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• **Supplementary File** •

Distributed self-triggered formation control for multi-agent systems

Jiantao SHI[1,2], Jun SUN[1,2], Yuhao YANG[1,2] & Donghua ZHOU[3*]

[1]*Nanjing Research Institute of Electronic Technology, Nanjing 210039, China;*
[2]*Key Laboratory of Intellisense Technology, CETC, Nanjing 210039, China;*
[3]*College of Electrical Engineering and Automation, Shandong University of Science and Technology, Qingdao 266590, China*

Appendix A Proof of the Theorems in this leter

(1) Proof of Theorem 1: Choose a Lyapunov function candidate for the MAS as follows:

$$V(t)=\xi^T(t)(I_N\otimes P)\xi(t), \tag{A1}$$

where $P>0$ is a symmetric positive definite matrix obtained from (4). Calculating the time derivative of $V(t)$, one has

$$\begin{aligned}\dot V(t)&=\dot\xi^T(t)(I_N\otimes P)\xi(t)+\xi^T(t)(I_N\otimes P)\dot\xi(t)\\&=\xi^T(t)\big(I_N\otimes(A^TP+PA)-2\mu H\otimes PBB^TP\big)\xi(t)-2\xi^T(t)(\mu I_N\otimes PBB^TP)\bar e(t)\\&=\xi^T(t)\big(I_N\otimes(A^TP+PA)-2\mu H\otimes PBB^TP\big)\xi(t)-2\mu\sum_{i=1}^N\xi_i(t)PBB^TP\bar e_i(t)\end{aligned} \tag{A2}$$

Let $\tilde\xi(t)\triangleq(U^T\otimes I_n)\xi(t)$, then it can be obtained that

$$\begin{aligned}&\xi^T(t)\big(I_N\otimes(A^TP+PA)-2\mu H\otimes PBB^TP\big)\xi(t)\\&=\sum_{i=1}^N\tilde\xi_i^T(t)(A^TP+PA-2\mu\lambda_iPBB^TP)\tilde\xi_i(t).\end{aligned} \tag{A3}$$

By choosing a sufficiently large μ such that $\mu\lambda_1\geqslant 1$, one has

$$A^TP+PA-2\mu\lambda_iPBB^TP\leqslant A^TP+PA-2PBB^TP=-\delta I, \tag{A4}$$

where the last equation is derived by using (4). It follows from (A3) that

$$\xi^T(t)\big(I_N\otimes(A^TP+PA)-2\mu H\otimes PBB^TP\big)\xi(t)\leqslant-\delta\|\xi(t)\|^2. \tag{A5}$$

From (A3-A5), it can be obtained that

$$\begin{aligned}\dot V(t)&\leqslant-\delta\sum_{i=1}^N\|\xi_i(t)\|^2-2\mu\sum_{i=1}^N\xi_i(t)PBB^TP\bar e_i(t)\\&\leqslant-\delta\sum_{i=1}^N\|\xi_i(t)\|^2+2\mu\|PBB^TP\|\sum_{i=1}^N\|\xi_i(t)\|\|\bar e_i(t)\|\\&\leqslant-\delta\sum_{i=1}^N\|\xi_i(t)\|^2+2\mu\|PBB^TP\|\sum_{i=1}^N\sum_{j\in N_i}a_{ij}\|\xi_i(t)\|\cdot\|\bar e_j(t)\|+2\mu\|PBB^TP\|\sum_{i=1}^N\sum_{j\in N_i}a_{ij}\|\xi_i(t)\|\cdot\|\bar e_i(t)\|\\&\leqslant-\delta\sum_{i=1}^N\|\xi_i(t)\|^2+\mu\|PBB^TP\|\sum_{i=1}^N\sum_{j\in N_i}a_{ij}(\theta\|\xi_i(t)\|^2+\frac{1}{\theta}\|\bar e_j(t)\|^2)+\mu\|PBB^TP\|\sum_{i=1}^N d_i(\theta\|\xi_i(t)\|^2+\frac{1}{\theta}\|\bar e_i(t)\|^2)\\&=\sum_{i=1}^N\big\{(-\delta+2\mu d_i\theta\|PBB^TP\|)\|\xi_i(t)\|^2+2\mu\frac{d_i}{\theta}\|PBB^TP\|\|\bar e_i(t)\|^2\big\},\end{aligned} \tag{A6}$$

