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Sequential fusion estimation for multisensor systems with non-Gaussian noises

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Abstract The sequential fusion estimation for multisensor systems disturbed by non-Gaussian but heavy-tailed noises is studied in this paper. Based on multivariate t-distribution and the approximate t-filter, the sequential fusion algorithm is presented. The performance of the proposed algorithm is analyzed and compared with the t-filter-based centralized batch fusion and the Gaussian Kalman filter-based optimal centralized fusion. Theoretical analysis and exhaustive experimental analysis show that the proposed algorithm is effective. As the generalization of the classical Gaussian Kalman filter-based optimal sequential fusion algorithm, the presented algorithm is shown to be superior to the Gaussian Kalman filter-based optimal centralized batch fusion and the optimal sequential fusion in estimation of dynamic systems with non-Gaussian noises.

Keywords state estimation, sequential fusion, non-Gaussian disturbance, heavy-tailed noise, multivariate t-distribution

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1 Introduction

Multisensor data fusion has been a hot topic in the last few years, primarily owing to its advantageous performance in high precision estimation, strong reliability and robustness [1, 2]. The aim of multisensor fusion estimation is to make the best use of the local measurements or local estimators generated from every single sensor, to get the fusion estimation, which has higher accuracy than any local estimator that barely uses single sensor's information [1, 3]. It is first studied in military applications and has been developed in many high-technology fields, such as aerospace, guidance, control, defense, navigation of intelligent vehicles, positioning of robotics, target tracking, monitoring and fault detection [1, 4–7].

Most of the literatures that study state filtering estimation or fusion estimation are based on the hypothesis that the process noise and the measurement noise meet Gaussian distribution. Ma et al. have done some good work on the filtering estimation. Ref. [8] concerns the variance-constrained distributed filtering problem for a class of time-varying systems subject to multiplicative noises, unknown but bounded disturbances and deception attacks over sensor networks, and Ref. [9] aims to construct a filter such that both the prespecified H_∞ requirement and the envelope constraint are guaranteed simultaneously over a finite horizon. While in some practical applications such as target tracking, where there are some observation outliers generated by unreliable sensors or in the influence of unknown disturbances, the

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process noise and the measurement noise are non-Gaussian. An effective way to model non-Gaussian noise is by using the student's t-distribution [10, 11].

To deal with the problem of filtering and smoothing for linear or nonlinear systems with heavy-tailed noises, some scholars have made many useful explorations and proposed some filters and smoothers based on multivariate t-distribution and the variational Bayesian approach [12–18]. However, the fusion estimation for systems with heavy-tailed noises is rarely considered. Ref. [10] proposes the centralized batch fusion for a kind of linear time-invariant system with heavy-tailed noises. Based on a cubature information filter, Ref. [11] presents the centralized fusion for a kind of nonlinear system with heavy-tailed noises.

It is well known that for the Gaussian noise-driven system, the centralized batch fusion is usually the globally optimal, which uses the original measurements directly by augmentation of measurement equations [1, 3]. However, it has several disadvantages: (1) because of augmentation of measurements and measurement matrices, etc., the computational burden is enlarged; (2) the batch fusion requires dealing with all the measurements simultaneously at the fusion center, which has low efficiency as all the measurements are unlikely to come to the fusion center at exactly the same time because of the network's limitation of transmitting information or other reasons. Therefore, sequential fusion is studied by a lot of literature for the Gaussian noise-driven systems, by which the observations of each sensor are processed, and the estimations are fused in a particular sequence in the fusion center [3, 19–25].

Through the above analysis, we find that there are many open problems for fusion estimation of dynamic systems disturbed by heavy-tailed noises.

- The sequential fusion for dynamic systems with heavy-tailed noises is not derived.
- The performance evaluation of different fusion algorithms for dynamic systems with heavy-tailed noises is not considered.

The main contributions of this paper are as follows:

- Because the characteristics of Gaussian and heavy-tailed noises are different, the formulation of the system should be different, and the extension of sequential fusion estimation from Gaussian noise systems to the heavy-tailed noise systems is non-trivial. In this article, we intend to present the sequential fusion estimation algorithm for multisensor dynamic systems with heavy-tailed noises that is formulated using multivariate t-distributions.
- The performance of different fusion algorithms will be analyzed theoretically and through simulation examples as well. It is compared with the t-filter-based centralized fusion and the Gaussian Kalman filter-based centralized fusion, in addition to the comparison between the local estimators that barely use a single sensor's information.

The rest of the paper is organized as follows. In Section 2, the problem is formulated. Section 3 derives the sequential fusion algorithm for the multisensor linear time-variant dynamic systems with heavy-tailed noises. Section 4 is the simulation and Section 5 draws the conclusion.

2 Problem formulation

The multisensor dynamic system that has N sensors observing a single target can be described as follows [1, 26, 27]:

$$x_{l+1} = F_l x_l + w_l, \quad l = 0, 1, \ldots, \tag{1}$$

$$y_{i,l} = H_{i,l} x_l + v_{i,l}, \quad i = 1, 2, \ldots, N, \tag{2}$$

where i denotes the i-th sensor. $x_l \in \mathbb{R}^n$ is the system state at the l-th time instant. $F_l \in \mathbb{R}^{n\times n}$ is the state transition matrix. $y_{i,l} \in \mathbb{R}^{m_i}$ is the measurement of sensor i at time l. $H_{i,l}$ is the measurement matrix of sensor i. The system noise w_l and the measurement noise $v_{i,l}$ are heavy-tailed noises, which can be modeled by the multivariate t-distribution as follows:

$$p(w_l) = \mathrm{St}(w_l; 0, Q_l, \nu_w), \tag{3}$$

$$p(v_{i,l}) = \mathrm{St}(v_{i,l}; 0, R_{i,l}, \nu_i), \quad i = 1, 2, \ldots, N, \tag{4}$$

where $\mathrm{St}(\cdot; \bar{x}, P, \nu)$ denotes a multivariate t-distribution whose mean is $\bar{x}$, scale matrix is P and degree of freedom (dof) is ν.

Similarly, it is assumed that the system state initial value x_0 is also heavy-tailed and meets the multivariate t-distribution with mean $\hat{x}_{0|0}$, the scale matrix $P_{0|0}$ and the dof ν_0, i.e.,

$$p(x_0) = \mathrm{St}(x_0; \hat{x}_{0|0}, P_{0|0}, \nu_0). \tag{5}$$

It is assumed that $x_0, v_{i,l}$ and w_l are uncorrelated.

