (26), it is easy to get that $\pi_i = \frac{\bar{\pi}_i m}{\sum_{j\in\mathcal{S}_S}\bar{\pi}_j m} = \bar{\pi}_i$. Thus, the condition (9) reduces to condition (27). Then, Corollary 2 follows from Theorem 1.

Furthermore, the subsystem transition probability p_{ij} is independent on the current subsystem mode $i \in \mathcal{S}_S$, i.e.,

$$p_{ij} = \mathbb{P}\big\{\sigma(T_{k+1}) = j \mid \sigma(T_k) = i\big\} = \mathbb{P}\big\{\sigma(T_{k+1}) = j\big\} = p_j,\ k \in \mathbb{N},\ j \in \mathcal{S}_S. \tag{28}$$

Then we can obtain the following Corollary 3.

Corollary 3 (Renewal process case). Under Assumption 1 and conditions (26) and (28), if there exist a function $V \in \mathcal{C}^2$, positive numbers c, p, μ_{ij}, and constants $\beta_i \geqslant 0, \lambda_i \in \mathbb{R}$ such that (5)–(8) and

$$\sum_{i\in\mathcal{S}_S} p_i \left(\lambda_i + \sum_{j\in\mathcal{S}_S} \frac{p_j}{m}\ln\mu_{ij} - 0.5\beta_i\right) < 0 \tag{29}$$

hold for all $i, j \in \mathcal{S}_S$, where $\mathrm{E}[\tau(1)] = m$, then stochastic system (1) switched by renewal process with synchronous impulsive jumps is exponentially stable a.s..

Proof. From conditions (26) and (28), we can see that the stationary distribution of impulsive switching signal $\sigma(t)$ is $\pi = (p_1, p_2, \ldots, p_r)$. Thus, the condition (9) reduces to condition (29). Then, Corollary 3 follows from Theorem 1.

Remark 2. (1) In Theorem 1, if there is no impulsive jump at switching instants and the estimate of Lyapunov function at switching instants is independent on the transition subsystem mode, i.e., $h(x(t^-), r(k)) = x(t^-)$ and $V(h(x(t^-), i, j), j) \leqslant \mu_i V(x(t^-), i)$, then Theorem 1 reduces to the Theorem 3.3 of [5].

(2) For system (1), if we do not take into account the effects of impulsive jumps and stochastic noise, i.e., $h(x(t^-), r(k)) = x(t^-)$ and $g(x(t), \sigma(t)) = 0$, then Theorem 1, Corollaries 1 and 3 reduce to Corollaries 1–3 of [6], respectively.

4 Exponential stability a.s. of semi-Markovian switched stochastic systems with asynchronous impulsive jumps

In this section, we consider the second situation that semi-Markovian switching and impulsive jumps occur asynchronously. We assume that $r(k)$, t_k, $\sigma(t)$ and $W(t)$ are mutually independent. In this case, we will study exponential stability a.s. for system (1). Firstly, we make some assumptions to describe the randomness of impulsive jumps, including the types of impulsive jumps and the impulsive arrival time sequence.

Assumption 2. The $\{r(k), k \in \mathbb{N}^+\}$ is an irreducible Markov chain taking values in $\mathcal{S}_J = \{1, 2, \ldots, q\}$.

Lemma 2 ([37]). Suppose that Assumption 2 holds, and then the Markov chain $\{r(k), k \in \mathbb{N}^+\}$ has a unique stationary distribution $\tilde{\pi} = (\tilde{\pi}_1, \tilde{\pi}_2, \ldots, \tilde{\pi}_q)$ and

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} I(r(k) = l)}{n} = \tilde{\pi}_l,\ \text{a.s.},\ \forall l \in \mathcal{S}_J. \tag{30}$$

Assumption 3. The $\{\xi(k), k \in \mathbb{N}^+\}$ is a family of nonnegative independent identical distributed random variables.

