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• **Supplementary File** •

Containment control for singular multi-agent systems based on internal model compensator

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Appendix A Importance

Theorem 1:

Proof. Based on the *Definition* 1, the stability of the pair (E_z, A_c) should be proved. Thus, there exists a matrix T such that A_c and E_z can be transformed into the following form:

$$\begin{aligned} \hat{A}_c &= TA_cT^{-1} \\ \hat{E}_z &= TE_zT^{-1}, \end{aligned}$$

with

$$\begin{aligned} \hat{A}_{ci} &= \begin{pmatrix} A_i + B_iK_{1i} & B_iK_{2i} \\ G_2C_i & G_1 \end{pmatrix} \\ \hat{E}_{zi} &= \begin{pmatrix} E_i & \mathbf{0} \\ \mathbf{0} & I_{ps_m} \end{pmatrix}, \\ i &= 1, 2, \ldots, N \end{aligned}$$

and $\hat{A}_c = \text{block diag}\Big\{\hat{A}_{c1}, \hat{A}_{c2}, \ldots, \hat{A}_{cN}\Big\}$. Let

$$\begin{aligned} T^{2k-1} &= \begin{pmatrix} I_N^k \otimes I_n & \mathbf{0} \end{pmatrix} \\ T^{2k} &= \begin{pmatrix} \mathbf{0} & I_N^k \otimes I_{ps_m} \end{pmatrix} \end{aligned}$$

in which $k = 1, 2, ..., N$ and I_N^k represents the k-row of identity matrix I_N. The transformational matrix is chosen as $T = \begin{pmatrix} (T^1)^T & (T^2)^T & \ldots & (T^{2N})^T \end{pmatrix}^T$. $\hat{A}_{ci}$ can be rewritten as $\hat{A}_{ci} = \mathfrak{A}_i + \mathfrak{B}_i\mathcal{K}_i$, with $\mathfrak{A}_i = \begin{pmatrix} A_i & \mathbf{0} \\ G_2C_i & G_1 \end{pmatrix}$, $\mathfrak{B}_i = \begin{pmatrix} B_i \\ \mathbf{0} \end{pmatrix}$. By *Lemma* 3, there exists the gain matrix $\mathcal{K}_i = (K_{1i}, K_{2i})$ such that $\left(\mathfrak{E}_i, \hat{A}_{ci}\right)$ is stable, in which $\mathfrak{E}_i = \hat{E}_{zi} = \begin{pmatrix} E_i & \mathbf{0} \\ \mathbf{0} & I_{ps_m} \end{pmatrix}$, one gets $\left(\hat{E}_{zi}, \hat{A}_{ci}\right)$ is stable, equivalently, (E_z, A_c) is stable.

Theorem 2:

Proof. Consider the following M equations:

$$\begin{aligned} E_zX_l\left(I_{2N} \otimes S\right) &= A_cX_l + B_{cl}, \\ l &\in \mathcal{M} \end{aligned} \tag{A1}$$

where, $\mathcal{M} = \{N+1, N+2, \ldots, N+M\}$, $X_l \in \mathbb{R}^{N(n+ps_m)\times 2Nq}$. Since (E_c, A_c) is stable, it has a unique solution X_l.

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Let

$$X_l = \begin{pmatrix} X_{l11} & X_{l12} \\ X_{l21} & X_{l22} \end{pmatrix},$$

with $X_{l11}, X_{l12} \in \mathbb{R}^{Nn\times Nq}$, $X_{l31}, X_{l32} \in \mathbb{R}^{Nps_m\times Nq}$. Then Eq.(A1) can be rewritten as follows:

$$\begin{aligned} X_{l21}\left(I_N\otimes S\right) &= \left(I_N\otimes G_1\right)X_{l21} \\ &\quad + \left(I_N\otimes G_2\right)\left(CX_{l11}\right) \end{aligned} \tag{A2}$$

$$\begin{aligned} X_{l22}\left(I_N\otimes S\right) &= \left(I_N\otimes G_1\right)X_{l22} + \left(I_N\otimes G_2\right) \\ &\quad \left(CX_{l12} - I_N\otimes \frac{1}{M}F_r\right). \end{aligned} \tag{A3}$$

Next, we will prove that $CX_{l11} = 0$. To begin with, let

$$X_{l21} = \begin{pmatrix} X^{l11} & \cdots & X^{l1N} \\ \vdots & & \vdots \\ X^{l(Np)1} & \cdots & X^{l(Np)N} \end{pmatrix}$$

$$CX_{l11} = \Omega_l = \begin{pmatrix} \Omega^{l11} & \cdots & \Omega^{l1N} \\ \vdots & & \vdots \\ \Omega^{l(Np)1} & \cdots & \Omega^{l(Np)N} \end{pmatrix}$$

with $X^{lij} \in \mathbb{R}^{s_m\times q}, \Omega^{lij} \in \mathbb{R}^{1\times q}, i = 1,\ldots,Np, j = 1,2,\ldots,N$.