Our study aims to estimate state x_l by sequentially using the multisensor observations, i.e., to find

$$\hat{x}_{l|l} = \mathrm{E}[x_l|Y_l] = \int x_l p(x_l|Y_l)\mathrm{d}x_l, \tag{6}$$

$$P_{l|l} = \frac{\nu-2}{\nu}\mathrm{E}[\tilde{x}_{l|l}\tilde{x}_{l|l}^{\mathrm{T}}|Y_l] = \frac{\nu-2}{\nu}\int \tilde{x}_{l|l}\tilde{x}_{l|l}^{\mathrm{T}} p(x_l|Y_l)\mathrm{d}x_l, \tag{7}$$

where

$$p(x_l|Y_l) = \mathrm{St}(x_l; \hat{x}_{l|l}, P_{l|l}, \nu), \tag{8}$$

$$Y_l = \{y_{i,t}, t = 1, 2, \ldots, l; i = 1, 2, \ldots, N\}. \tag{9}$$

3 The sequential fusion algorithm

In this section, it is assumed that the dofs of the initial state, the process noise, and the measurement noises are equal, namely, $\nu_w = \nu_i = \nu$ for $i = 0, 1, 2, \ldots, N$. The unequal case will be addressed in Remark 2.

Before presenting the sequential fusion algorithm, we introduce a lemma first. All the results of Lemma 1 can be found in [17, 28, 29], and the detailed proof of the last property of Lemma 1 can be found in [30].

Lemma 1. x meets the multivariate t-distribution $\mathrm{St}(x; \bar{x}, P, \nu)$ whose mean is $\bar{x}$, scale matrix is P, and dof is ν. Its probability density function (pdf) is

$$p(x) = \frac{\Gamma(\frac{\nu+2}{2})}{\Gamma(\frac{\nu}{2})}\frac{1}{(\nu\pi)^{\frac{d}{2}}}\frac{1}{\sqrt{\det(P)}}\left(1+\frac{\Delta^2}{\nu}\right)^{-\frac{\nu+2}{2}}, \tag{10}$$

where $\Delta^2 = (x-\bar{x})^{\mathrm{T}}P^{-1}(x-\bar{x})$.

It has the following properties:

- The covariance of x is $\Sigma = \frac{\nu}{\nu-2}P$.
- When ν tends to infinity, the distribution of x reduces to Gaussian.
- Let $y = Ax + b$, and then $p(y) = \mathrm{St}(y; A\bar{x} + b, APA^{\mathrm{T}}, \nu)$, where A and b are the matrix and vector of proper dimensions, respectively.
- If $x_1 \in \mathbb{R}^{d_1}$ and $x_2 \in \mathbb{R}^{d_2}$ meet joint t-distribution whose pdf is

$$p(x_1, x_2) = \mathrm{St}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \nu\right), \tag{11}$$

where $P_{ii} \in \mathbb{R}^{d_i\times d_i}$ is the scale matrix of x_i, $P_{ij} \in \mathbb{R}^{d_i\times d_j}$ is the joint scale matrix of x_i and x_j, $i = 1, 2$, then the marginal pdfs of x_1 and x_2 are

$$\begin{cases} p(x_1) = \mathrm{St}(x_1; \mu_1, P_{11}, \nu), \\ p(x_2) = \mathrm{St}(x_2; \mu_2, P_{22}, \nu). \end{cases} \tag{12}$$

The conditional pdf $p(x_1|x_2)$ is given by

$$p(x_1|x_2) = \mathrm{St}(x_1; \mu_{1|2}, P_{1|2}, \nu_{1|2}), \tag{13}$$

where

$$\nu_{1|2} = \nu + d_2, \tag{14}$$

$$\mu_{1|2} = \mu_1 + P_{12}P_{22}^{-1}(x_2 - \mu_2), \tag{15}$$

$$P_{1|2} = \frac{\nu + \Delta_2^2}{\nu + d_2}(P_{11} - P_{12}P_{22}^{-1}P_{12}^{\mathrm{T}}), \tag{16}$$

and $\Delta_2^2 = (x_2 - \mu_2)^{\mathrm{T}} P_{22}^{-1}(x_2 - \mu_2)$.

Let

$$y_l^a = \begin{bmatrix} y_{1,l} \\ y_{2,l} \\ \vdots \\ y_{N,l} \end{bmatrix}, \quad H_l^a = \begin{bmatrix} H_{1,l} \\ H_{2,l} \\ \vdots \\ H_{N,l} \end{bmatrix}, \quad v_l^a = \begin{bmatrix} v_{1,l} \\ v_{2,l} \\ \vdots \\ v_{N,l} \end{bmatrix}. \tag{17}$$

From (17), Eq. (2) has a new form:

$$y_l^a = H_l^a x_l + v_l^a. \tag{18}$$

From Lemma 1,

$$p(v_l^a) = \mathrm{St}(v_l^a; 0, R_l^a, \nu), \tag{19}$$

where

$$R_l^a = \mathrm{diag}\{R_{1,l}, R_{2,l}, \ldots, R_{N,l}\}. \tag{20}$$

For system (1) and (18), by the use of the properties listed in Lemma 1, the state estimation by the centralized fusion of sensors 1 to N can be computed by [17, 18, 31]

$$\begin{cases} \hat{x}_{c,l|l-1} = F_{l-1}\hat{x}_{c,l-1|l-1}, \\ P_{c,l|l-1} = F_{l-1}P_{c,l-1|l-1}F_{l-1}^{\mathrm{T}} + Q_{l-1}, \\ \hat{x}_{c,l|l} = \hat{x}_{c,l|l-1} + K_{c,l}\tilde{y}_{c,l}, \\ P_{c,l|l} = \frac{(\nu-2)(\nu+\Delta_{c,l}^2)}{\nu(\nu+m-2)}(I - K_{c,l}H_l^a)P_{c,l|l-1}, \\ K_{c,l} = P_{c,l|l-1}^{\tilde{x}\tilde{y}}(P_{c,l|l-1}^{\tilde{y}\tilde{y}})^{-1}, \\ P_{c,l|l-1}^{\tilde{y}\tilde{y}} = H_l^a P_{c,l|l-1} H_l^{a,\mathrm{T}} + R_l^a, \\ P_{c,l|l-1}^{\tilde{x}\tilde{y}} = P_{c,l|l-1}H_l^{\mathrm{T}}, \\ \Delta_{c,l}^2 = \tilde{y}_{c,l}^{\mathrm{T}}(P_{c,l|l-1}^{\tilde{y}\tilde{y}})^{-1}\tilde{y}_{c,l}, \\ \tilde{y}_{c,l} = y_l^a - \hat{y}_{c,l}, \\ \hat{y}_{c,l} = H_l^a\hat{x}_{c,l|l-1}, \end{cases} \tag{21}$$

where $m = \sum_{i=1}^N m_i$, and the subscript c denotes the centralized fusion.

