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• **Supplementary File** •

An enhanced anti-disturbance control law for systems with multiple disturbances

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Appendix A Proof of Theorem 1

The proof procedure is given as follows.

Define the Lyapunov functions as

$$\bar{V}_{0j} = \tilde{\xi}_j^T P_j \tilde{\xi}_j \tag{A1}$$

with $j = 1, 2, ..., n-1$.

According to (6), the time derivatives of $\bar{V}_{0j}$ can be computed as

$$\begin{aligned}\dot{\bar{V}}_{0j} &= \dot{\tilde{\xi}}_j^T P_j \tilde{\xi}_j + \tilde{\xi}_j^T P_j \dot{\tilde{\xi}}_j \\ &= \tilde{\xi}_j^T (W_j - L_j V_j)^T P_j \tilde{\xi}_j + \delta_j^T H_j^T P_j \tilde{\xi}_j + \tilde{\xi}_j^T P_j (W_j - L_j V_j) \tilde{\xi}_j + \tilde{\xi}_j^T P_j H_j \delta_j \\ &= \begin{bmatrix} \tilde{\xi}_j \\ \delta_j \end{bmatrix}^T \begin{bmatrix} P_j(W_j - L_j V_j) + (W_j - L_j V_j)^T P_j & P_j H_j \\ H_j^T P_j & 0 \end{bmatrix} \begin{bmatrix} \tilde{\xi}_j \\ \delta_j \end{bmatrix}.\end{aligned} \tag{A2}$$

Construct the auxiliary functions as

$$S_j = \bar{V}_{0j} + \int_0^t (\|z_j\|^2 - \gamma_j^2 \|\delta_j\|^2) dt. \tag{A3}$$

In view of (A2), the time derivatives of S_j can be derived as

$$\begin{aligned}\dot{S}_j &= \dot{\bar{V}}_{0j} + \|z_j\|^2 - \gamma_j^2 \|\delta_j\|^2 \\ &= \begin{bmatrix} \tilde{\xi}_j \\ \delta_j \end{bmatrix}^T \begin{bmatrix} (P_j W_j - P_{Lj} V_j) + (P_j W_j - P_{Lj} V_j)^T + T_j^T T_j & P_j H_j \\ H_j^T P_j & -\gamma_j^2 I \end{bmatrix} \begin{bmatrix} \tilde{\xi}_j \\ \delta_j \end{bmatrix}.\end{aligned} \tag{A4}$$

Hence, if Theorem 1 holds, we can get $S_j = \bar{V}_{0j} + \int_0^t (\|z_j\|^2 - \gamma_j^2 \|\delta_j\|^2) dt \leqslant 0$ under the zero initial condition based on $\dot{S}_j < 0$. In other words, $\int_0^t (\|z_j\|^2 - \gamma_j^2 \|\delta_j\|^2) dt \leqslant 0$ can be satisfied. That is, the robust H_∞ performance $\|z_j\|_2^2 \leqslant \gamma_j^2 \|\delta_j\|_2^2$ holds.

Appendix B Proof of Theorem 2

Based on the backstepping algorithm, the proof procedure of Theorem 2 is given as follows.

Step 1: Define $\varepsilon_1 = x_1 - x^d$ and consider the following Lyapunov function:

$$\bar{V}_1 = \frac{1}{2} \varepsilon_1^2. \tag{B1}$$

According to (11), the time-derivative of $\bar{V}_1$ can be given as

$$\begin{aligned}\dot{\bar{V}}_1 &= \varepsilon_1 \dot{\varepsilon}_1 \\ &= \varepsilon_1 (x_2 + \hat{d}_1 + \tilde{d}_1 - \dot{x}^d) \\ &= \varepsilon_1 (\varepsilon_2 + u_1^* + \hat{d}_1 + \tilde{d}_1 - \dot{x}^d),\end{aligned} \tag{B2}$$

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where $\varepsilon_2 = x_2 - u_1^*$, and u_1^* is a virtual control law which can be designed as

$$u_1^* = -k_1\varepsilon_1 - \hat{d}_1 + \dot{x}^d. \tag{B3}$$

As a result, the equation (B2) can be simplified as

$$\dot{\bar{V}}_1 = -k_1\varepsilon_1^2 + \varepsilon_1\varepsilon_2 + \varepsilon_1\tilde{d}_1. \tag{B4}$$