Remark 3. For the random impulsive signal, there are two ingredients, namely, the impulsive arrival time sequence and the types of impulsive jumps. In [8], the impulsive arrival time sequence is driven by a Poisson process, i.e., $\xi(k), k \in \mathbb{N}^+$ are independently exponentially distributed, and the types of jump maps are independently identically distributed. Compared with the case in [8], Assumptions 2 and 3 are more general. However, Assumption 2 is not suitable to describe the situation that each of the types of jump maps is driven by a different renewal process.

Lemma 3 (Renewal theorem [21]). Suppose that Assumption 3 holds, and then $\mathcal{N}_J(t)$ is a renewal process and

$$\lim_{t\to\infty} \frac{\mathcal{N}_J(t)}{t} = \frac{1}{\theta},\ \ \text{a.s.}, \tag{31}$$

where $\theta = \mathrm{E}[\xi(k)]$, θ is a positive constant.

Theorem 2. Under Assumptions 2 and 3, if there exist a function $V \in \mathcal{C}^2$, positive numbers c, p, μ_{ij}, α_l, and constants $\beta_i \geqslant 0, \lambda_i \in \mathbb{R}$ such that (5)–(7) and

$$V(x(t), j) \leqslant \mu_{ij} V(x(t), i), \tag{32}$$

$$V(h(x(t), l), i) \leqslant \alpha_l V(x(t), i), \tag{33}$$

$$\sum_{l \in \mathcal{S}_J} \tilde{\pi}_l \frac{\ln \alpha_l}{\theta} + \sum_{i \in \mathcal{S}_S} \pi_i \left(\lambda_i + \sum_{j \in \mathcal{S}_S} \frac{p_{ij}}{m_i} \ln \mu_{ij} - 0.5\beta_i \right) < 0, \tag{34}$$

hold for all $i, j \in \mathcal{S}_S, l \in \mathcal{S}_J$, then semi-Markovian switched stochastic system (1) with asynchronous impulsive jumps is exponentially stable a.s..

Proof. For any $t \in [T_k, T_{k+1})$ and $\sigma(t) = \sigma_k, \sigma(T_{k-1}) = \sigma_{k-1}$, $k \in \mathbb{N}^+$, let $\{\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_m\}$ be the impulse arrival time sequence in the time interval $[T_k, t)$ and $\{l_1, l_2, \ldots, l_m\}$ be the corresponding types of jump maps.

Combing (10) with (33) yields that for any $t \in [\bar{t}_m, T_{k+1})$,

$$\begin{aligned} \ln V(x(t), \sigma(t)) &= \ln V(x(\bar{t}_m), \sigma_k) + \int_{\bar{t}_m}^t \left[\Phi(s) - \frac{1}{2}\Psi(s)\right] \mathrm{d}s + \int_{\bar{t}_m}^t \Upsilon(s) \mathrm{d}W(s) \\ &\leqslant \ln \alpha_{l_m} + \ln V(x(\bar{t}_m^-), \sigma_k) + \int_{\bar{t}_m}^t \left[\Phi(s) - \frac{1}{2}\Psi(s)\right] \mathrm{d}s + \int_{\bar{t}_m}^t \Upsilon(s) \mathrm{d}W(s). \end{aligned} \tag{35}$$

Similarly, we have

$$\begin{aligned} \ln V(x(\bar{t}_m^-), \sigma_k) &= \ln V(x(\bar{t}_{m-1}), \sigma_k) + \int_{\bar{t}_{m-1}}^{\bar{t}_m} \left[\Phi(s) - \frac{1}{2}\Psi(s)\right] \mathrm{d}s + \int_{\bar{t}_{m-1}}^{\bar{t}_m} \Upsilon(s) \mathrm{d}W(s) \\ &\leqslant \ln \alpha_{l_{m-1}} + \ln V(x(\bar{t}_{m-1}^-), \sigma_k) + \int_{\bar{t}_{m-1}}^{\bar{t}_m} \left[\Phi(s) - \frac{1}{2}\Psi(s)\right] \mathrm{d}s + \int_{\bar{t}_{m-1}}^{\bar{t}_m} \Upsilon(s) \mathrm{d}W(s). \end{aligned}$$