To avoid augmentation of matrices and vectors, and to improve the efficiency of fusion estimation, we will deduce the sequential fusion algorithm in the sequel.

Theorem 1. For the linear dynamic system (1)–(5), the state estimation by sequential fusion of sensors 1 to N can be computed by

$$\begin{cases} \hat{x}_{s,l|l} = \hat{x}_{s,l|l-1} + \sum\limits_{i=1}^{N} K_{i,l}[y_{i,l} - H_{i,l}\hat{x}_{i-1,l|l}], \\ P_{s,l|l} = (\frac{\nu-2}{\nu})^N \prod\limits_{1}^{i=N} (\frac{\nu+\Delta_{i,l}^2}{\nu+m_i-2})(I - K_{i,l}H_{i,l})P_{s,l|l-1}, \end{cases} \tag{22}$$

where for $i = 1, 2, \ldots, N$,

$$\begin{cases} K_{i,l} = P_l^{\tilde{x}\tilde{y}_i}(P_l^{\tilde{y}_i\tilde{y}_i})^{-1}, \\ P_l^{\tilde{x}\tilde{y}_i} = P_{i-1,l|l}H_{i,l}^{\mathrm{T}}, \\ P_l^{\tilde{y}_i\tilde{y}_i} = H_{i,l}P_{i-1,l|l}H_{i,l}^{\mathrm{T}} + R_{i,l}, \\ \Delta_{i,l}^2 = \tilde{y}_{i,l}^{\mathrm{T}}(P_l^{\tilde{y}_i\tilde{y}_i})^{-1}\tilde{y}_{i,l}, \\ \tilde{y}_{i,l} = y_{i,l} - \hat{y}_{i,l}, \\ \hat{y}_{i,l} = H_{i,l}\hat{x}_{i-1,l|l}, \\ \hat{x}_{0,l|l} = \hat{x}_{s,l|l-1} = F_{l-1}\hat{x}_{s,l-1|l-1}, \\ P_{0,l|l} = P_{s,l|l-1} = F_{l-1}P_{s,l-1|l-1}F_{l-1}^{\mathrm{T}} + Q_{l-1}, \end{cases} \tag{23}$$

and $\prod_1^{i=N} D_i = D_N D_{N-1} \ \cdots \ D_1$ is the product of N terms from the largest index N to the smallest index 1. Note that $\prod_1^{i=N} D_i \neq \prod_{i=1}^{N} D_i = D_1 D_2 \ \cdots \ D_N$ because matrix multiplication does not commute.

Proof. Step 1: Time update. From Lemma 1 and (1)–(5), we have

$$p(x_{l-1}, w_{l-1}|Y_{l-1}) = \mathrm{St}\left(\begin{bmatrix} x_{l-1} \\ w_{l-1} \end{bmatrix}; \begin{bmatrix} \hat{x}_{s,l-1|l-1} \\ 0 \end{bmatrix}, \begin{bmatrix} P_{s,l-1|l-1} & 0 \\ 0 & Q_{l-1} \end{bmatrix}, \nu\right). \tag{24}$$

From (1) and Lemma 1, we have

$$p(x_l|Y_{l-1}) = \mathrm{St}(x_l; \hat{x}_{s,l|l-1}, P_{s,l|l-1}, \nu), \tag{25}$$

where

$$\hat{x}_{s,l|l-1} = F_{l-1}\hat{x}_{s,l-1|l-1}, \tag{26}$$

$$P_{s,l|l-1} = F_{l-1}P_{s,l-1|l-1}F_{l-1}^{\mathrm{T}} + Q_{l-1}. \tag{27}$$

Step 2: Measurement update—step by step. From

$$p(x_l, v_{1,l}|Y_{l-1}) = \mathrm{St}\left(\begin{bmatrix} x_l \\ v_{1,l} \end{bmatrix}; \begin{bmatrix} \hat{x}_{s,l|l-1} \\ 0 \end{bmatrix}, \begin{bmatrix} P_{s,l|l-1} & 0 \\ 0 & R_{1,l} \end{bmatrix}, \nu\right), \tag{28}$$

the following equation can be obtained:

$$p(x_l, y_{1,l}|Y_{l-1}) = \mathrm{St}\left(\begin{bmatrix} x_l \\ y_{1,l} \end{bmatrix}; \begin{bmatrix} \hat{x}_{s,l|l-1} \\ \hat{y}_{1,l} \end{bmatrix}, \begin{bmatrix} P_{s,l|l-1} & P_l^{\tilde{x}\tilde{y}_1} \\ P_l^{\tilde{x}\tilde{y}_1,T} & P_l^{\tilde{y}_1\tilde{y}_1} \end{bmatrix}, \nu\right), \tag{29}$$

where

$$\hat{y}_{1,l} = H_{1,l}\hat{x}_{s,l|l-1}, \tag{30}$$

$$P_l^{\tilde{x}\tilde{y}_1} = P_{s,l|l-1}H_{1,l}^{\mathrm{T}}, \tag{31}$$

$$P_l^{\tilde{y}_1\tilde{y}_1} = H_{1,l}P_{s,l|l-1}H_{1,l}^{\mathrm{T}} + R_{1,l}. \tag{32}$$

From the last property of Lemma 1, the conditional probability

$$p(x_l|Y_{l-1}, y_{1,l}) = \mathrm{St}(x_l; \hat{x}'_{1,l|l}, P'_{1,l|l}, \nu^{(1)}) \tag{33}$$

can be obtained by

$$\nu^{(1)} = \nu + m_1, \tag{34}$$

$$\hat{x}'_{1,l|l} = \hat{x}_{s,l|l-1} + K_{1,l}\tilde{y}_{1,l}, \tag{35}$$

$$P'_{1,l|l} = \frac{\nu + \Delta_{1,l}^2}{\nu + m_1}[P_{s,l|l-1} - P_l^{\tilde{x}\tilde{y}_1}(P_l^{\tilde{y}_1\tilde{y}_1})^{-1}P_l^{\tilde{x}\tilde{y}_1,\mathrm{T}}], \tag{36}$$

$$\Delta_{1,l}^2 = \tilde{y}_{1,l}^{\mathrm{T}}(P_l^{\tilde{y}_1\tilde{y}_1})^{-1}\tilde{y}_{1,l}, \tag{37}$$

$$\tilde{y}_{1,l} = y_{1,l} - \hat{y}_{1,l}. \tag{38}$$

Let

$$K_{1,l} = P_l^{\tilde{x}\tilde{y}_1}(P_l^{\tilde{y}_1\tilde{y}_1})^{-1}. \tag{39}$$