* Corresponding author (email: zdh@tsinghua.edu.cn)

where the third " $\leqslant$ " in (A6) comes from the following inequality

$$2\mu\|PBB^TP\|\sum_{i=1}^{N}\|\xi_i(t)\|\|\bar{e}_i(t)\| \leqslant 2\mu\|PBB^TP\|\sum_{i=1}^{N}\sum_{j\in N_i}a_{ij}\|\xi_i(t)\|\|\bar{e}_i(t)\|$$
$$\leqslant 2\mu\|PBB^TP\|\sum_{i=1}^{N}\sum_{j\in N_i}a_{ij}\|\xi_i(t)\|\cdot\|\bar{e}_i(t)\| + 2\mu\|PBB^TP\|\sum_{i=1}^{N}\sum_{j\in N_i}a_{ij}\|\xi_i(t)\|\cdot\|\bar{e}_j(t)\|$$

where θ is a positive scalar defined in Theorem 1. For each i, define the triggering condition as:

$$\|\bar{e}_i(t)\| \leqslant \frac{\gamma_i\lambda_i}{1+\gamma_i}\|\xi_i(t_k^i)\|,\ t\in[t_k^i, t_{k+1}^i). \tag{A7}$$

where γ_i is defined in Theorem 1. For simplicity and intuition, this triggering condition can be written as $\|\bar{e}_i(t)\| = \frac{\gamma_i\lambda_i}{1+\gamma_i}\|\xi_i(t_k^i)\|$ $(t\in[t_k^i, t_{k+1}^i))$.

It follows from (A7) that $\|\bar{e}_i(t)\| \leqslant \gamma_i\lambda_i\|\xi_i(t)\|$. Substituting this equation into (A6), and considering Lemma 1 and the definitions of $\gamma_i, \sigma_i, \theta$, one can get the following result:

$$\dot{V}(t) \leqslant \sum_{i=1}^{N}(\sigma_i-1)(\delta-2\mu d_i\theta\|PBB^TP\|)\|\xi_i(t)\|^2. \tag{A8}$$

Therefore, for any $0<\sigma_i<1$ and $0<\theta<\dfrac{\delta}{2\mu\max_i\{d_i\}\|PBB^TP\|}$, the inequality $\dot{V}(t)<0$ can be guaranteed.

In the following, the trigger condition will be analyzed. According to (4) and the definitions of $w_i(t)$ and $\eta_i(t)$, it can be obtained that

$$\|\dot{e}_i(t)\| = \|\dot{w}_i(t)\| = \lambda_i\|\dot{\eta}_i(t)\| \leqslant \|A\|\|e_i(t)\| + \lambda_i\|A\eta_i(t_k^i)\| + \lambda_i\|\mu BK[\sum_{j\in N_i}a_{ij}(\eta_i(t_k^i)-\eta_j(t_k^i)) + g_i\eta_i(t_k^i)]\|. \tag{A9}$$

So the evolution of $\|e_i(t)\|$ for $t\in[t_k^i, t_{k+1}^i)$ is bounded by the solution of the following equation

$$\|\dot{p}_i(t)\| = \|A\|\|p_i(t)\| + \lambda_i\|A\eta_i(t_k^i)\| + \lambda_i\|\mu BK[\sum_{j\in N_i}a_{ij}(\eta_i(t_k^i)-\eta_j(t_k^i)) + g_i\eta_i(t_k^i)]\|, \tag{A10}$$

with $p_i(t_k^i)=0$. The solution of (A10) is given in (A11).

$$\|p_i(t)\| = \frac{\lambda_i\|A\eta_i(t_k^i)\| + \lambda_i\|\mu BK[\sum_{j\in N_i}a_{ij}(\eta_i(t_k^i)-\eta_j(t_k^i)) + g_i\eta_i(t_k^i)]\|}{\|A\|}(e^{\|A\|(t-t_k^i)}-1). \tag{A11}$$

From (A7) and (A10), one has that an upper bound of the time for $e_i(t)$ to evolve from 0 to $\frac{\gamma_i\lambda_i}{1+\gamma_i}\|\eta_i(t_k^i)\|$ satisfies (A12).

$$\frac{\lambda_i\|A\eta_i(t_k^i)\| + \lambda_i\|\mu BK[\sum_{j\in N_i}a_{ij}(\eta_i(t_k^i)-\eta_j(t_k^i)) + g_i\eta_i(t_k^i)]\|}{\|A\|}(e^{\|A\|(t-t_k^i)}-1) = \frac{\gamma_i\lambda_i}{1+\gamma_i}\|\eta_i(t_k^i)\|. \tag{A12}$$

Moreover considering Assumption 3, the triggering time can be chosen as (5) in Theorem 1. This completes the proof.

