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• **Supplementary File** •

Adaptive outer synchronization between two delayed oscillator networks with cross couplings

Jianbao ZHANG[1,2], Ancai ZHANG[2], Jinde CAO[1*], Jianlong QIU[2,3] & Fuad E. ALSAADI[3]

[1]*School of Automation, Southeast University, Nanjing 210096, China;*
[2]*School of Information Science and Engineering, Linyi University, Linyi 276005, China;*
[3]*Department of Electrical and Computer Engineering, Faculty of Engineering, King Abdulaziz University, Jeddah 21589, Saudi Arabia*

Appendix A Proof of Theorem 1

In order to prove Theorem 1, we introduce the following two lemmas.

Lemma 1. [1] For any vectors $\bar{u}, u \in R^m$, the following inequality holds:

$$2\bar{u}^\top u \leqslant \bar{u}^\top \bar{u} + u^\top u.$$

Lemma 2. Denote the matrix

$$\Gamma = \begin{pmatrix} I_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \in R^{n\times m}, \tag{A1}$$

the following inequality holds for any $u \in R^m$ and $v \in R^n$,

$$2v^\top \Gamma u \leqslant u^\top \Gamma^\top \Gamma u + v^\top \Gamma\Gamma^\top v.$$

The proof of Lemma 2 is straightforward and omitted here. Now, we turn to the proof of Theorem 1.

Proof. Denote the parameter errors $\tilde{\alpha}(t) = \bar{\alpha}(t) - \alpha$ and $\tilde{\beta}(t) = \bar{\beta}(t) - \beta$, and consider the following Lyapunov function:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t),$$

where

$$V_1(t) = \frac{1}{2}\sum_{i=1}^{N}[e_{xi}^\top(t)e_{xi}(t) + e_{yi}^\top(t)e_{yi}(t)] + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}[p_{ij}(t) + \bar{c}_{ij} - c_{ij}]^2, \tag{A2}$$

$$V_2(t) = \frac{L_2||\alpha||}{2(1-\epsilon_x)}\sum_{i=1}^{N}\int_{t-\tau_x(t)}^{t} e_{xi}^\top(s)e_{xi}(s)ds + \frac{l_2||\beta||}{2(1-\epsilon_y)}\sum_{i=1}^{N}\int_{t-\tau_y(t)}^{t} e_{yi}^\top(s)e_{yi}(s)ds, \tag{A3}$$

$$V_3(t) = \frac{1}{2}\tilde{\alpha}^\top(t)\tilde{\alpha}(t) + \frac{1}{2}\tilde{\beta}^\top(t)\tilde{\beta}(t), \tag{A4}$$

$$V_4(t) = \sum_{i=1}^{N}[\frac{1}{2k_{xi}}(\eta_{xi}(t) - \eta_x^*)^2 + \frac{1}{2k_{yi}}(\eta_{yi}(t) - \eta_y^*)^2 + \frac{1}{2\xi_{xi}}(\gamma_{xi}(t) - \gamma_x^*)^2 + \frac{1}{2\xi_{yi}}(\gamma_{yi}(t) - \gamma_y^*)^2], \tag{A5}$$

where the positive constants $\eta_x^*, \eta_y^*, \gamma_x^*, \gamma_y^*$ will be determined later to make the derivative of $V(t)$ along the system (1)-(2) less than zero, and this will complete the proof. Briefly speaking, the first two terms of $V_1(t)$ are selected to prove outer synchronization, the third term is selected to compensate the topology difference of (1) and (2), $V_2(t)$ is selected to compensate the time varying delays existing in node dynamics, $V_3(t)$ is selected for the validity of the parameter update laws, and $V_4(t)$ is selected for global stability and convergence of the time varying control gains.

From the following inequality

$$\begin{aligned}\frac{d}{dt}[\int_{t-\tau_x(t)}^{t} e_{xi}^\top(s)e_{xi}(s)ds] &= e_{xi}^\top(t)e_{xi}(t) - e_{xi}^\top(t-\tau_x(t))e_{xi}(t-\tau_x(t))(1-\dot{\tau}_x(t)) \\ &\leqslant ||e_{xi}(t)||^2 - (1-\epsilon_x)||e_{xi}(t-\tau_x(t))||^2,\end{aligned} \tag{A6}$$