Substituting (39) and (31) to (36), we obtain

$$P'_{1,l|l} = \frac{\nu + \Delta_{1,l}^2}{\nu + m_1}(I - K_{1,l}H_{1,l})P_{s,l|l-1}. \tag{40}$$

To preserve the heavy-tailed property, by using the moment matching approach, we have the approximate t-distribution [17, 18, 31]:

$$p(x_l|Y_{l-1}, y_{1,l}) \approx \mathrm{St}(x_l; \hat{x}_{1,l|l}, P_{1,l|l}, \nu), \tag{41}$$

where

$$\hat{x}_{1,l|l} = \hat{x}'_{1,l|l} = \hat{x}_{s,l|l-1} + K_{1,l}[y_{1,l} - H_{1,l}\hat{x}_{s,l|l-1}], \tag{42}$$

$$\begin{aligned} P_{1,l|l} &= \frac{\nu^{(1)}(\nu-2)}{\nu(\nu^{(1)}-2)}P'_{1,l|l} \\ &= \frac{(\nu-2)(\nu+\Delta_{1,l}^2)}{\nu(\nu+m_1-2)}(I - K_{1,l}H_{1,l})P_{s,l|l-1}. \end{aligned} \tag{43}$$

Similarly, from

$$p(x_l, v_{2,l}|Y_{l-1}, y_{1,l}) = \mathrm{St}\left(\begin{bmatrix} x_l \\ v_{2,l} \end{bmatrix}; \begin{bmatrix} \hat{x}_{1,l|l} \\ 0 \end{bmatrix}, \begin{bmatrix} P_{1,l|l} & 0 \\ 0 & R_{2,l} \end{bmatrix}, \nu\right), \tag{44}$$

we have

$$p(x_l, y_{2,l}|Y_{l-1}, y_{1,l}) = \mathrm{St}\left(\begin{bmatrix} x_l \\ y_{2,l} \end{bmatrix}; \begin{bmatrix} \hat{x}_{1,l|l} \\ \hat{y}_{2,l} \end{bmatrix}, \begin{bmatrix} P_{1,l|l} & P_l^{\tilde{x}\tilde{y}_2} \\ P_l^{\tilde{x}\tilde{y}_2,T} & P_l^{\tilde{y}_2\tilde{y}_2} \end{bmatrix}, \nu\right), \tag{45}$$

where

$$\hat{y}_{2,l} = H_{2,l}\hat{x}_{1,l|l}, \tag{46}$$

$$P_l^{\tilde{x}\tilde{y}_2} = P_{1,l|l}H_{2,l}^{\mathrm{T}}, \tag{47}$$

$$P_l^{\tilde{y}_2\tilde{y}_2} = H_{2,l}P_{1,l|l}H_{2,l}^{\mathrm{T}} + R_{2,l}. \tag{48}$$

The conditional probability

$$p(x_l|Y_{l-1}, y_{1,l}, y_{2,l}) = \mathrm{St}(x_l; \hat{x}'_{2,l|l}, P'_{2,l|l}, \nu^{(2)}) \tag{49}$$

can be obtained by

$$\nu^{(2)} = \nu + m_2, \tag{50}$$

$$\hat{x}'_{2,l|l} = \hat{x}_{1,l|l} + K_{2,l}\tilde{y}_{2,l}, \tag{51}$$

$$P'_{2,l|l} = \frac{\nu + \Delta_{1,l}^2}{\nu + m_2}(I - K_{2,l}H_{2,l})P_{1,l|l}, \tag{52}$$

$$K_{2,l} = P_l^{\tilde{x}\tilde{y}_2}(P_l^{\tilde{y}_2\tilde{y}_2})^{-1}, \tag{53}$$

$$\Delta_{2,l}^2 = \tilde{y}_{2,l}^{\mathrm{T}}(P_l^{\tilde{y}_2\tilde{y}_2})^{-1}\tilde{y}_{2,l}, \tag{54}$$

$$\tilde{y}_{2,l} = y_{2,l} - \hat{y}_{2,l}. \tag{55}$$

By using the moment matching approach, the approximate t-distribution can be obtained by

$$p(x_l|Y_{l-1}, y_{1,l}, y_{2,l}) \approx \mathrm{St}(x_l; \hat{x}_{2,l|l}, P_{2,l|l}, \nu), \tag{56}$$

where

$$\hat{x}_{2,l|l} = \hat{x}'_{2,l|l} = \hat{x}_{1,l|l} + K_{2,l}[y_{2,l} - H_{2,l}\hat{x}_{1,l|l}], \tag{57}$$

$$\begin{aligned} P_{2,l|l} &= \frac{\nu^{(2)}(\nu-2)}{\nu(\nu^{(2)}-2)} P'_{2,l|l} \\ &= \frac{(\nu-2)(\nu+\Delta_{2,l}^2)}{\nu(\nu+m_2-2)}(I - K_{2,l}H_{2,l})P_{1,l|l}. \end{aligned} \tag{58}$$

Generally speaking, for $2 \leqslant i \leqslant N$, we have the approximate t-distribution:

$$p(x_l|Y_{l-1}, y_{1,l}, y_{2,l}, \ldots, y_{i,l}) \approx \mathrm{St}(x_l; \hat{x}_{i,l|l}, P_{i,l|l}, \nu), \tag{59}$$

where

$$\hat{x}_{i,l|l} = \hat{x}_{i-1,l|l} + K_{i,l}[y_{i,l} - H_{i,l}\hat{x}_{i-1,l|l}], \tag{60}$$

$$P_{i,l|l} = \frac{(\nu-2)(\nu+\Delta_{i,l}^2)}{\nu(\nu+m_i-2)}(I - K_{i,l}H_{i,l})P_{i-1,l|l}, \tag{61}$$

$$K_{i,l} = P_l^{\tilde{x}\tilde{y}_i}(P_l^{\tilde{y}_i\tilde{y}_i})^{-1}, \tag{62}$$

$$\Delta_{i,l}^2 = \tilde{y}_{i,l}^{\mathrm{T}}(P_l^{\tilde{y}_i\tilde{y}_i})^{-1}\tilde{y}_{i,l}, \tag{63}$$