(2) Proof of Theorem 2: Consider the Lyapunov function candidate

$$V(t) = \hat{w}^T(t)(I_N\otimes Q)\hat{w}(t), \tag{A13}$$

where Q is a symmetric positive definite matrix obtained from (8). Calculating the time derivative of $V(t)$, one has

$$\begin{aligned}\dot{V}(t) &= \dot{\hat{w}}^T(t)(I_N\otimes Q)\hat{w}(t) + \hat{w}^T(t)(I_N\otimes Q)\dot{\hat{w}}(t)\\ &= \hat{w}^T(t)[I_N\otimes(A^TQ+QA) - 2\mu H\otimes QBB^TQ]\hat{w}(t) + 2\mu\sum_{i=1}^{N}\sum_{j\in N_i}\hat{w}_i(t)QBB^TQ(\hat{e}_j(t)-\hat{e}_i(t)).\end{aligned} \tag{A14}$$

Similar to the analysis in Theorem 1, one has

$$\dot{V}(t) \leqslant \sum_{i=1}^{N}\{(-\varepsilon+2\mu d_i\theta\|QBB^TQ\|)\|\hat{w}_i(t)\|^2 + \frac{2\mu d_i}{\theta}\|QBB^TQ\|\|\hat{e}_i(t)\|^2\}. \tag{A15}$$

Triggering condition of agent i is designed as

$$\|\hat{e}_i(t)\| \leqslant \frac{\pi_i}{1+\pi_i}\|\hat{w}_i(t_k^i)\|, \tag{A16}$$

it follows that

$$\|\hat{e}_i(t)\| \leqslant \pi_i\|\hat{w}_i(t)\|. \tag{A17}$$

Substituting (A17) into (A15) yields

$$\dot{V}(t) \leqslant \sum_{i=1}^{N}(\sigma_i - 1)(\varepsilon - 2\mu\theta d_i\|QBB^TQ\|)\|\hat{w}_i(t)\|^2, \tag{A18}$$

then for any $0 < \sigma_i < 1$ and $0 < \theta < \dfrac{\varepsilon}{2\mu\max_i\{d_i\}\|PBB^TP\|}$, it can be obtained $\dot{V}(t) < 0$.

It follows from (A16) that

$$\|\hat{w}_i(t)\| \leqslant (1 + \frac{\pi_i}{1+\pi_i})\|\hat{w}_i(t_k^i)\|, \tag{A19}$$

which implies

$$\|\tilde{w}_i(t)\| \leqslant \|w_i(t)\| + (1 + \frac{\pi_i}{1+\pi_i})\|\hat{w}_i(t_k^i)\|. \tag{A20}$$

Furthermore, one has

$$\dot{w}_i(t) = Aw_i(t) - \mu BK\big[\sum_{j\in N_i} a_{ij}\big(\hat{w}_i(t_k^i) - \hat{w}_j(t_k^i)\big) + g_i\hat{w}_i(t_k^i)\big]. \tag{A21}$$

Then it can be obtained that

$$\|w_i(t)\| \leqslant \frac{1}{\|A\|}\|\mu BK\sum_{j\in N_i} a_{ij}\big(\hat{w}_i(t_k^i) - \hat{w}_j(t_k^i)\big) + g_i\hat{w}_i(t_k^i)\|(e^{\|A\|(t-t_k^i)} - 1). \tag{A22}$$

Moreover, one can obtain that

$$\|\dot{\hat{e}}_i(t)\| = \|A\|(\|\hat{e}_i(t)\| + \|\hat{w}_i(t_k^i)\|) + \|\mu BK\sum_{j\in N_i} a_{ij}\big(\hat{w}_i(t_k^i) - \hat{w}_j(t_k^i)\big)\| + \|LC\|\|\tilde{w}_i(t)\| \tag{A23}$$