* Corresponding author (email: jdcao@seu.edu.cn)

and the corresponding inequality with respect to y, we have

$$\dot{V}_2(t) \leqslant \sum_{i=1}^{N}[\frac{L_2||\alpha||}{2(1-\epsilon_x)}||e_{xi}(t)||^2 - \frac{L_2||\alpha||}{2}||e_{xi}(t-\tau_x(t))||^2 + \frac{l_2||\beta||}{2(1-\epsilon_y)}||e_{yi}(t)||^2 - \frac{l_2||\beta||}{2}||e_{yi}(t-\tau_y(t))||^2]. \quad \text{(A7)}$$

By the parameter update laws (3), the derivative of $V_3(t)$ reads,

$$\dot{V}_3(t) = -\sum_{i=1}^{N}\tilde{\alpha}^\top(t)f_2^\top(\bar{x}_i(t-\tau_x(t)))e_{xi}(t) - \sum_{j=1}^{N}\tilde{\beta}^\top(t)g_2^\top(\bar{y}_i(t-\tau_y(t)))e_{yi}(t). \quad \text{(A8)}$$

Similarly, denote $||e_{xi}(t)||_1 = e_{xi}(t)\text{sign}[e_{xi}(t)]$ and $||e_{yi}(t)||_1 = e_{yi}(t)\text{sign}[e_{yi}(t)]$, it follows from the controllers (4) that

$$\dot{V}_4(t) = \sum_{i=1}^{N}[(\eta_{xi}(t)-\eta_x^*)||e_{xi}(t)||^2 + (\eta_{yi}(t)-\eta_y^*)||e_{yi}(t)||^2 + (\gamma_{xi}(t)-\gamma_x^*)||e_{xi}(t)||_1 + (\gamma_{yi}(t)-\gamma_y^*)||e_{yi}(t)||_1]. \quad \text{(A9)}$$

The derivative of $V_1(t)$ along the system (1)-(2) reads,

$$\begin{aligned}\dot{V}_1(t) = & \sum_{i=1}^{N}e_{xi}^\top(t)[f_1(\bar{x}_i(t)) - f_1(x_i(t)) + f_2(\bar{x}_i(t-\tau_x(t)))\bar{\alpha}(t) - f_2(x_i(t-\tau_x(t)))\alpha] \\ & + \sum_{i=1}^{N}e_{yi}^\top(t)[g_1(\bar{y}_i(t)) - g_1(y_i(t)) + g_2(\bar{y}_i(t-\tau_y(t)))\bar{\beta}(t) - g_2(y_i(t-\tau_y(t)))\beta] \\ & + \sum_{i=1}^{N}[e_{xi}^\top(t)\Delta_x(t) + e_{yi}^\top(t)\Delta_y(t) + e_{xi}^\top(t)u_{xi}(t) + e_{yi}^\top(t)u_{yi}(t) - \sum_{j=1}^{N}p_{ij}(t)e_{yi}^\top(t)\Gamma\bar{x}_j(t) - c_{ij}e_{yi}^\top(t)\Gamma e_{xj}(t)].\end{aligned} \quad \text{(A10)}$$

It follows from Assumption 1 and Lemma 1 that

$$\begin{aligned}& e_{xi}^\top(t)[f_1(\bar{x}_i(t)) - f_1(x_i(t)) + f_2(\bar{x}_i(t-\tau_x(t)))\bar{\alpha}(t) - f_2(x_i(t-\tau_x(t)))\alpha] \\ \leqslant\ & L_1e_{xi}^\top(t)e_{xi}(t) + e_{xi}^\top(t)f_2(\bar{x}_i(t-\tau_x(t)))\tilde{\alpha}(t) + L_2||\alpha||||e_{xi}(t)||||e_{xi}(t-\tau_x(t))|| \\ \leqslant\ & (L_1 + \frac{L_2||\alpha||}{2})||e_{xi}(t)||^2 + \frac{L_2||\alpha||}{2}||e_{xi}(t-\tau_x(t))||^2 + e_{xi}^\top(t)f_2(\bar{x}_i(t-\tau_x(t)))\tilde{\alpha}(t).\end{aligned} \quad \text{(A11)}$$