Iterating the above procedure, we can see that for any $t \in [T_k, T_{k+1}), k \in \mathbb{N}$,

$$\begin{aligned} \ln V(x(t), \sigma(t)) &\leqslant \ln V(x(T_k), \sigma_k) + \int_{T_k}^t \left[\Phi(s) - \frac{1}{2}\Psi(s)\right] \mathrm{d}s + \int_{T_k}^t \Upsilon(s) \mathrm{d}W(s) + \sum_{n=1}^m \ln \alpha_{l_n} \\ &= \ln V(x(T_k), \sigma_k) + \int_{T_k}^t \left[\Phi(s) - \frac{1}{2}\Psi(s)\right] \mathrm{d}s + \int_{T_k}^t \Upsilon(s) \mathrm{d}W(s) \\ &\quad + \sum_{l \in \mathcal{S}_J} \ln \alpha_l \big(\mathcal{N}_J^l(t) - \mathcal{N}_J^l(T_k)\big), \end{aligned} \tag{36}$$

where $\mathcal{N}_J^l(t),\ l \in \mathcal{S}_J$ is the activated number of the lth impulsive jump map in the interval $(0, t]$. Similarly, one has

$$\begin{aligned} \ln V(x(T_k^-), \sigma_{k-1}) &\leqslant \ln V(x(T_{k-1}), \sigma_{k-1}) + \sum_{l \in \mathcal{S}_J} \ln \alpha_l \big(\mathcal{N}_J^l(T_k) - \mathcal{N}_J^l(T_{k-1})\big) \\ &\quad + \int_{T_{k-1}}^t \left[\Phi(s) - \frac{1}{2}\Psi(s)\right] \mathrm{d}s + \int_{T_{k-1}}^t \Upsilon(s) \mathrm{d}W(s),\ k \in \mathbb{N}^+. \end{aligned} \tag{37}$$

Combining (36) and (37) with (32) implies that

$$\begin{aligned} \ln V(x(t), \sigma_k) &\leqslant \ln \mu_{\sigma_{k-1}, \sigma_k} + \ln V(x(T_{k-1}), \sigma_{k-1}) + \sum_{l \in \mathcal{S}_J} \ln \alpha_l \big(\mathcal{N}_J^l(t) - \mathcal{N}_J^l(T_{k-1})\big) \\ &\quad + \int_{T_{k-1}}^t \left[\Phi(s) - \frac{1}{2}\Psi(s)\right] \mathrm{d}s + \int_{T_{k-1}}^t \Upsilon(s) \mathrm{d}W(s),\ t \in [T_k, T_{k+1}). \end{aligned} \tag{38}$$

Repeating the above procedure, we have

$$\ln V(x(t), \sigma(t)) \leqslant \ln V_0 + \sum_{i,j \in \mathcal{S}_S} \mathcal{N}_{ij}(t) \ln \mu_{ij} + \sum_{l \in \mathcal{S}_J} \ln \alpha_l \mathcal{N}_J^l(t) + \int_0^t \left[\Phi(s) - \frac{1}{2}\Psi(s)\right] \mathrm{d}s + M(t),\ t \geqslant 0. \tag{39}$$

With the similar analysis used in Theorem 1, we can obtain that for any $\varepsilon \in (0,1)$, there exists an integer $N_0 = N_0(\omega)$, such that $\forall N > N_0$, $0 \leqslant t \leqslant N$,

$$\ln V(x(t),\sigma(t)) \leqslant \ln V_0 + \frac{2}{\varepsilon}\ln N + \sum_{i,j\in\mathcal{S}_S}\mathcal{N}_{ij}(t)\ln\mu_{ij} + \sum_{l\in\mathcal{S}_J}\ln\alpha_l\mathcal{N}_J^l(t) + \sum_{i\in\mathcal{S}_S}\left(\lambda_i - \frac{1-\varepsilon}{2}\beta_i\right)\mathcal{T}_i(t), \text{ a.s..} \tag{40}$$