Substituting (A20) and (A22) into (A23), one has

$$\begin{aligned}
\|\dot{\hat{e}}_i(t)\| &\leqslant \|A\|(\|\hat{e}_i(t)\| + \|\hat{w}_i(t_k^i)\|) + \|\mu BK\sum_{j\in N_i} a_{ij}\big(\hat{w}_i(t_k^i) - \hat{w}_j(t_k^i)\big)\| + \|LC\|\big[\|w_i(t)\| + (1 + \frac{\pi_i}{1+\pi_i})\|\hat{w}_i(t_k^i)\|\big] \\
&\leqslant \|A\|\|\hat{e}_i(t)\| + (\|A\| + \|LC\|(1 + \frac{\pi_i}{1+\pi_i}))\|\hat{w}_i(t_k^i)\| + \|\mu BK\sum_{j\in N_i} a_{ij}\big(\hat{w}_i(t_k^i) - \hat{w}_j(t_k^i)\big)\| \\
&\quad + \|LC\|\big[\frac{1}{\|A\|}\|\mu BK\sum_{j\in N_i} a_{ij}\big(\hat{w}_i(t_k^i) - \hat{w}_j(t_k^i)\big)\|(e^{\|A\|(t-t_k^i)} - 1)\big] \\
&= \|A\|\|\hat{e}_i(t)\| + \|LC\|\big[\frac{1}{\|A\|}\|\mu BK\sum_{j\in N_i} a_{ij}\big(\hat{w}_i(t_k^i) - \hat{w}_j(t_k^i)\big)\|\big]e^{\|A\|(t-t_k^i)} + \big[\|A\| + \|LC\|(1 + \frac{\pi_i}{1+\pi_i})\big]\|\hat{w}_i(t_k^i)\| \\
&\quad + \|\mu BK\sum_{j\in N_i} a_{ij}\big(\hat{w}_i(t_k^i) - \hat{w}_j(t_k^i)\big)\| - \|LC\|\big[\frac{1}{\|A\|}\|\mu BK\sum_{j\in N_i} a_{ij}\big(\hat{w}_i(t_k^i) - \hat{w}_j(t_k^i)\big)\|\big].
\end{aligned} \tag{A24}$$

Then it follows that

$$\|\dot{\hat{e}}_i(t)\| \leqslant s_1\|\hat{e}_i(t)\| + s_2e^{s_1(t-t_k^i)} + s_3,\ \ t \in [t_k^i, t_{k+1}^i) \tag{A25}$$

where s_1, s_2, s_3 are defined in Theorem 2.

It is obvious that the evolution of $\hat{e}_i(t)$ for $t \in [t_k^i, t_{k+1}^i)$ is bounded by the solution of the following equation:

$$\|\dot{q}_i(t)\| \leqslant s_1\|q_i(t)\| + s_2e^{s_1(t-t_k^i)} + s_3. \tag{A26}$$

With $q(t_k^i) = 0$, the solution of (A26) is given by

$$\|q_i(t)\| = e^{s_1(t-t_k^i)}\big[\frac{s_3}{s_1} + s_2(t - t_k^i)\big] - \frac{s_3}{s_1}. \tag{A27}$$

From (A17) and (A27), one has that the upper bound of the time for $\|\hat{e}_i(t)\|$ to evolve from 0 to $\frac{\pi_i}{1+\pi_i}\|\hat{w}_i(t_k^i)\|$ satisfies the following equation:

$$e^{s_1(t-t_k^i)}\big[s_3 + s_1s_2(t - t_k^i)\big] - s_3 = \frac{s_1\pi_i}{1+\pi_i}\|\hat{w}_i(t_k^i)\|. \tag{A28}$$

which can be rewritten as

$$s_1s_2\varrho + s_3 = \varepsilon e^{-s_1\varrho}, \tag{A29}$$

where $\varrho = t - t_k^i,\ \varepsilon = s_3 + \frac{s_1\pi_i}{1+\pi_i}\|\hat{w}_i(t_k^i)\|$.

Considering the fact that $\varepsilon e^{-s_1\varrho}$ approaches zero and $s_1s_2\varrho$ approaches positive infinity as ϱ goes to infinity and $\varepsilon > s_3$. It can be obtained that there exists a positive scalar ϱ that solves this equation. Therefore, the triggering time instant could be chosen as $t_{k+1}^i = t_k^i + \varrho$. The proof is now completed.

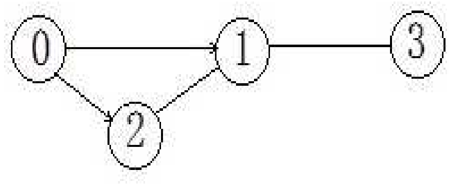

Figure B1 Communication topology of the MAS.