Similarly, there holds

$$\begin{aligned}& e_{yi}^\top(t)[g_1(\bar{y}_i(t)) - g_1(y_i(t)) + g_2(\bar{y}_i(t-\tau_y(t)))\bar{\beta}(t) - g_2(y_i(t-\tau_y(t)))\beta] \\ \leqslant\ & (l_1 + \frac{l_2||\beta||}{2})||e_{yi}(t)||^2 + \frac{l_2||\beta||}{2}||e_{yi}(t-\tau_y(t))||^2 + e_{yi}^\top(t)g_2(\bar{y}_i(t-\tau_y(t)))\tilde{\beta}(t).\end{aligned} \quad \text{(A12)}$$

Then, considering $\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) + \dot{V}_4(t)$ as a whole, we conclude from Assumption 3 that

$$\begin{aligned}\dot{V}(t) \leqslant & \sum_{i=1}^{N}[(\frac{L_2||\alpha||}{2(1-\epsilon_x)} + L_1 + \frac{L_2||\alpha||}{2} - \eta_x^*)||e_{xi}(t)||^2 + (\frac{l_2||\beta||}{2(1-\epsilon_y)} + l_1 + \frac{l_2||\beta||}{2} - \eta_y^*)||e_{yi}(t)||^2 \\ & + e_{xi}^\top(t)\Delta_x(t) - \gamma_x^*||e_{xi}(t)||_1 + e_{yi}^\top(t)\Delta_y(t) - \gamma_y^*||e_{yi}(t)||_1 + \sum_{j=1}^{N}c_{ij}e_{yi}^\top(t)\Gamma e_{xj}(t)].\end{aligned} \quad \text{(A13)}$$

With the help of Assumption2, Lemma 2 and the boundedness of the connection matrix, we have

$$\begin{aligned}\dot{V}(t) \leqslant & \sum_{i=1}^{N}[(\frac{L_2||\alpha||}{2(1-\epsilon_x)} + \frac{s + L_2||\alpha||}{2} + L_1 - \eta_x^*)||e_{xi}(t)||^2 \\ & + (\frac{l_2||\beta||}{2(1-\epsilon_y)} + \frac{s + l_2||\beta||}{2} + l_1 - \eta_y^*)||e_{yi}(t)||^2 + (\rho_x - \gamma_x^*)||e_{xi}(t)||_1 + (\rho_y - \gamma_y^*)||e_{yi}(t)||_1].\end{aligned} \quad \text{(A14)}$$

Select the constants $\eta_x^*, \eta_y^*, \gamma_x^*, \gamma_y^*$ large enough such that $\frac{L_2||\alpha||}{2(1-\epsilon_x)} + \frac{s + L_2||\alpha||}{2} + L_1 - \eta_x^* < 0$, $\frac{l_2||\beta||}{2(1-\epsilon_y)} + \frac{s + l_2||\beta||}{2} + l_1 - \eta_y^* < 0$, $\rho_x - \gamma_x^* < 0$ and $\rho_y - \gamma_y^* < 0$, Theorem 1 is thus proved based on Lyapunov stability theory.

Appendix B Theorem 2

Theorem 2 Consider the following network model

$$\begin{cases}\dot{\bar{x}}_i(t) = f_1(\bar{x}_i(t)) + f_2(\bar{x}_i(t-\tau_x(t)))\bar{\alpha}(t) + \Delta_x(t) + u_{xi}(t), \\ \dot{\bar{y}}_i(t) = g_1(\bar{y}_i(t)) + g_2(\bar{y}_i(t-\tau_y(t)))\bar{\beta}(t) + \sum_{j=1}^{N}\bar{c}_{ij}(t)\Gamma\bar{x}_j(t) + \Delta_y(t) + u_{yi}(t),\end{cases} \quad \text{(B1)}$$

where the meanings of all mathematical symbols are similar to the network (2) except $\bar{c}_{ij}(t)$ denoting topology identification update laws. Under Assumption 1, 2 and 3, outer synchronization is realized in the system (1)-(B1) with the parameter update laws (3) and the adaptive controllers

$$\begin{cases}u_{xi}(t) = -\eta_{xi}(t)e_{xi}(t) - \gamma_{xi}(t)\text{sign}[e_{xi}(t)], \\ u_{yi}(t) = -\eta_{yi}(t)e_{yi}(t) - \gamma_{yi}(t)\text{sign}[e_{yi}(t)], \\ \dot{\eta}_{xi}(t) = k_{xi}e_{xi}^\top(t)e_{xi}(t),\ k_{xi} > 0, \\ \dot{\eta}_{yi}(t) = k_{yi}e_{yi}^\top(t)e_{yi}(t),\ k_{yi} > 0, \\ \dot{\gamma}_{xi}(t) = \xi_{xi}e_{xi}(t)\text{sign}[e_{xi}(t)],\ \xi_{xi} > 0, \\ \dot{\gamma}_{yi}(t) = \xi_{yi}e_{yi}(t)\text{sign}[e_{yi}(t)],\ \xi_{yi} > 0, \\ \dot{\bar{c}}_{ij}(t) = -e_{yi}^\top(t)\Gamma\bar{x}_j(t),\end{cases} \quad \text{(B2)}$$