$$\tilde{y}_{i,l} = y_{i,l} - \hat{y}_{i,l}, \tag{64}$$

$$\hat{y}_{i,l} = H_{i,l}\hat{x}_{i-1,l|l}. \tag{65}$$

When $i = N$, the state estimation by sequential fusion of sensors $1, \ldots, N$ can be deduced:

$$\begin{aligned} \hat{x}_{s,l|l} &= \hat{x}_{N,l|l} = \hat{x}_{N-1,l|l} + K_{N,l}[y_{N,l} - H_{N,l}\hat{x}_{N-1,l|l}] \\ &= \hat{x}_{s,l|l-1} + \sum_{i=1}^{N} K_{i,l}[y_{i,l} - H_{i,l}\hat{x}_{i-1,l|l}], \end{aligned} \tag{66}$$

$$\begin{aligned} P_{s,l|l} = P_{N,l|l} &= \frac{(\nu-2)(\nu+\Delta_{N,l}^2)}{\nu(\nu+m_N-2)}(I - K_{N,l}H_{N,l})P_{N-1,l|l} \\ &= \left(\frac{\nu-2}{\nu}\right)^N \prod_{1}^{i=N}\left(\frac{\nu+\Delta_{i,l}^2}{\nu+m_i-2}\right)(I - K_{i,l}H_{i,l})P_{s,l|l-1}, \end{aligned} \tag{67}$$

where

$$\begin{cases} \hat{x}_{0,l|l} = \hat{x}_{s,l|l-1}, \\ P_{0,l|l} = P_{s,l|l-1}. \end{cases} \tag{68}$$

This completes the proof.

It is well known that for the state estimation of Gaussian driven linear dynamic systems, the Kalman filter-based centralized batch fusion and the optimal sequential fusion are equivalent in the sense of least mean square error (LMSE) [1, 20, 32]. For the t-distribution-based filter, to preserve the heavy-tailed property, a moment matching technique is used to generate the state estimation, so it is actually an approximate filter. Therefore, unlike the Gaussian driven system, for the system with heavy-tailed noises, the approximate t-filter-based centralized fusion estimation given by (21) and the sequential fusion estimation computed by Theorem 1 are inequivalent. In fact, we have the following theorem.

Theorem 2. The sequential fusion estimation given in Theorem 1 is not equivalent to the centralized fusion estimation result given in (21) in the sense of LMSE, and any of them could be better, which is determined by the dof of the noise, the dimension of the measurements, and the quantity of the residuals.

Proof. Let

$$a_{c,l} = \frac{(\nu-2)(\nu+\Delta_{c,l}^2)}{\nu(\nu+m-2)}, \tag{69}$$

$$a_{s,l} = \left(\frac{\nu-2}{\nu}\right)^N \prod_{1}^{i=N} \left(\frac{\nu+\Delta_{i,l}^2}{\nu+m_i-2}\right). \tag{70}$$

Similar to the proof of the equivalence property of Kalman filter-based optimal centralized fusion and the optimal sequential fusion [1, 20, 32], it can be easily verified that

$$\frac{1}{a_{c,l}} P_{c,l|l} = \frac{1}{a_{s,l}} P_{s,l|l}. \tag{71}$$

Therefore,

$$P_{s,l|l} = \frac{a_{s,l}}{a_{c,l}} P_{c,l|l}. \tag{72}$$

Let

$$b_{sc,l} = \frac{a_{s,l}}{a_{c,l}}. \tag{73}$$

Substituting (69) and (70) into (73), we have

$$b_{sc,l} = \left(\frac{v-2}{v}\right)^{N-1} \prod_{i=1}^{N} \left(\frac{v+\Delta_{i,l}^2}{v+m_i-2}\right) \left(\frac{\nu+m-2}{\nu+\Delta_{c,l}^2}\right). \tag{74}$$

From (72)–(74), it can be easily seen that when $b_{sc,l} > 1$, $P_{s,l|l} > P_{c,l|l}$; otherwise, when $b_{sc,l} < 1$, $P_{s,l|l} < P_{c,l|l}$. Only when $b_{sc,l} = 1$, we have $P_{s,l|l} = P_{c,l|l}$. Thus, which of $P_{s,l|l}$ and $P_{c,l|l}$ is larger depends on the dof of the noise ν, m_i (i.e., the dimension of $y_{i,l}$), and the residual of the measurements (i.e., $\tilde{y}_{i,l}$ and $\tilde{y}_{c,l}$).

Remark 1. It can be easily verified that when ν tends to infinity, the algorithm given in (21) and Theorem 1 reduce to the classical Gaussian driven Kalman filter-based optimal centralized fusion and optimal sequential fusion, respectively. Therefore, the algorithms derived in this study are the generalization of the traditional ones that based on Gaussian distribution. It is well known that the t-distribution is quite similar to Gaussian when the dof is large enough. So, to better formulate the heavy-tailed noises, the dof of the t-distribution should not be too large. That is part of the reason why we use the moment matching approach in generating the approximate t-filter and fusion algorithms for systems with heavy-tailed noises.

Remark 2. If the dof of the process noise, the measurement noises, and the initial state are different in the problem formulation, we may use the moment matching method to realize the centralized fusion and the sequential fusion algorithms. For example, under the following formulation:

$$\begin{cases} p(x_0) = \mathrm{St}(x_0; \hat{x}_{0|0}, P_{0|0}, \nu_0), \\ p(w_l) = \mathrm{St}(w_l; 0, Q_l, \nu_w), \\ p(v_{i,l}) = \mathrm{St}(v_{i,l}; 0, R_{i,l}, \nu_i), \ i = 1, 2, \ldots, N, \end{cases} \tag{75}$$

to best preserve the heavy-tailed property, let $\nu = \min\{\nu_i, i = w, 0, 1, 2, \ldots, N\}$ [17]. By the use of the moment matching approach, $p(x_0)$, $p(w_l)$ and $p(v_{i,l})$ can be approximated by

$$\begin{cases} p(x_0^\star) = \mathrm{St}(x_0; \hat{x}_{0|0}, P_{0|0}^\star, \nu), \\ p(w_l^\star) = \mathrm{St}(w_l^\star; 0, Q_l^\star, \nu), \\ p(v_{i,l}^\star) = \mathrm{St}(v_{i,l}^\star; 0, R_{i,l}^\star, \nu), \end{cases} \tag{76}$$

where $P^{\star}_{0|0} = \frac{(\nu-2)\nu_0}{(\nu_0-2)\nu} P_{0|0}$, $Q^{\star}_l = \frac{(\nu-2)\nu_w}{(\nu_w-2)\nu} Q_l$ and $R^{\star}_{i,l} = \frac{(\nu-2)\nu_i}{(\nu_i-2)\nu} R_{i,l}$. Then, $w^{\star}_l$ and w_l, $v^{\star}_{i,l}$ and $v_{i,l}$, $x^{\star}_0$ and x_0 have the same mean and covariance, respectively. By using $Q^{\star}_l$, $P^{\star}_{0|0}$ and $R^{\star}_{i,l}$ to replace Q_l, $P_{0|0}$ and $R_{i,l}$, respectively, in (21) and Theorem 1, we obtain the t-distribution filter-based centralized fusion and sequential fusion algorithms, respectively, for systems with heavy-tailed noises of different dofs.