Consequently, if $N-1 \leqslant t \leqslant N$ and $N \geqslant N_0$, we can get that

$$\begin{aligned}\frac{1}{t}\ln V(x(t),\sigma(t)) \leqslant\ & \frac{1}{N-1}\left(\ln V_0 + \frac{2}{\varepsilon}\ln N\right) + \sum_{l\in\mathcal{S}_J}\ln\alpha_l\frac{\mathcal{N}_J^l(t)}{t} \\ & + \sum_{i,j\in\mathcal{S}_S}\frac{\mathcal{N}_{ij}(t)}{t}\ln\mu_{ij} + \sum_{i\in\mathcal{S}_S}\left(\lambda_i - \frac{1-\varepsilon}{2}\beta_i\right)\frac{\mathcal{T}_i(t)}{t}, \quad \text{a.s..}\end{aligned} \tag{41}$$

By (30), we have that

$$\lim_{t\to\infty}\frac{\sum_{k=1}^{\mathcal{N}_J(t)} I(r(k)=l)}{\mathcal{N}_J(t)} = \tilde{\pi}_l, \quad \text{a.s., } l\in\mathcal{S}_J.$$

Combining this with (31), we obtain

$$\lim_{t\to\infty}\frac{\mathcal{N}_J^l(t)}{t} = \frac{\tilde{\pi}_l}{\theta}, \quad \text{a.s., } l\in\mathcal{S}_J. \tag{42}$$

Substituting (3), (20), (42) into (41), we have

$$\limsup_{t\to\infty}\frac{1}{t}\ln V(x(t),\sigma(t)) \leqslant \sum_{l\in\mathcal{S}_J}\tilde{\pi}_l\frac{\ln\alpha_l}{\theta} + \sum_{i\in\mathcal{S}_S}\pi_i\left(\lambda_i + \sum_{j\in\mathcal{S}_S}\frac{p_{ij}}{m_i}\ln\mu_{ij} - \frac{1-\varepsilon}{2}\beta_i\right), \text{ a.s..}$$

Letting $\varepsilon \to 0$ and then by conditions (5) and (34), we get that

$$\limsup_{t\to\infty}\frac{1}{t}\ln\|x(t;x_0,\sigma_0)\| < 0, \text{ a.s..}$$

Thus, the semi-Markovian switched stochastic system (1) with asynchronous impulsive jumps is exponentially stable a.s..

Remark 4. Notice that there are three sources of randomness (i.e., stochastic noise, semi-Markovian switching and randomly impulsive jumps) and two kinds of discrete-time random signals in system (1). The effects of impulses, switches and noises are coupled and quantitated in (34). In other words, we provided a design approach to balance impulses, switches and stochastic noises in order to guarantee system stability. Moreover, Eq. (34) is different from these in existing results. For example, only switches or impulses were discussed in [5–8], and switched systems with synchronous impulses were studied in [16, 18, 19].

Remark 5. Theorems 1 and 2 discussed the exponential stability a.s. for semi-Markovian switched systems with synchronous impulsive jumps and asynchronous impulsive jumps, respectively. For the case that switches and impulses occur synchronously, the impulsive interval is equivalent to the sojourn time of subsystem between consecutive impulses, which means that $\xi(k), k \in \mathbb{N}^+$ may obey dependent and different probability distribution. For the case that switches and impulses occur asynchronously, the impulsive signal and switching signal are mutually independent; that is, there may be zero or multiple impulsive jumps during the activation time of a subsystem. However, the impulsive intervals $\xi(k), k \in \mathbb{N}^+$ are independently and identically distributed. Thus, these two cases cannot be determined that which is more general.

In addition, Theorem 2 also can be used to analyze the exponential stability a.s. of the following randomly impulsive systems with multiple jumps:

$$\begin{cases} \mathrm{d}x(t) = f(x(t))\mathrm{d}t + g(x(t))\mathrm{d}W(t), & t \neq t_k, \\ x(t) = h(x(t^-), r(k)), & t = t_k, \ k\in\mathbb{N}^+, \end{cases} \tag{43}$$

where t_k and $r(k)$ are the same as in system (1).

