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On feedback invariant subspace of Boolean control networks

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Dear editor,

Subspace theory is important in the study of nonlinear systems. The model of Boolean networks (BNs) was firstly established by Kauffman for the regulation of genes. Boolean networks with inputs and outputs are called Boolean control networks (BCNs). In the last decade, a semi-tensor product (STP) method has been proposed for the study of BNs and BCNs [1]. Consequently, many novel results have been presented for the analysis and control of BNs [2, 3]. Moreover, STP method has also been applied to other related fields [4, 5].

Particularly, using STP, Cheng and Qi [6] established the state-space structure of BNs, including coordinate transformation, regular subspace and invariant subspace. The state-space structure is very important for disturbance decoupling and Kalman decomposition of BCNs [7, 8]. Zhu and Jü [9] further investigated some problems about regular subspace and invariant subspace in BCNs. However, for a given regular subspace, how to find a state feedback control and a coordinate transformation such that the regular subspace becomes an invariant subspace is still untilled. This problem is called feedback invariant subspace of BCNs.

We investigate the verification of feedback invariant subspace of BCNs. The main contributions of this study are two-fold. (i) The concept of feedback invariant subspace is firstly proposed for BCNs in this study. Based on the algebraic form of BCNs, the verification of feedback invariant subspace is converted to the solvability of matrix equation. (ii) Two new methods, namely, column stacking method and method of undetermined coefficients, are proposed for the solvability of matrix equation. Both methods can obtain all possible state feedback gain matrices for feedback invariant subspace of BCNs.

Some notations are listed below. $\mathcal{D} := \{0, 1\}$. δ_n^i denotes the i-th column of the identity matrix I_n. $\Delta_n := \{\delta_n^i : i = 1, \ldots, n\}$. A matrix $L \in \mathbb{R}^{m\times n}$ is called a logical matrix, if its columns belong to Δ_m. $\mathcal{L}_{m\times n}$ denotes the set of $m \times n$ logical matrices. A logical matrix $L = [\delta_m^{i_1}, \ldots, \delta_m^{i_n}]$ is briefly written as $L = \delta_m[i_1, \ldots, i_n]$. Symbols "$\otimes$" and "$\ltimes$" represent Kronecker product and STP of matrices, respectively. $W_{[m,n]}$ and R_k^p denote swap matrix and power-reducing matrix, respectively. $V_c(A)$ denotes the column stacking form of matrix A. For details, please refer to Ref. [1].

Consider the following Boolean control network:

$$\begin{cases} x_1(t+1) = f_1(X(t), U(t)), \\ \quad\vdots \\ x_n(t+1) = f_n(X(t), U(t)), \end{cases} \tag{1}$$

where $X(t) = (x_1(t), \ldots, x_n(t)) \in \mathcal{D}^n$ denotes the state variables of system (1) at time t, $U(t) = (u_1(t), \ldots, u_m(t)) \in \mathcal{D}^m$ denotes the control variables of system (1) at time t, and $f_i : \mathcal{D}^{m+n} \to \mathcal{D}$, $i = 1, \ldots, n$ are Boolean functions.

Firstly, we give the definition of feedback invariant subspace of system (1).

Definition 1. Given a regular subspace $\mathcal{Z} =$

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$F_\ell\{Z^1\}$, $Z^1=\{z_1^1,\dots,z_s^1\}$. $\mathcal{Z}$ is said to be a feedback invariant subspace of system (1), if there exist $Z^2=\{z_1^2,z_2^2,\dots,z_{n-s}^2\}$ and state feedback control in the form of

$$\begin{cases} u_1(t+1)=g_1(X(t)),\\ \quad\vdots\\ u_m(t+1)=g_m(X(t)),\end{cases}\tag{2}$$

where $g_i:\mathcal{D}^n\to\mathcal{D}$, $i=1,\dots,m$ are Boolean functions to be determined, such that $T:(x_1,\dots,x_n)\mapsto Z=(Z^1,Z^2)$ is a logical coordinate transformation. Under the coordinate transformation and the control (2), system (1) becomes the following form:

$$\begin{cases} Z^1(t+1)=F^1(Z^1(t)),\\ Z^2(t+1)=F^2(Z(t)).\end{cases}\tag{3}$$

Using the vector form of Boolean variables, system (1) can be converted into the following algebraic form:

$$x(t+1)=Lu(t)x(t),\tag{4}$$

where $x(t)=\ltimes_{i=1}^n x_i(t)$, $u(t)=\ltimes_{j=1}^m u_j(t)$, and $L\in\mathcal{L}_{2^n\times 2^{m+n}}$ is the state transition matrix.