Step 2: Consider the Lyapunov function:

$$\bar{V}_2 = \bar{V}_1 + \frac{1}{2}\varepsilon_2^2. \tag{B5}$$

Computing its time-derivative yields

$$\begin{aligned}
\dot{\bar{V}}_2 &= \dot{\bar{V}}_1 + \varepsilon_2\dot{\varepsilon}_2 \\
&= -k_1\varepsilon_1^2 + \varepsilon_1\varepsilon_2 + \varepsilon_1\tilde{d}_1 + \varepsilon_2(\dot{x}_2 - \dot{u}_1^*) \\
&= -k_1\varepsilon_1^2 + \varepsilon_1\varepsilon_2 + \varepsilon_1\tilde{d}_1 + \varepsilon_2(x_3 + \hat{d}_2 + \tilde{d}_2 - \tfrac{\partial u_1^*}{\partial x_1}\dot{x}_1 - \tfrac{\partial u_1^*}{\partial \hat{\xi}_1}\dot{\hat{\xi}}_1 - \tfrac{\partial u_1^*}{\partial x^d}\dot{x}^d - \tfrac{\partial u_1^*}{\partial \dot{x}^d}\ddot{x}^d) \\
&= -k_1\varepsilon_1^2 + \varepsilon_1\varepsilon_2 + \varepsilon_1\tilde{d}_1 + \varepsilon_2(\varepsilon_3 + u_2^* + \hat{d}_2 + \tilde{d}_2 - \tfrac{\partial u_1^*}{\partial x_1}(x_2 + \hat{d}_1 + \tilde{d}_1) - \tfrac{\partial u_1^*}{\partial \hat{\xi}_1}(W_1\hat{\xi}_1 + L_1\tilde{d}_1) \\
&\quad - \tfrac{\partial u_1^*}{\partial x^d}\dot{x}^d - \tfrac{\partial u_1^*}{\partial \dot{x}^d}\ddot{x}^d),
\end{aligned} \tag{B6}$$

where $\varepsilon_3 = x_3 - u_2^*$, and u_2^* is a virtual control law. Similarly, the virtual control law u_2^* can be designed as

$$u_2^* = -k_2\varepsilon_2 - \varepsilon_1 - \hat{d}_2 + \frac{\partial u_1^*}{\partial x_1}(x_2 + \hat{d}_1) + \frac{\partial u_1^*}{\partial \hat{\xi}_1}W_1\hat{\xi}_1 + \frac{\partial u_1^*}{\partial x^d}\dot{x}^d + \frac{\partial u_1^*}{\partial \dot{x}^d}\ddot{x}^d. \tag{B7}$$

Substituting (B7) into (B6) yields

$$\dot{\bar{V}}_2 = -k_1\varepsilon_1^2 - k_2\varepsilon_2^2 + \varepsilon_2\varepsilon_3 + \varepsilon_1\tilde{d}_1 + \varepsilon_2\tilde{d}_2 - \varepsilon_2(\frac{\partial u_1^*}{\partial x_1}\tilde{d}_1 + \frac{\partial u_1^*}{\partial \hat{\xi}_1}L_1\tilde{d}_1). \tag{B8}$$

Step 3: Consider the Lyapunov function:

$$\bar{V}_3 = \bar{V}_2 + \frac{1}{2}\varepsilon_3^2. \tag{B9}$$

It can be obtained that

$$\begin{aligned}
\dot{\bar{V}}_3 &= \dot{\bar{V}}_2 + \varepsilon_3\dot{\varepsilon}_3 \\
&= -k_1\varepsilon_1^2 - k_2\varepsilon_2^2 + \varepsilon_2\varepsilon_3 + \varepsilon_1\tilde{d}_1 + \varepsilon_2\tilde{d}_2 - \varepsilon_2(\tfrac{\partial u_1^*}{\partial x_1}\tilde{d}_1 + \tfrac{\partial u_1^*}{\partial \hat{\xi}_1}L_1\tilde{d}_1) + \varepsilon_3(\dot{x}_3 - \dot{u}_2^*) \\
&= -k_1\varepsilon_1^2 - k_2\varepsilon_2^2 + \varepsilon_2\varepsilon_3 + \varepsilon_1\tilde{d}_1 + \varepsilon_2\tilde{d}_2 - \varepsilon_2(\tfrac{\partial u_1^*}{\partial x_1}\tilde{d}_1 + \tfrac{\partial u_1^*}{\partial \hat{\xi}_1}L_1\tilde{d}_1) + \varepsilon_3(\varepsilon_4 + u_3^* + \hat{d}_3 + \tilde{d}_3 \\
&\quad - (\sum_{q=1}^{2}\tfrac{\partial u_2^*}{\partial x_q}(x_{q+1} + \hat{d}_q + \tilde{d}_q) + \sum_{q=1}^{2}\tfrac{\partial u_2^*}{\partial \hat{\xi}_q}(W_q\hat{\xi}_q + L_q\tilde{d}_q) + \sum_{q=1}^{3}\tfrac{\partial u_2^*}{\partial x^{d(q-1)}}x^{d(q)})),
\end{aligned} \tag{B10}$$

where $\varepsilon_4 = x_4 - u_3^*$, and the virtual control law u_3^* is designed as

$$u_3^* = -k_3\varepsilon_3 - \varepsilon_2 - \hat{d}_3 + \sum_{q=1}^{2}\frac{\partial u_2^*}{\partial x_q}(x_{q+1} + \hat{d}_q) + \sum_{q=1}^{2}\frac{\partial u_2^*}{\partial \hat{\xi}_q}(W_q\hat{\xi}_q) + \sum_{q=1}^{3}\frac{\partial u_2^*}{\partial x^{d(q-1)}}x^{d(q)}. \tag{B11}$$

Substituting (B11) into (B10), one obtains

$$\dot{\bar{V}}_3 = -k_1\varepsilon_1^2 - k_2\varepsilon_2^2 - k_3\varepsilon_3^2 + \varepsilon_3\varepsilon_4 + \varepsilon_1\tilde{d}_1 + \varepsilon_2\tilde{d}_2 + \varepsilon_3\tilde{d}_3 - \varepsilon_2(\frac{\partial u_1^*}{\partial x_1} + \frac{\partial u_1^*}{\partial \hat{\xi}_1}L_1)\tilde{d}_1 - \varepsilon_3(\sum_{q=1}^{2}\frac{\partial u_2^*}{\partial x_q}\tilde{d}_q + \sum_{q=1}^{2}\frac{\partial u_2^*}{\partial \hat{\xi}_q}L_q\tilde{d}_q). \tag{B12}$$

Step j($4 \leqslant j \leqslant n-1$): Consider the following Lyapunov function

$$\bar{V}_j = \bar{V}_{j-1} + \frac{1}{2}\varepsilon_j^2, \tag{B13}$$

where $\varepsilon_j = x_j - u_{j-1}^*$.