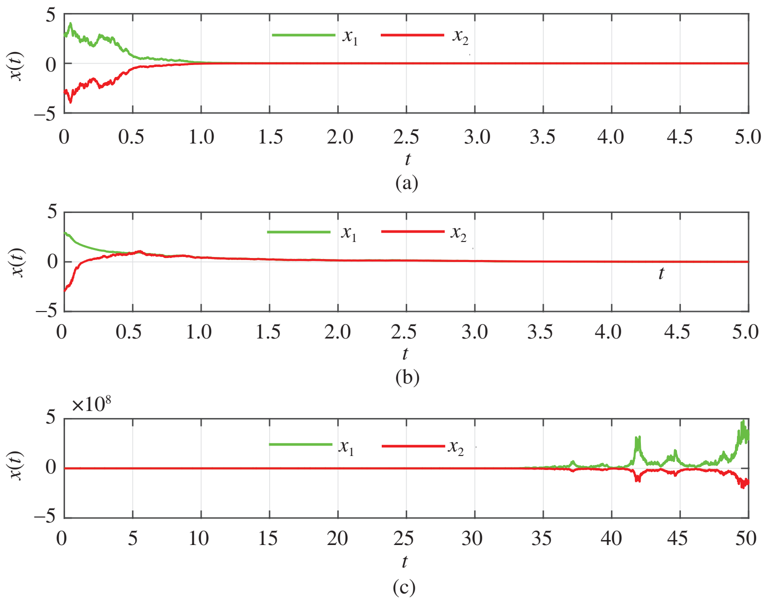

Figure 1 (Color online) The state trajectories of the three subsystems of system (1). (a) $\sigma(t) = 1$; (b) $\sigma(t) = 2$; (c) $\sigma(t) = 3$.

Corollary 4. Under Assumptions 2 and 3, if there exist a function $V \in \mathcal{C}^2$, positive numbers c, p, α_l, and constants $\beta \geqslant 0, \lambda \in \mathbb{R}$ such that

$$c\|x(t)\|^p \leqslant V(x(t)), \tag{44}$$
$$\mathcal{L}V(x(t)) \leqslant \lambda V(x(t)), \tag{45}$$
$$|\mathcal{H}V(x(t))|^2 \geqslant \beta V^2(x(t), i), \tag{46}$$
$$V(h(x(t), l)) \leqslant \alpha_l V(x(t)), \tag{47}$$
$$\sum_{l \in \mathcal{S}_J} \tilde{\pi}_l \frac{\ln \alpha_l}{\theta} + \lambda - 0.5\beta < 0 \tag{48}$$

hold, then randomly impulsive system (43) with multiple jumps is exponentially stable a.s..

Proof. In Theorem 2, if $f(x(t), \sigma(t)) \equiv f(x(t))$ and $g(x(t), \sigma(t)) \equiv g(x(t))$, that is, there is no subsystem mode switching in system (1), then system (1) reduces to system (43). In this case, conditions (5)–(7), (33), (34) become to conditions (44)–(48), respectively. Thus, Corollary 4 can be easily obtained by using Theorem 2.

5 Numerical example

Example 1. Consider system (1) with the subsystems' parameters:

$$f(x(t), 1) = (-2x_1,\ x_1 - 1.5x_2)^{\mathrm{T}}, \quad g(x(t), 1) = (x_1,\ x_2)^{\mathrm{T}},$$
$$f(x(t), 2) = (-3x_1 + 2x_2,\ 4x_1 - 5x_2)^{\mathrm{T}}, \quad g(x(t), 2) = (0.5x_2 \cos x_1,\ x_1 \sin x_2)^{\mathrm{T}},$$
$$f(x(t), 3) = (0.5x_1 - 0.125x_2,\ 0.5x_2 \sin^2 x_1 - 0.125x_1)^{\mathrm{T}}, \quad g(x(t), 3) = (x_1 \cos x_2,\ x_2)^{\mathrm{T}}.$$

Choose Lyapunov functions: $V(x, 1) = 0.5x_1^2 + x_2^2, V(x, 2) = x_1^2 + 0.5x_2^2, V(x, 3) = 0.5(x_1^2 + x_2^2)$. We can calculate that $\lambda_1 = -1, \lambda_2 = -1.5, \lambda_3 = 2.25$ and $\beta_1 = 4, \beta_2 = 0, \beta_3 = 4$. We can see that there are both stable subsystems ($\sigma(t) = 1, 2$) and an unstable subsystem ($\sigma(t) = 3$) in the whole system. The state trajectories of three subsystems are given in Figure 1.