Similarly, the state feedback control (2) has the following algebraic form:

$$u(t)=Gx(t),\tag{5}$$

where $G\in\mathcal{L}_{2^m\times 2^n}$ is the state feedback gain matrix.

Putting (5) into (4), we obtain the following closed-loop system:

$$x(t+1)=LGR_{2^n}^p x(t):=\widetilde{L}x(t).\tag{6}$$

According to [6], $\mathcal{Z}=F_\ell\{Z^1\}$ is a feedback invariant subspace of system (1), if and only if there exist $G\in\mathcal{L}_{2^m\times 2^n}$ and $H\in\mathcal{L}_{2^s\times 2^s}$ such that

$$Q\widetilde{L}=HQ,\tag{7}$$

that is,

$$QLGR_{2^n}^p=HQ.\tag{8}$$

In the following, we will investigate how to design $G\in\mathcal{L}_{2^m\times 2^n}$ and $H\in\mathcal{L}_{2^s\times 2^s}$ such that (8) holds.

Given a regular subspace $\mathcal{Z}=F_\ell\{Z^1\}$, $Z^1=\{z_1^1,\dots,z_s^1\}$, in order to obtain $G\in\mathcal{L}_{2^m\times 2^n}$ and $H\in\mathcal{L}_{2^s\times 2^s}$ such that $\mathcal{Z}$ is a feedback invariant subspace, we need to solve matrix equation (8). We solve matrix equation (8) via column stacking method. For applications of the column stacking approach, please see Ref. [2].

Taking column stacking form of each side in (8), we have

$$V_c(QLGR_{2^n}^p)=V_c(HQ).\tag{9}$$

For the right hand side of (9), one can obtain

$$V_c(HQ)=Q^{\mathrm{T}}\ltimes V_c(H)=(Q^{\mathrm{T}}\otimes I_{2^s})V_c(H).\tag{10}$$

Set

$$A=Q^{\mathrm{T}}\otimes I_{2^s},\ \alpha=V_c(H)=(\alpha_1,\dots,\alpha_{2^{2s}})^{\mathrm{T}}.\tag{11}$$

Then, Eq. (10) can be rewritten as

$$V_c(HQ)=A\alpha.\tag{12}$$

For the left hand side of (9), we have

$$\begin{aligned}&V_c(QLGR_{2^n}^p)\\&=V_c(QL(G\otimes I_{2^n})R_{2^n}^p)\\&=[(R_{2^n}^p)^{\mathrm{T}}\otimes(QL)]V_c[G\otimes I_{2^n}]\\&=[(R_{2^n}^p)^{\mathrm{T}}\otimes(QL)]W_{[2^n,2^n]}W_{[2^{m+n},2^n]}W_{[2^{2n},2^{m+n}]}\\&\quad\ltimes V_c(I_{2^n})V_c(G).\end{aligned}\tag{13}$$

Let

$$\begin{aligned}B=&[(R_{2^n}^p)^{\mathrm{T}}\otimes(QL)]W_{[2^n,2^n]}W_{[2^{m+n},2^n]}\\&\ltimes W_{[2^{2n},2^{m+n}]}V_c(I_{2^n}),\end{aligned}\tag{14}$$

and

$$\beta=V_c(G)=(\beta_1,\dots,\beta_{2^{m+n}})^{\mathrm{T}}.\tag{15}$$

Then, Eq. (13) can be expressed as

$$V_c(QLGR_{2^n}^p)=B\beta.\tag{16}$$

From (12) and (16), matrix equation (9) can be converted into the following linear equation:

$$A\alpha=B\beta,\tag{17}$$

where A and α are given in (11), B is given in (14), and β is given in (15).

Noticing that $G\in\mathcal{L}_{2^m\times 2^n}$ and $H\in\mathcal{L}_{2^s\times 2^s}$ are logical matrices, we can obtain the following constraints:

$$\begin{cases}\alpha_{(i-1)2^s+j}\in\{0,1\},\ i=1,\dots,2^s,\\ \qquad\qquad j=1,\dots,2^s,\\ \displaystyle\sum_{j=1}^{2^s}\alpha_{(i-1)2^s+j}=1,\ i=1,\dots,2^s,\\ \beta_{(i-1)2^m+j}\in\{0,1\},\ i=1,\dots,2^n,\\ \qquad\qquad j=1,\dots,2^m,\\ \displaystyle\sum_{j=1}^{2^m}\beta_{(i-1)2^m+j}=1,\ i=1,\dots,2^n.\end{cases}\tag{18}$$

Based on the above analysis, we have the following result.