Its time-derivative can be derived as

$$\begin{aligned}
\dot{\bar{V}}_j &= \dot{\bar{V}}_{j-1} + \varepsilon_j\dot{\varepsilon}_j \\
&= -\sum_{q=1}^{j-1}k_q\varepsilon_q^2 + \varepsilon_{j-1}\varepsilon_j + \sum_{q=1}^{j-1}\varepsilon_q\tilde{d}_q - \sum_{\lambda=2}^{j-1}(\varepsilon_\lambda(\sum_{s=1}^{\lambda-1}\tfrac{\partial u_{\lambda-1}^*}{\partial x_s}\tilde{d}_s + \sum_{s=1}^{\lambda-1}\tfrac{\partial u_{\lambda-1}^*}{\partial \hat{\xi}_s}L_s\tilde{d}_s)) + \varepsilon_j(\dot{x}_j - \dot{u}_{j-1}^*) \\
&= -\sum_{q=1}^{j-1}k_q\varepsilon_q^2 + \varepsilon_{j-1}\varepsilon_j + \sum_{q=1}^{j-1}\varepsilon_q\tilde{d}_q - \sum_{\lambda=2}^{j-1}(\varepsilon_\lambda(\sum_{s=1}^{\lambda-1}\tfrac{\partial u_{\lambda-1}^*}{\partial x_s}\tilde{d}_s + \sum_{s=1}^{\lambda-1}\tfrac{\partial u_{\lambda-1}^*}{\partial \hat{\xi}_s}L_s\tilde{d}_s)) + \varepsilon_j(\varepsilon_{j+1} + u_j^* + \hat{d}_j + \tilde{d}_j \\
&\quad - \sum_{q=1}^{j-1}\tfrac{\partial u_{j-1}^*}{\partial x_q}(x_{q+1} + \hat{d}_q + \tilde{d}_q) - \sum_{q=1}^{j-1}\tfrac{\partial u_{j-1}^*}{\partial \hat{\xi}_q}(W_q\hat{\xi}_q + L_q\tilde{d}_q) - \sum_{q=1}^{j}\tfrac{\partial u_{j-1}^*}{\partial x^{d(q-1)}}x^{d(q)}),
\end{aligned} \tag{B14}$$

where $\varepsilon_{j+1} = x_{j+1} - u_j^*$.

By designing u_j^* as

$$u_j^* = -k_j\varepsilon_j - \varepsilon_{j-1} - \hat{d}_j + \sum_{q=1}^{j-1}\frac{\partial u_{j-1}^*}{\partial x_q}(x_{q+1} + \hat{d}_q) + \sum_{q=1}^{j-1}\frac{\partial u_{j-1}^*}{\partial \hat{\xi}_q}W_q\hat{\xi}_q + \sum_{q=1}^{j}\frac{\partial u_{j-1}^*}{\partial x^{d(q-1)}}x^{d(q)}. \tag{B15}$$

The equation (B14) can be transformed into

$$\dot{\bar{V}}_j = -\sum_{q=1}^{j} k_q \varepsilon_q^2 + \varepsilon_j \varepsilon_{j+1} + \sum_{q=1}^{j} \varepsilon_q \tilde{d}_q - \sum_{\lambda=2}^{j} (\varepsilon_\lambda (\sum_{s=1}^{\lambda-1} \frac{\partial u_{\lambda-1}^*}{\partial x_s} \tilde{d}_s + \sum_{s=1}^{\lambda-1} \frac{\partial u_{\lambda-1}^*}{\partial \hat{\xi}_s} L_s \tilde{d}_s)). \tag{B16}$$

Step n: Construct the Lyapunov function

$$\bar{V}_n = \bar{V}_{n-1} + \frac{1}{2}\varepsilon_n^2. \tag{B17}$$

We can obtain the time-derivative of $\bar{V}_n$ as

$$\begin{aligned} \dot{\bar{V}}_n &= \dot{\bar{V}}_{n-1} + \varepsilon_n \dot{\varepsilon}_n \\ &= -\sum_{q=1}^{n-1} k_q \varepsilon_q^2 + \varepsilon_{n-1}\varepsilon_n + \sum_{q=1}^{n-1} \varepsilon_q \tilde{d}_q - \sum_{\lambda=2}^{n-1} (\varepsilon_\lambda (\sum_{s=1}^{\lambda-1} \frac{\partial u_{\lambda-1}^*}{\partial x_s} \tilde{d}_s + \sum_{s=1}^{\lambda-1} \frac{\partial u_{\lambda-1}^*}{\partial \hat{\xi}_s} L_s \tilde{d}_s)) + \varepsilon_n (\dot{x}_n - \dot{u}_{n-1}^*) \\ &= -\sum_{q=1}^{n-1} k_q \varepsilon_q^2 + \varepsilon_{n-1}\varepsilon_n + \sum_{q=1}^{n-1} \varepsilon_q \tilde{d}_q - \sum_{\lambda=2}^{n-1} (\varepsilon_\lambda (\sum_{s=1}^{\lambda-1} \frac{\partial u_{\lambda-1}^*}{\partial x_s} \tilde{d}_s + \sum_{s=1}^{\lambda-1} \frac{\partial u_{\lambda-1}^*}{\partial \hat{\xi}_s} L_s \tilde{d}_s)) + \varepsilon_n (f_0(x_1, x_2, ..., x_n) \\ &\quad + \hat{x}_{n+1} + b_0 u + e_2 - \dot{u}_{n-1}^*), \end{aligned} \tag{B18}$$