Firstly, we consider the case that the semi-Markovian switching and impulsive jumps occur synchronously. The types of jumps are given by $h(x(t), 2, 1) = h(x(t), 3, 1) = (\sqrt{2}x_1,\ \frac{\sqrt{2}}{2}(x_1 - x_2))^{\mathrm{T}}$, $h(x(t), 1, 2) = h(x(t), 3, 2) = (\frac{\sqrt{6}}{2}x_2,\ \sqrt{3}x_1)^{\mathrm{T}}$ and $h(x(t), 1, 3) = h(x(t), 2, 3) = (\frac{1}{2}x_1,\ \frac{\sqrt{2}}{2}x_2)^{\mathrm{T}}$. For the impulsive jumps, we can get that $\mu_{12} = 3, \mu_{13} = 0.25, \mu_{21} = 2, \mu_{23} = 0.5, \mu_{31} = 4, \mu_{32} = 3$ based on (8).

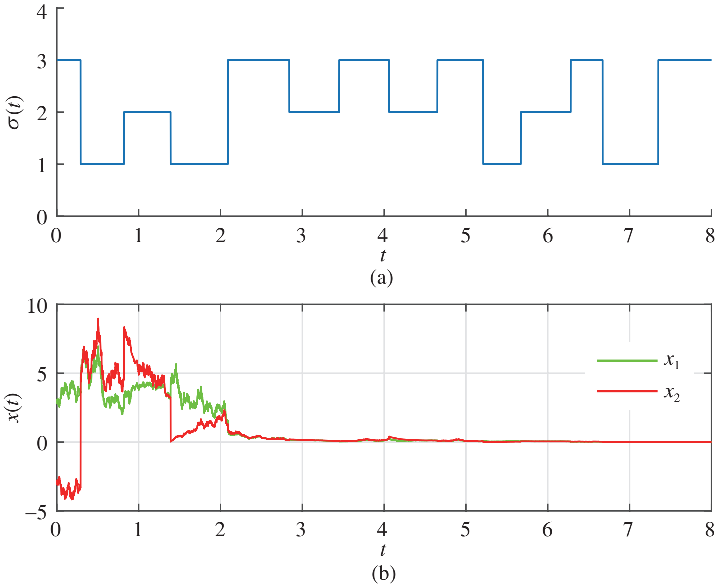

Figure 2 (Color online) The switching signals (a) and state trajectories (b) of semi-Markovian switched system (1) with synchronous impulsive jumps.

For the semi-Markovian switching signal $\sigma(t)$, let the transition probability matrix of σ_k be

$$P = \begin{bmatrix} 0 & 0.2 & 0.8 \\ 0.6 & 0 & 0.4 \\ 0.7 & 0.3 & 0 \end{bmatrix},$$

and $m_1 = 0.5, m_2 = 0.6$ and $m_3 = 0.8$. Then, we calculate that the stationary distribution of $\sigma(t)$ is $\pi = [0.3125, 0.1875, 0.5]$. Thus, we have $\sum_{i\in\mathcal{S}_S} \pi_i(\lambda_i + \sum_{j\in\mathcal{S}_S} \frac{p_{ij}}{m_i} \ln \mu_{ij} - 0.5\beta_i) = -0.7938 < 0$ and the semi-Markovian switched stochastic system (1) with synchronous impulsive jumps is exponentially stable a.s.. The switching signals and state trajectories are shown in Figure 2.