Appendix B Simulation results

In order to illustrate the effectiveness of the obtained theoretical results, the proposed self-triggered formation control scheme is applied to an example system by a numerical simulation. Consider a MAS of 4 agents indexed by 0, 1, 2 and 3, in which 0 is referred as the leader and agents indexed by 1, 2 and 3 are the followers. The topology of the MAS is shown in Figure B1. The individual dynamics of agents have the following parameters:

$$A = \begin{bmatrix} 0.8 & -0.2 \\ -0.25 & 0.75 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Choose $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, $\sigma_3 = 0.3$, $\delta = 0.01$, $\theta = 0.003$. By solving the Riccati function, on can get that $P = \begin{bmatrix} 1.0367 & -0.1431 \\ -0.1431 & 0.0268 \end{bmatrix}$, $K = [0.8936\ \ -0.1163]$. Choosing $L = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.43 \end{bmatrix}$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, $\sigma_3 = 0.3$, $\alpha = 0.2$, $\theta = 0.003$ for the observer-based control protocol. Using the control schemes under the self-triggered condition given in this letter, the formation trajectory and the observation trajectory profiles of the MAS using the observers of the MAS are shown in Figure B2. Moreover, the formation errors of the MAS using the self-triggered static output feedback control and the observer-based self-triggered control are shown in Figure B3 and Figure B4, respectively. Moreover, the triggering instants of each agent in the formation are shown in Figure B5.

In the simulation results, it can be observed that the scheduled formation has been reached and maintained for the MAS under the proposed protocols. Specially, the formation performance can be shown clearly by the formation errors' evolution curves given in Figure B3 and B4, from which it can be observed that the formation errors approach zero as time goes to infinity. That is, the desired formation configuration is achieved.

Remark. Comparing Figure B2(a) with Figure B2(b), one can find that the performance of the static output-based formation controller is better than the observer-based one. Because, in this simulation, the measurement matrix C is an identity matrix, which means that the states of agents can be obtained, however, with respect to the observer-based formation control, the state estimation errors exist inevitably. Such that the formation control performance is affected. However, the application conditions of static output-based formation control are not easy to meet. Usually, observer-based formation control schemes are widely used. Therefore, it is very important to improve the performance of observers. According to the literature on observer-based control and estimation, choosing the appropriate observer gain matrix and initial values of observers can improve the performance of observers. This problem will be investigated in our future work.

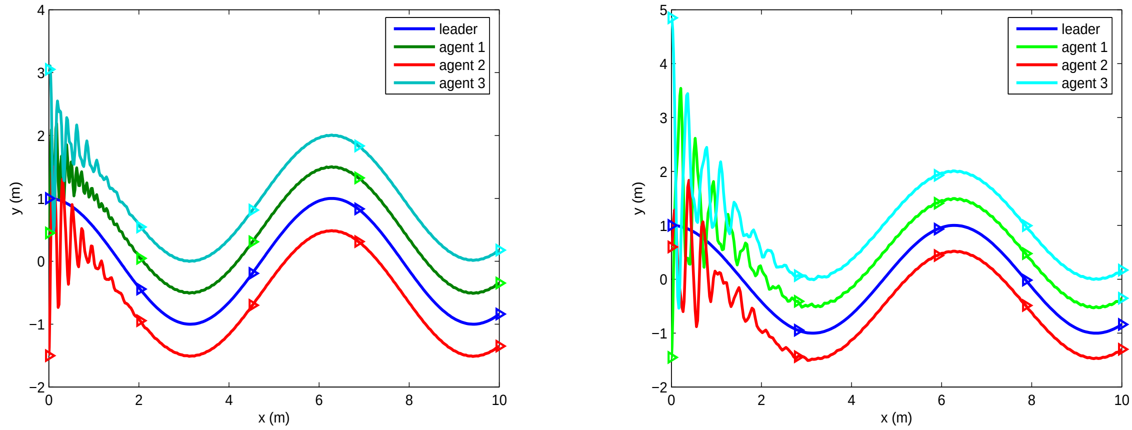

Figure B2 Trajectory profiles of the MAS. (a) Trajectory profiles of the MAS using the self-triggered static output feedback control; (b) Trajectory profiles of the MAS using the observer-based self-triggered control.