Note that the following equation holds:

$$\dot{u}_{n-1}^* = \sum_{q=1}^{n-1} \frac{\partial u_{n-1}^*}{\partial x_q} (x_{q+1} + \hat{d}_q + \tilde{d}_q) + \sum_{q=1}^{n-1} \frac{\partial u_{n-1}^*}{\partial \hat{\xi}_q} (W_q \hat{\xi}_q + L_q \tilde{d}_q) + \sum_{q=1}^{n} \frac{\partial u_{n-1}^*}{\partial x^{d(q-1)}} x^{d(q)}. \tag{B19}$$

Then, based on (12), we can derive that

$$\dot{\bar{V}}_n = -\sum_{q=1}^{n} k_q \varepsilon_q^2 + \sum_{q=1}^{n-1} \varepsilon_q \tilde{d}_q + \varepsilon_n e_2 - \sum_{\lambda=2}^{n} (\varepsilon_\lambda (\sum_{s=1}^{\lambda-1} \frac{\partial u_{\lambda-1}^*}{\partial x_s} \tilde{d}_s + \sum_{s=1}^{\lambda-1} \frac{\partial u_{\lambda-1}^*}{\partial \hat{\xi}_s} L_s \tilde{d}_s)). \tag{B20}$$

For the sake of simplicity, define $\Delta_\lambda = \sum_{s=1}^{\lambda-1} \frac{\partial u_{\lambda-1}^*}{\partial x_s} \tilde{d}_s + \sum_{s=1}^{\lambda-1} \frac{\partial u_{\lambda-1}^*}{\partial \hat{\xi}_s} L_s \tilde{d}_s$. One can obtains

$$\begin{aligned} \dot{\bar{V}}_n &= -\sum_{q=1}^{n} k_q \varepsilon_q^2 + \sum_{q=1}^{n-1} \varepsilon_q \tilde{d}_q + \varepsilon_n e_2 - \sum_{\lambda=2}^{n} \varepsilon_\lambda \Delta_\lambda \\ &\leqslant -k_1 \varepsilon_1^2 - \sum_{q=2}^{n-1} k_q \varepsilon_q^2 - k_n \varepsilon_n^2 + \|\varepsilon_1\| \left\|\tilde{d}_1\right\| + \sum_{q=2}^{n-1} \|\varepsilon_q\| \left\|\tilde{d}_q\right\| + \|\varepsilon_n\| \|e_2\| + \sum_{\lambda=2}^{n-1} \|\varepsilon_\lambda\| \|\Delta_\lambda\| + \|\varepsilon_n\| \|\Delta_n\| \\ &\leqslant -k_1 \varepsilon_1^2 - \sum_{q=2}^{n-1} k_q \varepsilon_q^2 - k_n \varepsilon_n^2 + \frac{\varepsilon_1^2}{2} + \frac{\tilde{d}_1^2}{2} + \sum_{q=2}^{n-1} (\frac{\varepsilon_q^2}{2} + \frac{\tilde{d}_q^2}{2}) + \frac{\varepsilon_n^2}{2} + \frac{e_2^2}{2} + \sum_{\lambda=2}^{n-1} (\frac{\varepsilon_\lambda^2}{2} + \frac{\Delta_\lambda^2}{2}) + \frac{\varepsilon_n^2}{2} + \frac{\Delta_n^2}{2} \\ &= -(k_1 - \tfrac{1}{2})\varepsilon_1^2 - \sum_{q=2}^{n} (k_q - 1)\varepsilon_q^2 + \sum_{q=1}^{n-1} \frac{\tilde{d}_q^2}{2} + \sum_{\lambda=2}^{n} \frac{\Delta_\lambda^2}{2} + \frac{e_2^2}{2}. \end{aligned} \tag{B21}$$

