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• **Supplementary File** •

Optimal containment control of continuous-time multi-agent systems with unknown disturbances using data-driven approach

Zhinan PENG[1], Jiefu ZHANG[1], Jiangping HU[1*], Rui HUANG[1] & Bijoy Kumar GHOSH[1,2]

[1]*School of Automation Engineering, University of Electronic Science and Technology of China, Chengdu* 611731*, China;*
[2]*Department of Mathematics and Statistics, Texas Tech University, Lubbock* 79409*, USA*

Appendix A The proof of Theorem 1

Proof. Assuming that u_i^*, v_i^* and V_i^* are the optimal control policy and performance indices in (10), (11) and (8). Then, u_i^* will be proved to be the solution of the original system (1), which means system (1) can be asymptotically stabilized by u_i^* with uncertainties.

Taking the derivative of V_i^*, one has

$$\begin{aligned}\dot{V}_i^* =&(\frac{\partial V_i^*}{\partial e_i})^T \dot{e}_i\\ =&(\frac{\partial V_i^*}{\partial e_i})^T [Ae_i + (d_i + \sum_{k=n+1}^{n+m} g_i^k)B_i u_i^* + (d_i + \sum_{k=n+1}^{n+m} g_i^k)\\ &\times D_i\omega_i - \sum_{j\in N_i} a_{ij}B_j u_j^* - \sum_{j\in N_i} a_{ij}D_j\omega_j]\end{aligned} \tag{A1}$$

Adding and subtracting the term $(d_i + \sum_{k=n+1}^{n+m} g_i^k)[B_i u_i^* + (I - B_i B_i^\dagger)D_i v_i^*]$ and $\sum_{j\in N_i} a_{ij}[B_j u_j^* + (I - B_j B_j^\dagger)D_j v_j^*]$ simultaneously, (A1) becomes

$$\begin{aligned}\dot{V}_i^* =& (\frac{\partial V_i^*}{\partial e_i})^T(\dot{e}_i)(Ae_i + (d_i + \sum_{k=n+1}^{n+m} g_i^k)[B_i u_i^* + (I - B_i B_i^\dagger)\\ &\times D_i v_i^*]) - \sum_{j\in N_i}[B_j u_j^* + (I - B_j B_j^\dagger)D_j v_j^*] + (\frac{\partial V_i^*}{\partial e_i})^T\\ &\times \Big((d_i + \sum_{k=n+1}^{n+m} g_i^k)B_i B_i^\dagger D_i\omega_i + (d_i + \sum_{k=n+1}^{n+m} g_i^k)(I - B_i B_i^\dagger)D_i(\omega_i - v_i^*)\\ &+ \sum_{j\in N_i} a_{ij}(I - B_j B_j^\dagger)D_j v_j^* - \sum_{j\in N_j} a_{ij}D_j\omega_j\Big)\end{aligned} \tag{A2}$$

According to (9), we have

$$\begin{aligned}&(\frac{\partial V_i^*}{\partial e_i})^T\big(Ae_i + (d_i + \sum_{k=n+1}^{n+m} g_i^k)[B_i u_i^* + (I - B_i B_i^\dagger)\\ &\times D_i v_i^*]) - \sum_{j\in N_i}[B_j u_j^* + (I - B_j B_j^\dagger)D_j v_j^*]\big)\\ =& -\|\rho\|^2 - \zeta^2\|p_{ii}\|^2\tau_b^2 - e_i^T Q_{ii} e_i + u_i^{*T} R_{ii} u_i^*\\ &- \zeta^2 v_i^{*T} P_{ii} v_i^* - \sum_{j\in N_i} u_j^{*T} R_{ij} u_j^* - \sum_{j\in N_i} \zeta^2 v_j^{*T} P_{ij} v_j^*\end{aligned} \tag{A3}$$

* Corresponding author (email: hujp@uestc.edu.cn)

Then, considering (10) and (11), we have

$$(d_i + \sum_{k=n+1}^{n+m} g_i^k) B_i^T \frac{\partial V_i^*}{\partial e_i} + 2R_{ii} u_i^* = 0 \tag{A4}$$

and

$$(d_i + \sum_{k=n+1}^{n+m} g_i^k) D_i^T (I - B_i B_i^\dagger) \frac{\partial V_i^*}{\partial e_i} + 2\zeta^2 P_{ii} v_i^* = 0 \tag{A5}$$