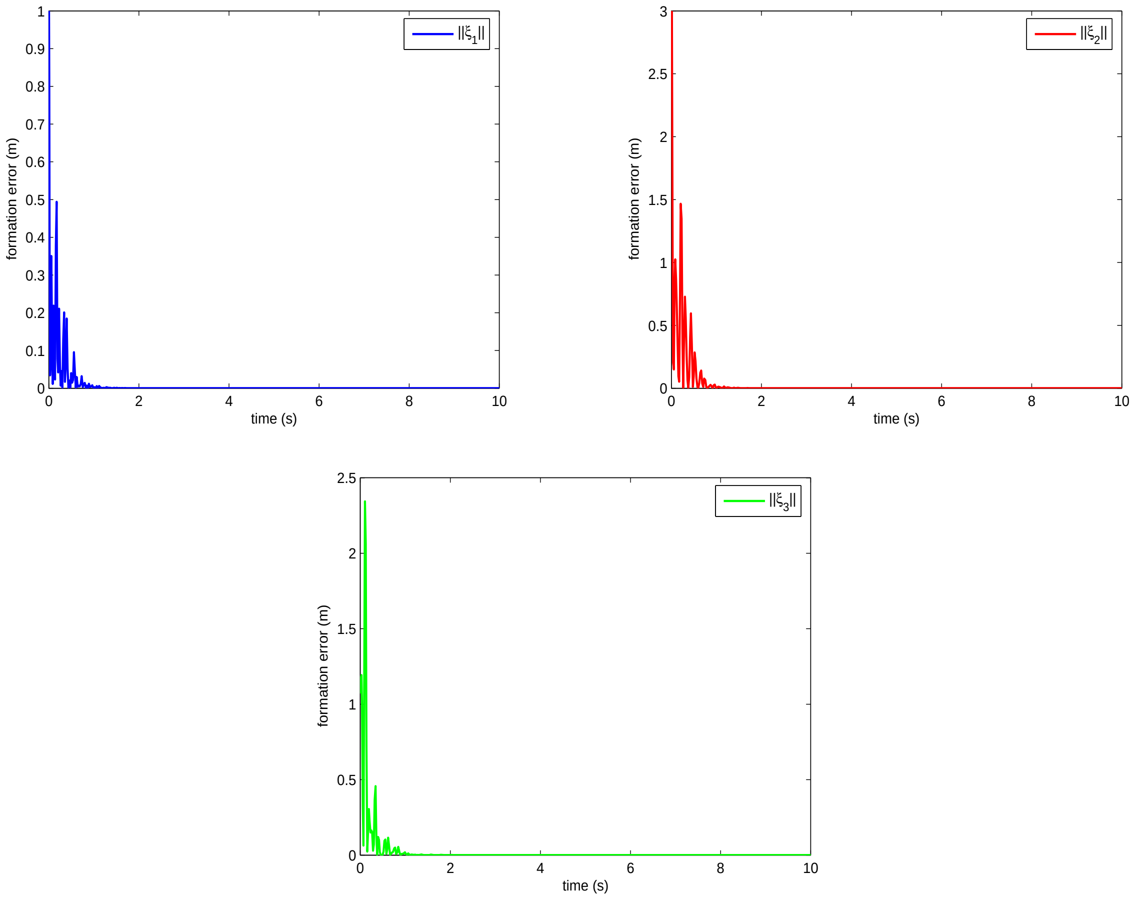

Figure B3 The formation errors of the MAS using the self-triggered static output feedback control. (a) The formation error of agent 1; (b) The formation error of agent 2; (c) The formation error of agent 3