4 Numerical example

An example is used to show the effectiveness and the robustness of the presented algorithms in this section.

Consider a two-dimensional linear target tracking system observed by three sensors [20]:

$$x_{l+1} = Fx_l + w_l, \tag{77}$$

$$y_{i,l} = H_i x_l + v_{i,l}, \ i = 1, 2, 3, \tag{78}$$

where

$$F = \begin{bmatrix} 0.95 & T \\ 0 & 0.95 \end{bmatrix}. \tag{79}$$

T denotes the sampling interval that takes value as 1 s. State vector $x_l = [s_l \ \dot{s}_l]^{\mathrm{T}}$, where s_l and $\dot{s}_l$ denote position and velocity of the target, respectively. $H_1 = [1 \ 1]$, $H_2 = [0.9 \ 0.7]$, $H_3 = [0.8 \ 0.5]$. The initial state and the scale matrix are $\hat{x}_{0|0} = [10 \ 0]^{\mathrm{T}}$ and $P_{0|0} = \mathrm{diag}[2 \ 2]$. In this example, the heavy-tailed process noise w_l and measurement noise $v_{i,l}$ are generated according to

$$p(w_l) = \mathrm{St}(w_l; 0, Q, \nu), \tag{80}$$

$$p(v_{i,l}) = \mathrm{St}(v_{i,l}; 0, R_i, \nu), \ i = 1, 2, 3, \tag{81}$$

where $Q = \mathrm{diag}[1 \ 1]$, $R_1 = 8$, $R_2 = 16$, $R_3 = 20$ and $\nu = 3$. To analyze the filtering performance, the root mean square errors (RMSEs) of position and velocity are utilized:

$$\begin{cases} \mathrm{RMSE_p} = \frac{1}{L}\sqrt{\sum_{i=1}^{L}(s_l - \hat{s}_{l|l})^2}, \\ \mathrm{RMSE_v} = \frac{1}{L}\sqrt{\sum_{i=1}^{L}(\dot{s}_l - \hat{\dot{s}}_{l|l})^2}. \end{cases} \tag{82}$$

To get the RMSE of the state estimates, $L = 200$ Monte Carlo simulations are run.

Figure 1 shows the true values and the estimates of position and velocity by using different fusion algorithms in one Monte Carlo simulation, where the Gaussian centralized fusion (G-CF) denotes the state estimate obtained by using the centralized batch fusion based on Kalman filter that regards the heavy-tailed noises as Gaussian noises with covariances being $Q' = \frac{\nu}{\nu-2}Q$ and $R'_i = \frac{\nu}{\nu-2}R_i$, respectively. The centralized fusion (CF) denotes the estimate by using the heavy-tailed centralized fusion algorithm for linear systems and the sequential fusion (SF) denotes the estimate by using the sequential fusion proposed in Section 3. From Figure 1, one can get that no matter in position or in velocity, the estimate by the presented SF best matches the original signal, followed by the CF, and then the G-CF, which shows the superiority of the presented sequential fusion among different fusion algorithms.

Figures 2 and 3 show the RMSEs of the position and the velocity by using single sensors and different fusion algorithms. Figures 2(a) and 3(a) compare the RMSEs of position and velocity of different fusion algorithms, respectively. From these two subgraphs, we can find that the presented sequential fusion has higher estimation accuracy on both position and velocity compared to the G-CF, and the CF shows nearly the same performance as the SF. Figures 2(b) and 3(b) compare the RMSEs of the position and the velocity between single sensors and the SF algorithm. There are three sensors observing the target with different accuracy. The proposed SF has the smallest RMSEs in position and velocity than any single sensor and the effectiveness of the presented sequential fusion algorithm has been illustrated.

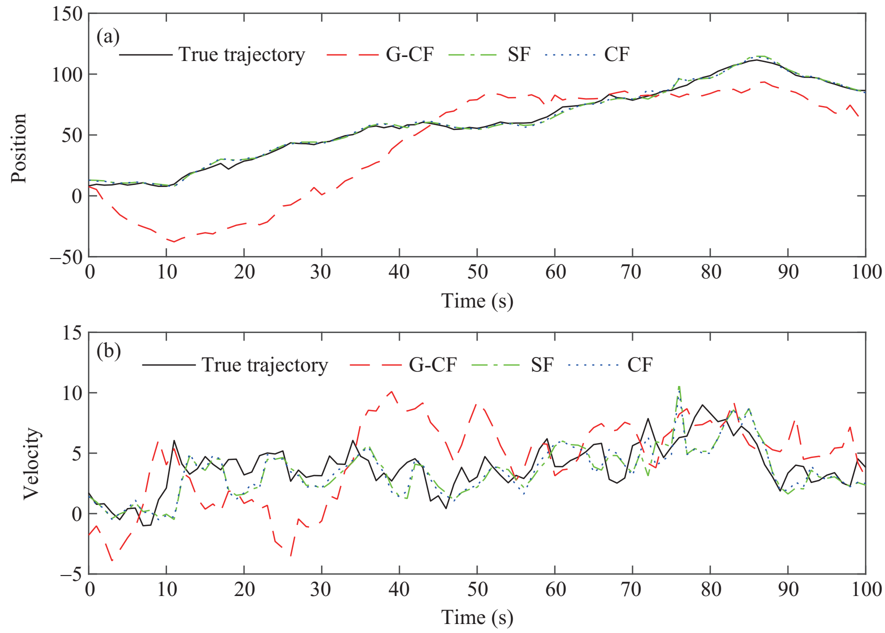

Figure 1 (Color online) True values and the fusion estimates of (a) position and (b) velocity.