where $i, j = 1, 2, \cdots, N$.
Proof. Choose the following Lyapunov function:

$$V(t) = W_1(t) + V_2(t) + V_3(t) + V_4(t),$$

where $V_2(t), V_3(t), V_4(t)$ are defined in the proof of Theorem 1, and

$$W_1(t) = \frac{1}{2}\sum_{i=1}^{N}[e_{xi}^{\top}(t)e_{xi}(t) + e_{yi}^{\top}(t)e_{yi}(t)] + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(\bar{c}_{ij}(t) - c_{ij})^2. \tag{B3}$$

The remainder of the argument is analogous to that of Theorem 1, and it is too long to give here.

For decades, a large number of papers have been carried out to discuss synchronization of complex networks with certain topology. However, it is usually hard to get the exact topology of complex networks in practical situations. Therefore, the identification of network topology has become an important issue in the research of complex networks. Theorem 2 provides us adaptive identification laws of uncertain topology of complex networks with cross couplings.

Appendix C Simulation examples

Consider a concrete example of the network (1)-(2), the topology structure of which is shown in Figure 1. Select the oscillators x_i as Lorenz oscillators,

$$\dot{x}_i(t) = f_1(x_i(t)) + f_2(x_i(t))\alpha$$

and select the oscillators y_i as cellular neural networks,

$$\dot{y}_i(t) = g_1(y_i(t)) + g_2(y_i(t))\beta,$$

where

$$f_1(x_i(t)) = \begin{pmatrix} 0 \\ -x_{i3}(t)x_{i1}(t) - x_{i2}(t) \\ x_{i1}(t)x_{i2}(t) \end{pmatrix}, f_2(x_i(t)) = \begin{pmatrix} x_{i2}(t) - x_{i1}(t) & 0 & 0 \\ 0 & x_{i1}(t) & 0 \\ 0 & 0 & -x_{i3}(t) \end{pmatrix},$$

$$g_1(y_i(t)) = \begin{pmatrix} 1.25 & -3.2 & -3.2 \\ -3.2 & 1.1 & -4.4 \\ -3.2 & 4.4 & 1.0 \end{pmatrix}\begin{pmatrix} h(y_{i1}(t)) \\ h(y_{i2}(t)) \\ h(y_{i3}(t)) \end{pmatrix}, g_2(y_i(t)) = \begin{pmatrix} y_{i1}(t) & 0 & 0 \\ 0 & y_{i2}(t) & 0 \\ 0 & 0 & y_{i3}(t) \end{pmatrix},$$

where $h(s) = (|s+1| - |s-1|)/2$, $i = 1, \cdots, 6$. Suppose $\alpha = (10, 28, 8/3)^{\top}$ and $\beta = -(1,1,1)^{\top}$ are the true values of the unknown system parameters, and the initial estimate of the system parameters $\bar{\alpha}(0)$ and $\bar{\beta}(0)$ are selected randomly in the interval $[0, 1]$. Take the time delays $\tau_x(t) = 0.1$ and $\tau_y(t) = 0.2$, the disturbances $\Delta_x(t) = (\cos t, 2\sin t, 3\sin t)^{\top}$ and $\Delta_y(t) = (-3\cos t, \sin t, -2\sin t)^{\top}$. By using MATLAB software, we perform the following simulations by using MATLAB software.

Numerical simulations for Theorem 1

Set the elements of the connection matrix $\bar{C} = (\bar{c}_{ij})_{N\times N}$ of the system (2) randomly in the interval $[-10, 10]$, three figures are plotted to show the validity of Theorem 1 and its corollaries.

The first simulation is presented to show outer synchronization of the system (1)-(2). As indicated in Figure C1, the synchronization errors of the six pairs of nodes converge to zero, which implies that outer synchronization is achieved. Then, we present another simulation to illustrate the effectiveness of the parameter update laws in Figure C2. Figure C3 shows the convergence of the parameter update laws. It can be seen that the time evolutions of the parameter update laws (3) are in good agreement with the true value of the unknown system parameter.