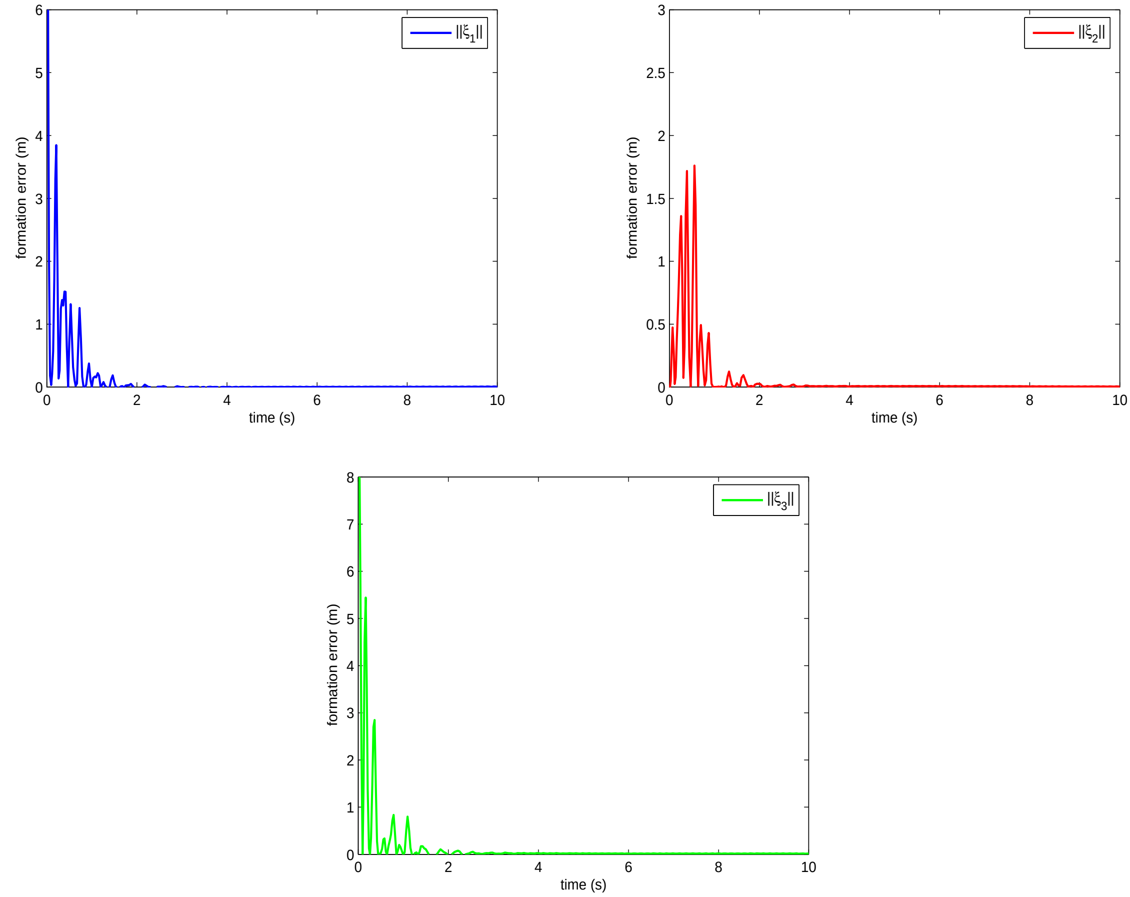

Figure B4 The formation errors of the MAS using the observer-based self-triggered control. (a) The formation error of agent 1; (b) The formation error of agent 2; (c) The formation error of agent 3

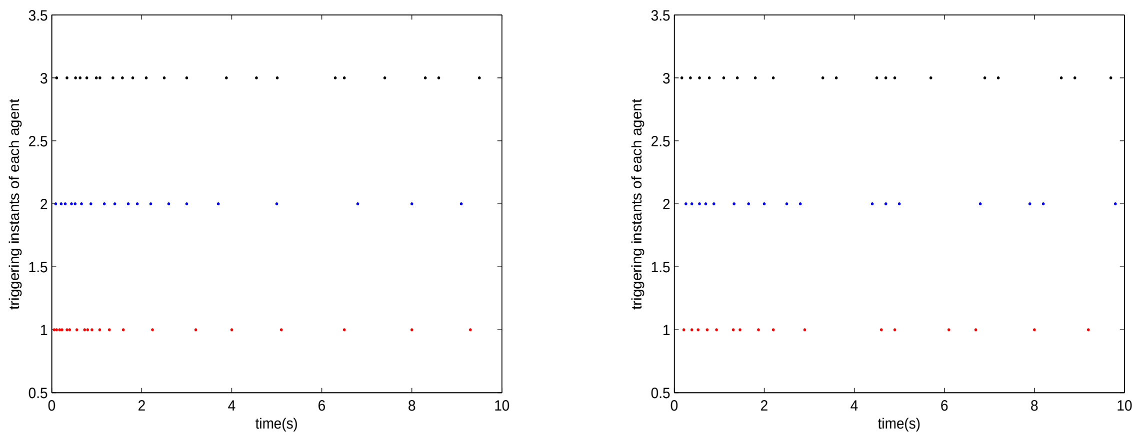

Figure B5 The triggering instants of each agent in the formation. (a) Triggering instants in self-triggered static output feedback control; (b) Triggering instants in observer-based self-triggered control.