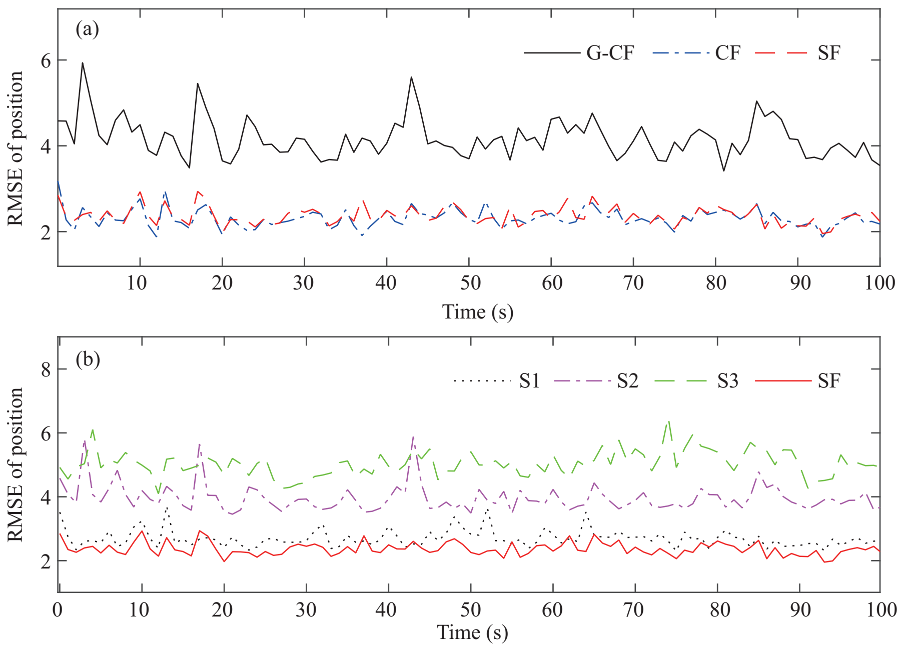

Figure 2 (Color online) RMSEs of the position by using the presented SF algorithm compared with (a) other fusion algorithms and (b) single sensors.

To better compare the performance of different fusion algorithms, the time average RMSEs ($\mathrm{RMSE_p}$ for position and $\mathrm{RMSE_v}$ for velocity) by single sensors and by different fusion algorithms are listed in Table 1.

Table 1 shows that the proposed SF is most effective in RMSE of the velocity, and the CF shows a better performance in RMSE of the position. The G-CF gives the worst results among these three fusion algorithms. From the first row of Table 1, one finds that the average $\mathrm{RMSE_p}$ by using G-CF is larger than that by single sensor S1. Because the RMSEs by single sensors are obtained by using the approximate *t*-filter, which is more effective than Gaussian Kalman filter when dealing with the state estimation problem for linear systems with heavy-tailed noises. Thus, the sensor with the highest accuracy is more reasonable

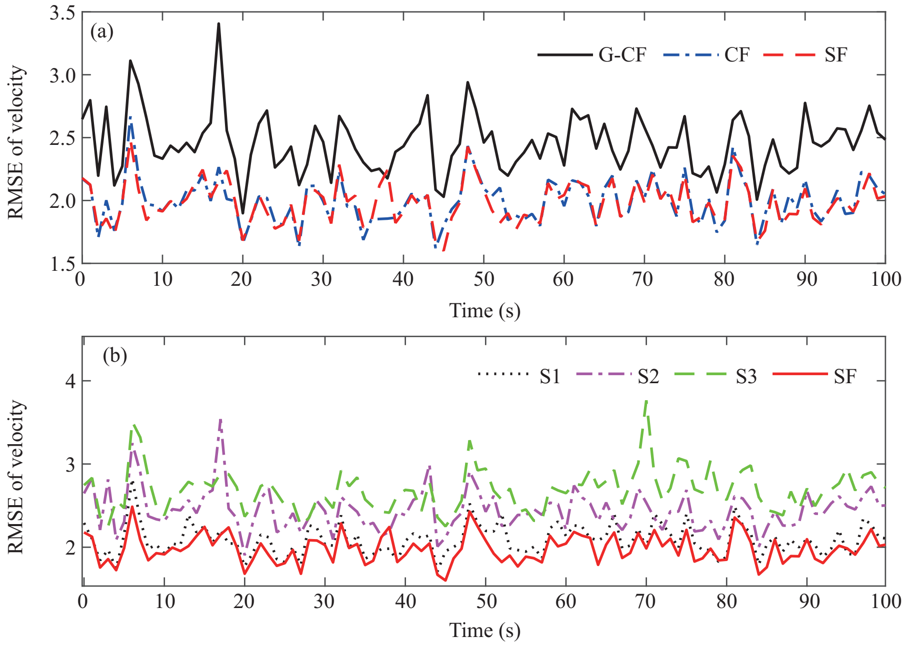

Figure 3 (Color online) RMSEs of the velocity by using the presented SF algorithm compared with (a) other fusion algorithms and (b) single sensors.

Table 1 Average RMSEs by using single sensors and different fusion algorithms

Sensor	$RMSE_p$	$RMSE_v$	Algorithm	$RMSE_p$	$RMSE_v$
S1	2.6945	2.0883	G-CF	4.1684	2.4537
S2	3.9596	2.4450	CF	2.3128	1.9949
S3	5.0342	2.6874	SF	2.3677	1.9840

Table 2 Average CPU time per Monte Carlo run of single sensors and different fusion algorithms

Sensor	CPU time (ms)	Algorithm	CPU time (ms)
S1	1.94	G-CF	5.24
S2	1.93	CF	9.52
S3	1.96	SF	5.87

to show superiority than the Gaussian Kalman filter-based centralized fusion in state estimation. The CF and the SF based on approximate t-filter show better performance than all single sensors.

To make a further performance evaluation of different fusion algorithms, we show the CPU time of three single sensors and three fusion algorithms in Table 2. It is shown from Table 2 that the G-CF has the fastest computation speed among the three fusion algorithms, followed by the proposed SF with a little bit more computing time. The CF is the most time consuming. These results are consistent with our theory analysis that the computation of heavy-tailed algorithms is a little bit more complex than that of the Gaussian algorithms, so it is reasonable for it to take more computation time. From Tables 1 and 2, one can find that the proposed SF is the best among these three fusion algorithms in considering of both estimation accuracy and computation efficiency.