Substitute (A3), (A4) and (A5) into (A1), we have

$$\begin{aligned}
\dot{V}_i^* =& -\|\rho\|^2 - \zeta^2 \|p_{ii}\|^2 \tau_b^2 - e_i^T Q_{ii} e_i - u_i^{*T} R_{ii} u_i^* \\
& - \zeta^2 v_i^{*T} P_{ii} v_i^* - \sum_{j \in N_i} u_j^{*T} R_{ij} u_j^* - \sum_{j \in N_i} \zeta^2 v_j^{*T} P_{ij} v_j^* \\
& - 2 u_i^{*T} R_{ii} B_i^\dagger D_i \omega_i - 2\zeta^2 v_i^* P_{ii} (\omega_i - v_i^*) \\
& + (\frac{\partial V_i^*}{\partial e_i})^T \sum_{j \in N_i} a_{ij} (I - B_j B_j^\dagger) D_j v_j^* \\
& + (\frac{\partial V_i^*}{\partial e_i})^T \sum_{j \in N_i} a_{ij} D_j \omega_j
\end{aligned} \tag{A6}$$

Assuming that $R_{ii} = r_{ii}\ r_{ii}^T$, we can obtain

$$-u_i^{*T} R_{ii} u_i^* - 2 u_i^{*T} R_{ii} B_i^+ D_i \omega_i = -\|r_{ii}^T u_i^* + r_{ii}^T B_i^\dagger D_i \omega_i\|^2 + \|r_{ii}^T B^\dagger D_i \omega_i\|^2 \tag{A7}$$

Furthermore, according to (11), one has

$$(\frac{\partial V_i^*}{\partial e_i})^T \sum_{j \in N_i} a_{ij} (I - B_j B_j^\dagger) D_j v_j^* = -2\zeta^2 (d_i + \sum_{k=n+1}^{n+m} g_i^k)^{-1} \sum_{j \in N_i} a_{ij} v_j^{*T} P_{ij} v_j^* \tag{A8}$$

Therefore, we can rewrite (A6) as

$$\begin{aligned}
\dot{V}_i^* =& -\|\rho\|^2 - \zeta^2 \|p_{ii}\|^2 \tau_b^2 - e_i^T Q_{ii} e_i \\
& - \zeta^2 v_i^{*T} P_{ii} v_i^* - \sum_{j \in N_i} u_j^{*T} R_{ij} u_j^* - \sum_{j \in N_i} \zeta^2 v_j^{*T} P_{ij} v_j^* \\
& - 2\zeta^2 v_i^* P_{ii} (\omega_i - v_i^*) - \|r_{ii}^T u_i^* + r_{ii}^T B_i^\dagger D_i \omega_i\|^2 \\
& + \|r_{ii}^T B^\dagger D_i \omega_i\|^2 - 2\zeta^2 (d_i + \sum_{k=n+1}^{n+m} g_i^k)^{-1} \\
& \times \sum_{j \in N_i} a_{ij} v_j^{*T} P_{ij} v_j^* - (\frac{\partial V_i^*}{\partial e_i})^T \sum_{j \in N_i} a_{ij} D_j \omega_j
\end{aligned} \tag{A9}$$

Considering (A8) and (A9), we have

$$\begin{aligned}
& - \sum_{j \in N_i} u_j^{*T} R_{ij} u_j^* + 2 \sum_{j \in N_i} a_{ij} (d_i + \sum_{k=n+1}^{n+m} g_i^k)^{-1} u_j^* R_{ij} D_j \omega_j \\
= & - \sum_{j \in N_i} \|r_{ij}^T u_j^* - r_{ij}^T a_{ij} (d_i + \sum_{k=n+1}^{n+m} g_i^k)^{-1} D_j \omega_j\|^2 \\
& + \sum_{j \in N_i} \|r_{ij}^T a_{ij} (d_i + \sum_{k=n+1}^{n+m} g_i^k)^{-1} D_j \omega_j\|^2
\end{aligned} \tag{A10}$$

and

$$\begin{aligned}
& - \sum_{j \in N_i} \zeta^2 v_j^{*T} P_{ij} v_j^* + 2\zeta^2 a_{ij} (d_i + \sum_{k=n+1}^{n+m} g_i^k) v_j^* P_{ij} \omega_j \\
= & - \zeta^2 \sum_{j \in N_i} \|p_{ij}^T v_j^* - p_{ij}^T a_{ij} (d_i + \sum_{k=n+1}^{n+m} g_i^k)^{-1} \omega_j\|^2 \\
& + \zeta^2 \sum_{j \in N_i} \|p_{ij}^T a_{ij} (d_i + \sum_{k=n+1}^{n+m} g_i^k)^{-1} \omega_j\|^2
\end{aligned} \tag{A11}$$