From Theorem 1, it can be seen that Δ_λ and $\tilde{d}_q$ are bounded. Meanwhile, according to [1,2], by properly selecting the parameters β_1 and β_2 in ESO, the estimation errors e_1 and e_2 can be limited small enough. Then, one can obtain that $\sum_{q=1}^{n-1} \frac{\tilde{d}_q^2}{2} + \sum_{\lambda=2}^{n} \frac{\Delta_\lambda^2}{2} + \frac{e_2^2}{2}$ is bounded. Hence, according to [3], the variables $\varepsilon_p \, (p = 1, 2, ..., n)$ will converge to the neighborhoods of zeros asymptotically by selecting $k_1 > \frac{1}{2}$ and $k_q > 1$. Thus, x_1 will converge to the neighborhood of x^d asymptotically. This completes the proof.

Appendix C Numerical simulation

In this section, in order to demonstrate the effectiveness of proposed methods, numerical simulations are conducted. Without loss of generality, a second order control system is considered as follows:

$$\begin{cases} \dot{x}_1 = x_2 + d_1, \\ \dot{x}_2 = f_0(x_1, x_2) + bu + w. \end{cases} \tag{C1}$$

Assume that $d_1 = a \sin(\omega_0 t + \varphi)$, where the frequency information $\omega_0 = 2$ is known. Set $a = 0.5$, $\varphi = \frac{\pi}{3}$, $H_1 = I$, and δ is a bounded random signal ($\|\delta\| \leqslant 0.5$) in this simulation. Then, we can get $W_1 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, V_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

The desired trajectory is selected as $x^d = \sin(t)$. The nonlinear function $f_0(x_1, x_2) = x_1^2 + x_2$, the parameters $b_0 = 1$, and $\Delta b = 0.2$ are adopted. The disturbance w is assumed to be $w = 0.2 + \sin(0.1t + \frac{\pi}{6})$.

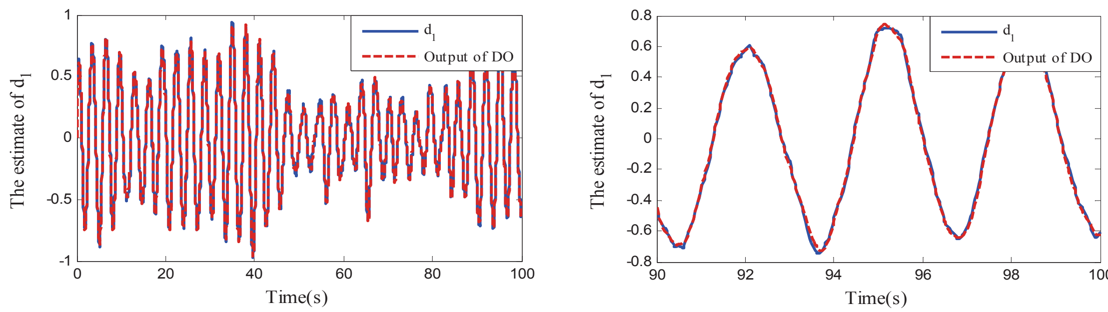
For the above second order system, the EADC can be written as

$$u = -\frac{1}{b_0}((1 + k_1 k_2)x_1 + (k_1 + k_2)x_2 - (1 + k_1 k_2)x^d + (k_1 + k_2)\hat{d}_1 - (k_1 + k_2)\dot{x}^d + f_0(x_1, x_2) + \hat{x}_3 + V_1 W_1 \hat{\xi}_1 - \ddot{x}^d). \tag{C2}$$

The parameters for the DO, ESO, and controller are selected as $L_1 = [9.20, 7.69]^T$, $\alpha = 0.1$, $\beta_1 = -5$, and $\beta_2 = -50$. The initial conditions of x_1 and x_2 are 1 and 0.5, respectively. Based on these parameters, the simulation results are shown as follows.