5 Conclusion

The sequential fusion estimation algorithm for linear multisensor dynamic systems with heavy-tailed process noise and measurement noise is presented. Through theoretical proof and experimental analysis, the following conclusion can be drawn: for the heavy-tailed multisensor system, (1) the presented sequential fusion algorithm is effective, which is superior to the Gaussian-based classical optimal centralized batch fusion; (2) the t-filter-based sequential fusion algorithm and the centralized batch fusion algorithm are not equivalent, and either of them could be better; (3) the traditional optimal sequential fusion algorithm that is based on the classical Kalman filter under Gaussian assumption is a special case of the given algorithm. Thus, the proposed algorithm has promising application values in many fields, such as target tracking, defense, control systems, the surveillance, robotics and localization.

There are still many good research topics concerning the fusion estimation for the systems with heavy-tailed noises that can be studied in the future, such as the fusion estimation for nonlinear systems with heavy-tailed noises, the fusion problem in multirate multisensor systems, the filter and the fusion estimation for the systems with correlated heavy-tailed noises. Solving these problems will be more practical.

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References

1 Bar-Shalom Y, Li X R, Kirubarajan T. Estimation With Applications to Tracking and Navigation: Theory Algorithm and Software. Hoboken: John Wiley & Sons, Inc., 2001

2 Khaleghi B, Khamis A, Karray F O, et al. Multisensor data fusion: a review of the state-of-the-art. Inf Fusion, 2013, 14: 28–44

3 Deng Z, Zhang P, Qi W, et al. Sequential covariance intersection fusion Kalman filter. Inf Sci, 2012, 189: 293–309

4 Edward W, James L. Multisensor Data Fusion. Boston: Artech House, 1990

5 Yan L, Li X R, Xia Y, et al. Modeling and estimation of asynchronous multirate multisensor system with unreliable measurements. IEEE Trans Aerosp Electron Syst, 2015, 51: 2012–2026

6 Hu J, Wang Z, Chen D, et al. Estimation, filtering and fusion for networked systems with network-induced phenomena: new progress and prospects. Inf Fusion, 2016, 31: 65–75

7 Yan L, Xia Y, Fu M. Optimal fusion estimation for stochastic systems with cross-correlated sensor noises. Sci China Inf Sci, 2017, 60: 120205

8 Ma L F, Wang Z D, Han Q L, et al. Variance-constrained distributed filtering for time-varying systems with multiplicative noises and deception attacks over sensor networks. IEEE Sens J, 2017, 17: 2279–2288

9 Ma L F, Wang Z D, Lam H K, et al. Envelope-constrained h_∞ filtering for nonlinear systems with quantization effects: the finite horizon case. Automatica, 2018, 93: 527–534

10 Zhu H, Leung H, He Z. A variational Bayesian approach to robust sensor fusion based on student-t distribution. Inf Sci, 2013, 221: 201–214

11 Shen C, Li J, Huang W. Robust centralized multi-sensor fusion using cubature information filter. In: Proceedings of the 30th Chinese Control and Decision Conference (2018 CCDC), 2018. 3297–3302

12 Wang X, Liang Y, Pan Q, et al. Nonlinear Gaussian smoothers with colored measurement noise. IEEE Trans Automat Contr, 2015, 60: 870–876

13 Piché R, Särkkä S, Hartikainen J. Recursive outlier-robust filtering and smoothing for nonlinear systems using the multivariate student-t-distribution. In: Proceedings of IEEE International Workshop on Machine Learning for Signal Processing, Santander, 2012. 1–6

14 Huang Y, Zhang Y, Li N, et al. A robust gaussian approximate filter for nonlinear systems with heavy tailed measurement noises. In: Proceedings of IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), 2016. 4209–4213

15 Huang Y, Zhang Y, Li N, et al. A robust student's t based cubature filter. In: Proceedings of the 19th International Conference on Information Fusion, Heidelberg, 2016. 9–16

16 Amor N, Kahlaoui S, Chebbi S. Unscented particle filter using student-t distribution with non-Gaussian measurement noise. In: Proceedings of 2018 International Conference on Advanced Systems and Electric Technologies, 2018. 34–38

17 Roth M, Ožkan E, Gustafsson F. A student's filter for heavy tailed process and measurement noise. In: Proceedings of International Conference on Acoustics, Speech and Signal Processing (ICASSP), 2013. 5770–5774

18 Huang Y, Zhang Y, Li N, et al. Robust student's t based nonlinear filter and smoother. IEEE Trans Aerosp Electron Syst, 2016, 52: 2586–2596

19 Zhang W A, Shi L. Sequential fusion estimation for clustered sensor networks. Automatica, 2018, 89: 358–363

20 Lin H, Sun S. Optimal sequential fusion estimation with stochastic parameter perturbations, fading measurements, and correlated noises. IEEE Trans Signal Process, 2018, 66: 3571–3583

21 Wang Y, Jing Z L, Hu S Q, et al. On the sensor order in sequential integrated probability data association filter. Sci China Inf Sci, 2012, 55: 491–500

22 Wen C, Cai Y, Wen C, et al. Optimal sequential Kalman filtering with cross-correlated measurement noises. Aerospace Sci Tech, 2013, 26: 153–159

23 Yan L, Li X R, Xia Y, et al. Optimal sequential and distributed fusion for state estimation in cross-correlated noise. Automatica, 2013, 49: 3607–3612

24 Ge Q, Zheng Z, Zhang S, et al. A data fusion approach based on sequential and strong tracking filters with nonlinear dynamic systems. In: Proceedings of 2006 the 1st International Symposium on Systems and Control in Aerospace and

Astronautics, Harbin, 2006. 1344–1349
25 Duan Z, Li X R, Han C, et al. Sequential unscented Kalman filter for radar target tracking with range rate measurements. In: Proceedings of the 7th International Conference on Information Fusion (FUSION 2005), 2005. 130–137
26 Ristic B, Arulampalam S, Gordon N. Beyond the Kalman Filter. London: Artech House, 2004
27 Yan L, Xia Y, Liu B, et al. Multisensor Optimal Estimation Theory and Its Application. Beijing: The Science Publishing House, 2015
28 DeGroot M H. Optimal Statistical Decisions. Beijing: Publishing House of Tsinghua University, 2006
29 Koop G M. Bayesian Econometrics. Hoboken: John Wiley & Sons Inc., 2003
30 Roth M. On the Multivariate t Distribution. Technical Report, Division of Automatic Control, Linköping University Electronic Press, 2013
31 Agamennoni G, Nieto J I, Nebot E M. Approximate inference in state-space models with heavy-tailed noise. IEEE Trans Signal Process, 2012, 60: 5024–5037
32 Kailath T, Sayed A H, Hassibi B. Linear Estimation. Upper Saddle River: Prentice–Hall, 2000