Further, because $P_{ii} = p_{ii}\ p_{ii}^T$, we have

$$-2\zeta^2 v_i^{*T} P_{ii} \omega_i \leqslant \zeta^2 (\|p_{ii}^T v_i^*\|^2 + \|p_{ii}^T \omega_i\|^2) \tag{A12}$$

Therefore, we can obtain the derivative of V_i^* by substituting (A10), (A11) and (A12) into (A9) as

$$\begin{aligned}\dot{V}_i^* \leqslant & -\|\rho\|^2 - \zeta^2\|p_{ii}\|^2\tau_b^2 - e_i^TQ_{ii}e_i \\ & - \zeta^2 v_i^{*T}P_{ii}v_i^* - \|r_{ii}^Tu_i^* + r_{ii}^TB_i^\dagger D_i\omega_i\|^2 \\ & - \sum_{j\in N_i}\|r_{ij}^Tu_j^* - r_{ij}^Ta_{ij}(d_i + \sum_{k=n+1}^{n+m}g_i^k)^{-1}D_j\omega_j\|^2 \\ & - \zeta^2\sum_{j\in N_i}\|p_{ij}^Tv_j^* - p_{ij}^Ta_{ij}(d_i+\sum_{k=n+1}^{n+m}g_i^k)^{-1}\omega_j\|^2 + \|r_{ii}^TB_i^\dagger D_i\omega_i\|^2 \\ & + \zeta^2\sum_{j\in N_i}\|p_{ij}^Ta_{ij}(d_i+\sum_{k=n+1}^{n+m}g_i^k)^{-1}\omega_j\|^2 + 2\zeta^2v_i^{*T}P_{ii}v_i^* \\ & + \sum_{j\in N_i}\|r_{ij}^Ta_{ij}(d_i+\sum_{k=n+1}^{n+m}g_i^k)^{-1}D_j\omega_j\|^2 + \zeta^2\|p_{ii}^Tv_i^*\|^2 + \zeta^2\|p_{ii}^T\omega_i\|^2 \\ & -2\zeta^2(d_i+\sum_{k=n+1}^{n+m}g_i^k)^{-1}\times\sum_{j\in N_i}a_{ij}v_j^{*T}P_{ij}v_j^*\end{aligned} \tag{A13}$$

Then, the further simplified form of (A13) can be represented as

$$\begin{aligned}\dot{V}_i^* \leqslant & -\Big(\|\rho\|^2 - \|r_{ii}^TB^\dagger D_i\omega_i\|^2 - \sum_{j\in N_i}\|r_{ij}^Ta_{ij}(d_i+\sum_{k=n+1}^{n+m}g_i^k)^{-1}D_j\omega_j\|^2 \\ & -\zeta^2\sum_{j\in N_i}\|p_{ij}^T(d_i+\sum_{k=n+1}^{n+m}g_i^k)^{-1}\omega_j\|^2\Big) \\ & -\zeta^2\|p_{ii}^T\|^2(\tau_b^2-\omega_i^2) - e_i^TQ_{ii}e_i + 2\zeta^2v_I^*P_{ii}v_i^*\end{aligned} \tag{A14}$$

It is note that when $\|\rho\|^2$ satisfies

$$\begin{aligned}\|\rho\|^2 \geqslant & \|r_{ii}^TB_i^\dagger D_i\omega_m\|^2 + \sum_{j\in N_i}\|r_{ij}^Ta_{ij}(d_i+\sum_{k=n+1}^{n+m}g_i^k)^{-1}D_j\tau_b\|^2 \\ & + \sum_{j\in N_i}\zeta^2\|p_{ij}^T(d_i+\sum\nolimits_{k=n+1}^{n+m}g_i^k)^{-1}\tau_b\|^2\end{aligned} \tag{A15}$$

then (A14) becomes

$$\dot{V}_i^* \leqslant 2\zeta^2(v_i^{*T}P_{ii}v_i^* - e_i^TQ_{ii}e_i) - (1-2\zeta^2)e_i^TQ_{ii}e_i \tag{A16}$$

Therefore if the conditions $0<\zeta<\frac{\sqrt{2}}{2}$ and $e_i^TQ_{ii}e_i > v_i^{*T}P_{ii}v_i^*$ are satisfied, we have $\dot{V}_i^* < 0$ for any $e_i \neq 0$, which means optimal control input u_i^* and v_i^* can asymptotically stabilize system (5), that is $\|e_i\| \to 0$ as $t\to\infty$. Thus, the designed optimal control policy u_i^* can makes agent achieve containment control with unknown disturbances. This completes the proof.