Theorem 1. Given a regular subspace $\mathcal{Z} = F_\ell\{Z^1\}$, $Z^1 = \{z_1^1, \ldots, z_s^1\}$. $\mathcal{Z}$ is a feedback invariant subspace of system (1), if and only if the following linear equation has solution:

$$\begin{cases} A\alpha - B\beta = 0, \\ \alpha_{(i-1)2^s+j} \in \{0,1\},\ i = 1,\ldots,2^s, \\ \qquad\qquad j = 1,\ldots,2^s, \\ \sum_{j=1}^{2^s} \alpha_{(i-1)2^s+j} = 1,\ i = 1,\ldots,2^s, \\ \beta_{(i-1)2^m+j} \in \{0,1\},\ i = 1,\ldots,2^n, \\ \qquad\qquad j = 1,\ldots,2^m, \\ \sum_{j=1}^{2^m} \beta_{(i-1)2^m+j} = 1,\ i = 1,\ldots,2^n, \end{cases} \tag{19}$$

where A and α are given in (11), B is given in (14), and β is given in (15).

The main feature of Theorem 1 is that one can convert matrix equation (8) into linear equation (19), which facilitates the solvability of matrix equation (8). However, because most of elements in α and β are zero, linear equation (19) is somewhat tedious. In the following, we propose a method of undetermined coefficients for the solvability of matrix equation (8).

Given a regular subspace $\mathcal{Z} = F_\ell\{Z^1\}$, $Z^1 = \{z_1^1, \ldots, z_s^1\}$. For the matrix equation (8), one can see that Q and L are known, denoted by

$$Q = \delta_{2^s}[q_1, \ldots, q_{2^n}],$$

and

$$L = \delta_{2^n}[l_1, \ldots, l_{2^{n+m}}],$$

while H and G are unknown and to be determined, denoted by

$$H = \delta_{2^s}[h_1, \ldots, h_{2^s}],$$

and

$$G = \delta_{2^m}[g_1, \ldots, g_{2^n}].$$

Then, a simple calculation shows that

$$QL = \delta_{2^s}[q_{l_1}, \ldots, q_{l_{2^{m+n}}}], \tag{20}$$

and

$$HQ = \delta_{2^s}[h_{q_1}, \ldots, h_{q_{2^n}}]. \tag{21}$$

Divide $QL = \delta_{2^s}[q_{l_1}, \ldots, q_{l_{2^{m+n}}}]$ into 2^m equal blocks as

$$QL = [\mathrm{Blk}_1(QL), \ldots, \mathrm{Blk}_{2^m}(QL)],$$

where $\mathrm{Blk}_i(QL) = \delta_{2^s}[q_{l_{(i-1)2^n+1}}, \ldots, q_{l_{i2^n}}]$, $i = 1, \ldots, 2^m$. Then $QLG = [\mathrm{Blk}_{g_1}(QL), \ldots, \mathrm{Blk}_{g_{2^n}}(QL)]$.

Noticing that

$$R_{2^n}^p = \delta_{2^{2n}}[1, \ldots, (j-1)2^n + j, \ldots, (2^n-1)2^n + 2^n],$$

we have

$$QLGR_{2^n}^p = \delta_{2^s}[q_{l_{(g_1-1)2^n+1}}, \ldots, q_{l_{(g_j-1)2^n+j}}, \ldots, q_{l_{(g_{2^n}-1)2^n+2^n}}]. \tag{22}$$

For the matrix equation $QLGR_{2^n}^p = HQ$, one can see from (21) and (22) that

$$q_{l_{(g_j-1)2^n+j}} = h_{q_j},\ j = 1, 2, \ldots, 2^n. \tag{23}$$

Based on the above analysis, we have the following result on the verification of feedback invariant subspace.

Theorem 2. Given a regular subspace $\mathcal{Z} = F_\ell\{Z^1\}$, $Z^1 = \{z_1^1, \ldots, z_s^1\}$. $\mathcal{Z}$ is a feedback invariant subspace of system (1), if and only if the following index equation has solution:

$$\begin{cases} q_{l_{(g_1-1)2^n+1}} = h_{q_1}, \\ q_{l_{(g_2-1)2^n+2}} = h_{q_2}, \\ \qquad \vdots \\ q_{l_{(g_{2^n}-1)2^n+2^n}} = h_{q_{2^n}}, \\ g_j \in \{1, \ldots, 2^m\},\ j = 1, 2, \ldots, 2^n, \\ h_{q_j} \in \{1, \ldots, 2^s\},\ j = 1, 2, \ldots, 2^n. \end{cases} \tag{24}$$

Compared with the column stacking method, the method of undetermined coefficients is more effective in reducing the computational burden.

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