By the definition of G_1 and G_2 in (6), and $\gamma_i = \gamma$, $\sigma_i = \sigma$ in (8), Eq.(A2) is equivalent as

$$X^{lij}S = \gamma X^{lij} + \sigma\Omega^{lij}, \tag{A4}$$

Let

$$X^{lij} = \begin{pmatrix} X_1^{lij} \\ X_2^{lij} \\ \vdots \\ X_{s_m}^{lij} \end{pmatrix}$$

with $X_k^{lij} \in \mathbb{R}^{1\times q}, k = 1,\ldots,s_m$ being the k-th row of X^{lij}. Combining (8) and (A4) provides:

$$\begin{pmatrix} X_1^{lij} \\ X_2^{lij} \\ \vdots \\ X_{s_m}^{lij} \end{pmatrix} S = \gamma \begin{pmatrix} X_1^{lij} \\ X_2^{lij} \\ \vdots \\ X_{s_m}^{lij} \end{pmatrix} + \sigma\Omega^{lij},$$

which can be written as

$$X_k^{lij}S = X_{k+1}^{lij},\ \ k = 1,2,\ldots,s_m-1, \tag{A5}$$

$$X_{s_m}^{lij}S + a_{s_m}X_1^{lij} + a_{s_m-1}X_2^{lij} + \ldots + a_1X_{s_m}^{lij} = \Omega^{lij}, \tag{A6}$$

Substituting A5 into A6:

$$\Omega^{lij} = X_1^{lij}\left(S^{s_m} + a_1S^{s_m-1} + \ldots + a_{s_m I_q}\right),$$

Since the matrix S has a minimal polynomial which divides the characteristic polynomial of G_1, $S^{s_m} + a_1S^{s_m-1} + \ldots + a_{s_m I_q} = 0$ and $\Omega^{lij} = \mathbf{0}, i = 1,\ldots,Np, j = 1,2,\ldots,N$.i.e.

$$CX_{l11} = \mathbf{0}. \tag{A7}$$

Similarly, from the equation (A3), one has:

$$CX_{l12} - I_N\otimes\frac{1}{M}F_r = \mathbf{0}. \tag{A8}$$

The compact form of $\tilde{e}_i = C_ix_i - F_r\eta_i$ is $\tilde{e} = Cx - \left(I_N\otimes F_r\right)\eta$. The error e can be rewritten as

$$\begin{aligned} e &= \left(\mathcal{H}\otimes I_p\right)Cx - \sum_{l\in\mathcal{M}}\left(\mathcal{H}_l\otimes F_r\right)\bar{\omega}_l \\ &= \left(\mathcal{H}\otimes I_p\right)Cx - \left(\mathcal{H}\otimes I_p\right)\left(I_N\otimes F_r\right)\eta \\ &\quad + \left(\mathcal{H}\otimes I_p\right)\left(I_N\otimes F_r\right)\eta - \sum_{l\in\mathcal{M}}\left(\mathcal{H}_l\otimes F_r\right)\bar{\omega}_l \\ &= \left(\mathcal{H}\otimes I_p\right)\tilde{e} + \left(\mathcal{H}\otimes F_r\right)\left(\eta - \sum_{l\in\mathcal{M}}\left(\mathcal{H}^{-1}\mathcal{H}_l\otimes I_q\right)\bar{\omega}_l\right) \end{aligned}$$

$$= \left(\mathcal{H}\otimes I_p\right)\tilde{e} + \left(\mathcal{H}\otimes F_r\right)\tilde{\eta}, \tag{A9}$$

Consider $\tilde{e}_i = C_i x_i - F_r \eta_i$, then:

$$\begin{aligned}\tilde{e} &= Cx - \left(I_N \otimes F_r\right)\eta \\ &= C_c x_c - \sum_{l\in\mathcal{M}} D_{cl} v_l,\end{aligned} \tag{A10}$$

where $C_c = \begin{pmatrix} C & \mathbf{0} \end{pmatrix}$, $D_{cl} = \begin{pmatrix} \mathbf{0} & I_N \otimes \frac{1}{M} F_r \end{pmatrix}$. Therefore, combining (A7) and (A8) acquires

$$\begin{aligned}C_c X_l - D_{cl} &= \begin{pmatrix} C_c X_{l11} & C_c X_{l12} \end{pmatrix} \\ &\quad - \begin{pmatrix} \mathbf{0} & I_N \otimes \frac{1}{M} F_r \end{pmatrix} \\ &= \begin{pmatrix} C_c X_{l11} & C_c X_{l12} - I_N \otimes \frac{1}{M} F_r \end{pmatrix} \\ &= \mathbf{0}.\end{aligned} \tag{A11}$$