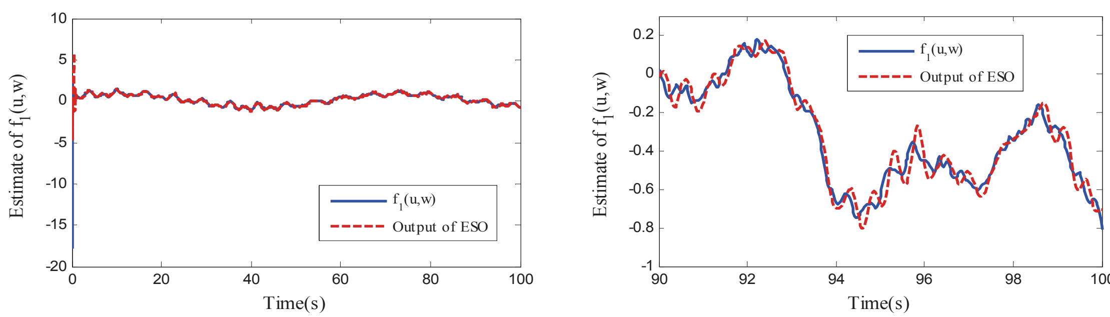
Figure C1 shows the performance of DO in estimating the unmatched disturbance d_1, from which we can see that the proposed DO can estimate the actual value of d_1 precisely and rapidly. Similarly, it can be concluded from Figure C2 that the proposed ESO can estimate the matched disturbance $f_1(u,w)$ effectively. The two figures have demonstrated the effectiveness of proposed DO and ESO in disturbance estimation.

On the basis of disturbance estimation, the controller (C2) is designed. It can be seen from Figure C3 that the state x_1 can track the desired trajectory x^d precisely under the proposed EADC. Moreover, comparisons with linear ADRC method proposed in [2] are given. The bandwidth of ESO for the linear ADRC is selected as 20. From Figure C4 we can see that, compared to the linear ADRC method, the control accuracy has been significantly improved by the proposed EADC.



(a) Estimate of d_1 (b) Partial enlargement

Figure C1 The performance of DO in estimating d_1.



(a) Estimate of $f_1(u,w)$ (b) Partial enlargement

Figure C2 The performance of ESO in estimating $f_1(u,w)$.

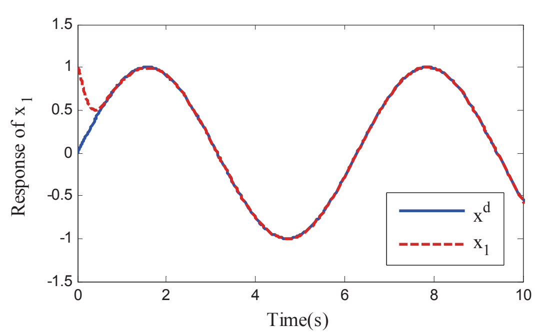


Figure C3 The performance of EADC.

References

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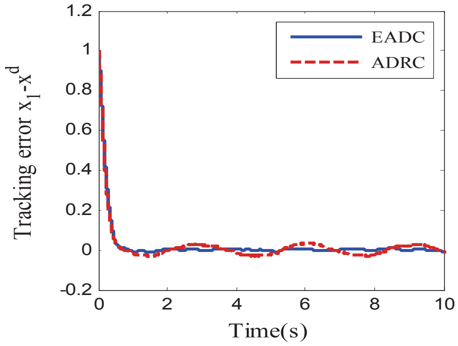
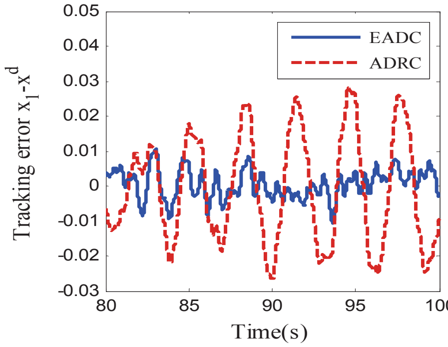


(a) Transient response



(b) Steady-state error

Figure C4 Performance comparisons with ADRC.

3 Xia Y Q, Zhu Z, Fu M Y, et al. Attitude tracking of rigid spacecraft with bounded disturbances. IEEE Trans Ind electron, 2011, 58(2): 647-659.