Next, we study the case that the semi-Markovian switching and impulsive jumps occur asynchronously. The switching signal, Lyapunov functions and subsystem parameters are the same as those above. We can get that $\mu_{12} = 2, \mu_{13} = 1, \mu_{21} = 2, \mu_{23} = 1, \mu_{31} = 2, \mu_{32} = 2$ based on (32). For the asynchronous impulsive jumps, let $r(k)$ be a discrete-time Markov chain taking values in $\mathcal{S}_J = \{1, 2, 3\}$ and the types of jumps are given by $h(x(t), 1) = (\frac{\sqrt{2}}{2}x_1,\ \frac{\sqrt{2}}{2}x_2)^{\rm T}$, $h(x(t), 2) = (\sqrt{2}x_1,\ \sqrt{2}x_2)^{\rm T}$ and $h(x(t), 3) = (\sqrt{3}x_1,\ \sqrt{3}x_2)^{\rm T}$. A simple calculation shows that $\alpha_1 = 0.5, \alpha_2 = 2$ and $\alpha_3 = 3$. Let the transition probability matrix of $r(k)$ be

$$\Pi = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

and have a unique invariant distribution $\tilde{\pi} = [\frac{21}{62}, \frac{23}{62}, \frac{18}{62}]$. The impulsive interval in the mean is $\theta = 1.5$. Thus, we can get $\sum_{l\in\mathcal{S}_J} \tilde{\pi}_l \frac{\ln \alpha_l}{\theta} + \sum_{i\in\mathcal{S}_S} \pi_i(\lambda_i + \sum_{j\in\mathcal{S}_S} \frac{p_{ij}}{m_i} \ln \mu_{ij} - 0.5\beta_i) = -0.2164 < 0$ and the semi-Markovian switched stochastic system (1) with asynchronous impulsive jumps is exponentially stable a.s.. The switching signals, impulsive jumps and state trajectories of system (1) are shown in Figure 3.

6 Conclusion

The exponential stability a.s. was discussed for semi-Markovian switched systems with synchronous impulsive jumps and asynchronous impulsive jumps, respectively. By applying the multiple Lyapunov function approach and the ergodic properties of semi-Markovian process and discrete-time Markov chain, sufficient conditions of exponential stability a.s. were obtained. Theorem 1 presents a unified framework

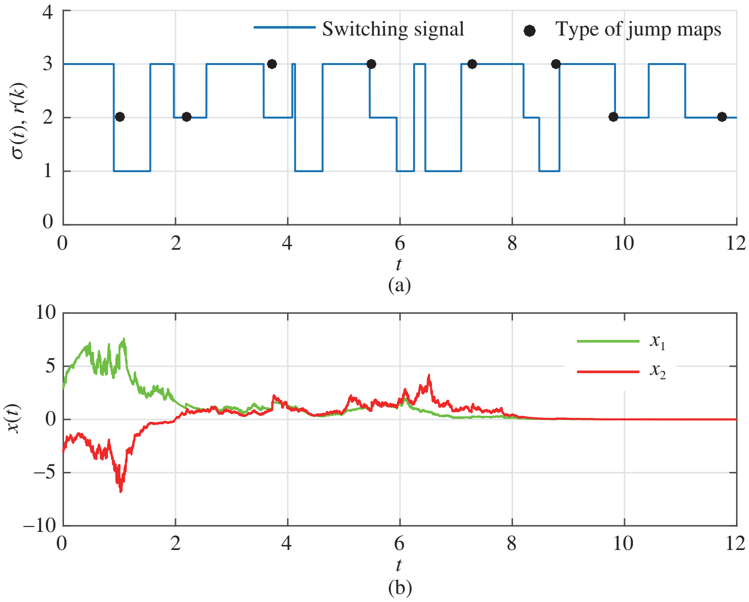

Figure 3 (Color online) The switching signals and impulsive jumps (a), and state trajectories (b) of semi-Markovian switched system (1) with asynchronous impulsive jumps.

to study exponential stability a.s. for impulsive switched stochastic systems driven by Markovian process or renewal process. Theorem 2 can be regarded as a general result of the semi-Markovian switched stochastic systems and randomly impulsive systems. Future work will focus on applying our results to real-world complicated systems such as networked systems [28].

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