Remark 1. From Theorem 1, we can obtain that the system (1) and (7) is equivalent, and the containment control can be achieved for the original multi-agent systems (1) with designed u_i^*. Thus such model transformation is feasible. In this paper, the way to ensure the condition is always satisfied is to select appropriate weighting matrices and parameter.

Appendix B Numerical experiments

In this section, a simulation is given to validate the proposed optimal containment control algorithm. Consider a multi-agent system with the communication topology as shown in Fig. B1. Nodes 1,2,3, 4 represent the followers while nodes 5,6,7 are the leaders. From Fig. B1, one has $a_{14}=a_{21}=a_{32}=a_{43}=1$, $g_1^5=g_2^6=g_3^7=1$, $d_1=d_2=d_3=d_4=1$.

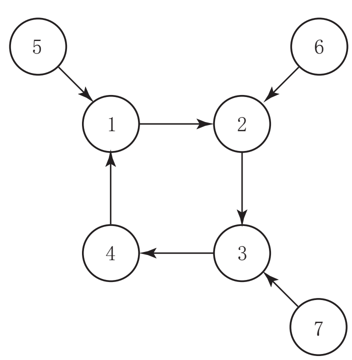

Figure B1 The communication topology

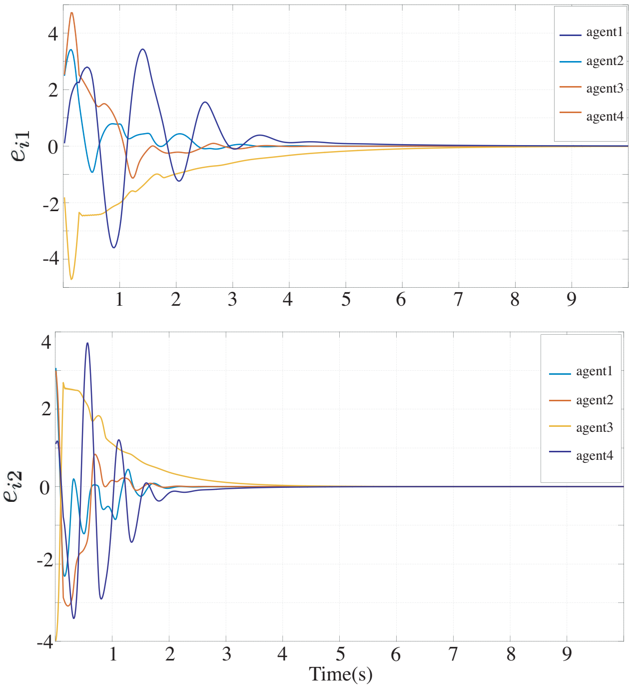

Figure B2 Dynamic of local neighborhood tracking errors.

The system parameter matrices of the multi-agent systems are given as $A = \begin{bmatrix} 0.9950 & 0.0998 \\ -0.0998 & 0.9950 \end{bmatrix}$ $B_1 = \begin{bmatrix} 0 \\ 0.0900 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ 0.1000 \end{bmatrix}$, $B_3 = \begin{bmatrix} 0 \\ 0.1500 \end{bmatrix}$ and $B_4 = \begin{bmatrix} 0 \\ 0.2000 \end{bmatrix}$, $D_1 = \begin{bmatrix} 0.0900 \\ 0 \end{bmatrix}$, $D_2 = \begin{bmatrix} 0.1000 \\ 0 \end{bmatrix}$, $D_3 = \begin{bmatrix} 0.1500 \\ 0 \end{bmatrix}$ and $D_4 = \begin{bmatrix} 0.2000 \\ 0 \end{bmatrix}$, $\omega_i(t) = \alpha_{i1} x_{i1} cos(1/(x_{i2} + \alpha_{i2}))$ where $x_i = [x_{i1}, x_{i2}]^T$, $\alpha_{i1} \in [-0.2, 0.1]$, $\alpha_{i2} \in [-10, 100]$. In fact, $B_i^{\dagger} = (B_i^T B_i)^{-1} B_i^T$. Then, we can obtain the converted multi-agent systems according to (7).