Let $\tilde{x}_c = x_c - \sum\limits_{l\in\mathcal{M}} X_l v_l$, then the following is satisfied

$$\begin{aligned}E_z \dot{\tilde{x}}_c &= E_z \dot{x}_c - E_z \sum_{l\in\mathcal{M}} X_l \dot{v}_l \\ &= A_c x_c + \sum_{l\in\mathcal{M}} B_{cl} v_l \\ &\quad - E_z \sum_{l\in\mathcal{M}} X_l \begin{pmatrix} \left(I_N \otimes S\right)\bar{\omega}_l \\ \Phi \end{pmatrix},\end{aligned}$$

where $\Phi = \dot{\eta} = \left(I_N \otimes S + \mu\left(\mathcal{H}\otimes I_q\right)\right)\eta - \mu \sum\limits_{l\in\mathcal{M}} \left(\mathcal{H}_l \otimes I_q\right)\bar{\omega}_l$, and the fact

$$\begin{pmatrix} \left(I_N \otimes S\right)\bar{\omega}_l \\ \Phi \end{pmatrix} = \begin{pmatrix} \left(I_N \otimes S\right)\bar{\omega}_l \\ \left(I_N \otimes S\right)\eta \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mu\left(\mathcal{H}\otimes I_q\right)\tilde{\eta} \end{pmatrix},$$

yields:

$$\begin{aligned}E_z \dot{\tilde{x}}_c &= A_c x_c + \sum_{l\in\mathcal{M}} B_{cl} v_l - E_z \sum_{l\in\mathcal{M}} X_l \begin{pmatrix} \left(I_N \otimes S\right)\bar{\omega}_l \\ \left(I_N \otimes S\right)\eta \end{pmatrix} \\ &\quad - E_z \sum_{l\in\mathcal{M}} \begin{pmatrix} \mathbf{0} \\ \mu\left(\mathcal{H}\otimes I_q\right)\tilde{\eta} \end{pmatrix} \\ &= A_c x_c + \sum_{l\in\mathcal{M}} B_{cl} v_l - \sum_{l\in\mathcal{M}} E_z X_l \left(I_{2N} \otimes S\right) v_l \\ &\quad - E_z \sum_{l\in\mathcal{M}} X_l \begin{pmatrix} \mathbf{0} \\ \mu\left(\mathcal{H}\otimes I_q\right)\tilde{\eta} \end{pmatrix},\end{aligned} \tag{A12}$$

According to (A2), one gets:

$$E_z \dot{\tilde{x}}_c = A_c \tilde{x}_c + \Theta\tilde{\eta}, \tag{A13}$$

in which, $\Theta = -E_z \sum\limits_{l\in\mathcal{M}} X_l \begin{pmatrix} \mathbf{0} \\ \mu\left(\mathcal{H}\otimes I_q\right) \end{pmatrix}$.

Let $\varphi = \left(\tilde{x}_c^T, \tilde{\eta}^T\right)^T$, then

$$E_d \dot{\varphi} = \begin{pmatrix} A_c & \Theta \\ 0 & \left(I_N \otimes S\right) + \mu\left(\mathcal{H}\otimes I_q\right) \end{pmatrix} \varphi, \tag{A14}$$

in which, $E_d = \begin{pmatrix} E_z & \mathbf{0} \\ \mathbf{0} & I_{Nq} \end{pmatrix}$. From the proof that $\left(E_z, A_c\right)$ is stable, and $\left(I_N \otimes S\right) + \mu\left(\mathcal{H}\otimes I_q\right)$ is Hurwitz. Then, it can be concluded that $\lim\limits_{t\to\infty} \varphi = 0$. Thus, we have $\lim\limits_{t\to\infty} \tilde{x}_c = 0$.

Additionally, the error $\tilde{e}$ in (A10) can be written as

$$\begin{aligned}\tilde{e} &= C_c x_c - \sum_{l\in\mathcal{M}} D_{cl} v_l \\ &= C_c \left(\tilde{x}_c + \sum_{l\in\mathcal{M}} X_l v_l\right) - \sum_{l\in\mathcal{M}} D_{cl} v_l \\ &= C_c \tilde{x}_c + \sum_{l\in\mathcal{M}} \left(C_c X_l - D_{cl}\right) v_l \\ &= C_c \tilde{x}_c.\end{aligned} \tag{A15}$$

Therefore, $\lim\limits_{t\to\infty} \tilde{e} = 0$, means $\lim\limits_{t\to\infty} e = 0$, and the containment control problem for singular multi-agent systems has been solved.