The weighting matrices in the performance index functions (9) are given by $R_{11} = R_{14} = R_{21} = R_{22} = R_{32} = R_{33} = R_{43} = R_{44} = 1$, $P_{11} = P_{22} = P_{33} = P_{44} = 0.01$, $p_{ii} = 0.1$ for $i = 1, 2, 3, 4$, $P_{14} = P_{21} = P_{32} = P_{43} = 1$. $Q_{11} = Q_{22} = Q_{33} = Q_{44} = I_{2\times 2}$. The parameter is set as $\zeta = 0.05$. The learning rates are selected as $\beta_a = \beta_c = 0.005$. Take the initial states of the followers as $x_1 = (2, 2)^T, x_2 = (2.5, 2.5)^T, x_3 = (0.5, -0.5)^T, x_4 = (0.6, 0.6)^T$, and the initial states of the leaders are randomly. For the critic-actor network, repeat the weight update process until $\| \sum \omega_{ci}^{l+1} - \sum \omega_{ci}^{l} \| < \epsilon$, where ω_{ci}^{l} is the critic weight at the lth iteration, and $\epsilon = 1 \times 10^{-5}$.

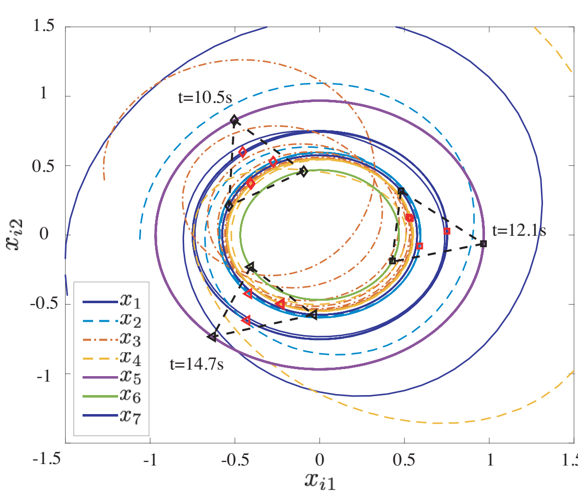

Figure B3 2-D phase plane plot.

The evolution of the local neighbor tracking errors of the converted system is shown in Fig. B2. Then, we apply the optimal control into the original system with disturbance. From Fig. B3, it can be observed that all the followers move into the interior of the convex hull formed by the leaders as $t = 10.5, 12.1, 14.7$ (s). The state trajectories of the followers and the leaders are shown in Fig. B4, where the solid lines denote the envelopes formed by the trajectories of the leaders. It can be seen that the trajectories of the followers stay in the envelope formed by those leaders after $t = 12$ (s). The simulation results show that the proposed algorithm can achieve containment control of multi-agent systems with unknown disturbances.

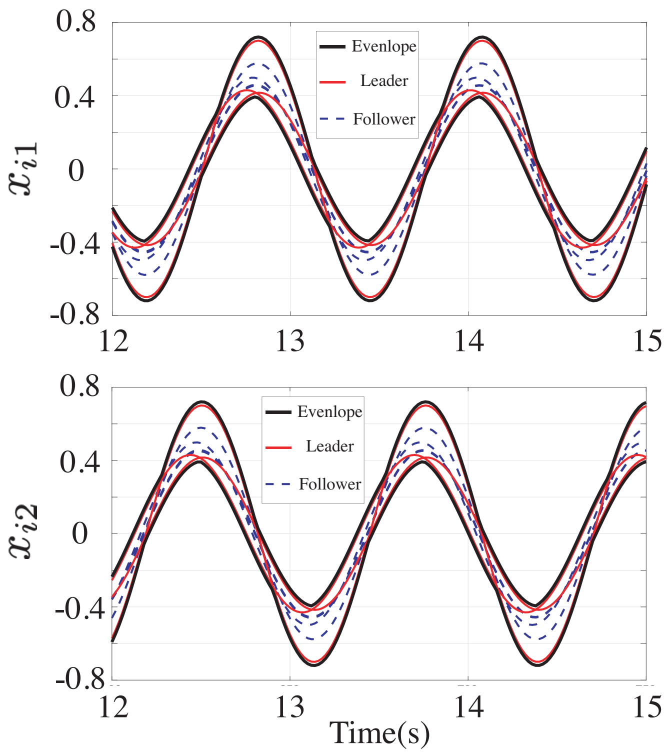

Figure B4 The state trajectory of agents.