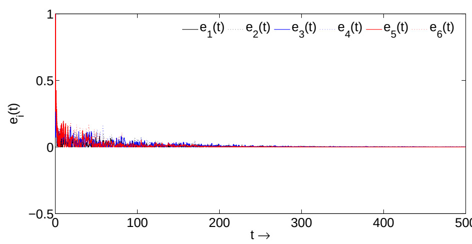

Figure C1 Time evolutions of the synchronization errors $e_i(t)$ of the system (1)-(2), where $e_i(t) = ||e_{xi}(t)|| + ||e_{yi}(t)||$.

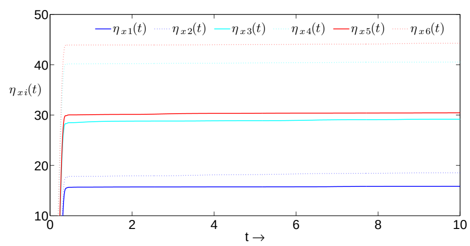

Figure C2 Time evolutions of the functions $\eta_{xi}(t)$ in the adaptive controllers (A3).

Figure C3 indicates that the adaptation rates of the parameter update laws (3) are very slow, and this could be due to the poor selection of the true values of the unknown system parameters α and β. For example, choose $\alpha = (100, 280, 80/3)^{\top}$

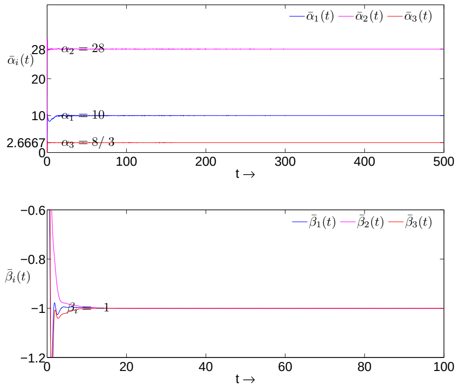

Figure C3 Parameter identification of the system (1)-(2) by choosing $\alpha = (10, 28, 8/3)^\top$ and $\beta = -(1, 1, 1)^\top$. The parameter update laws (3) converge to the true values of the unknown system parameters slowly.

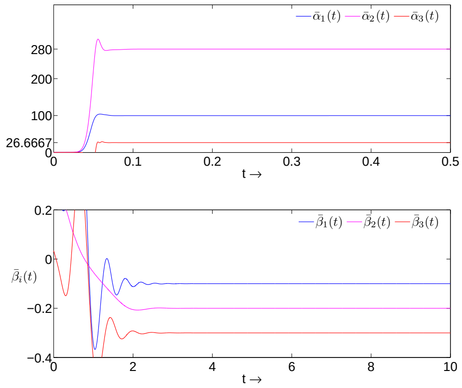

Figure C4 Parameter identification of the system (1)-(2) by choosing $\alpha = (100, 280, 80/3)^\top$ and $\beta = -(0.1, 0.2, 0.3)^\top$. The adaptation rates of the parameter update laws (3) have been promoted significantly.

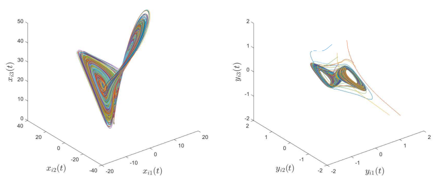

Figure C5 The trajectories of the 6 coupled dynamical nodes. Each node consists of a Lorenz oscillator and a cellular neural network.

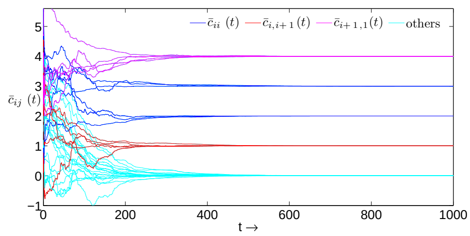

Figure C6 Topology identification of the system (1)-(B1) with the adaptive controllers (B3): $\lim_{t\to\infty} \bar{c}_{ij}(t) = c_{ij}$.

and $\beta = -(0.1, 0.2, 0.3)^\top$, Figure C4 is plotted. As can be seen, the adaptation rates have been promoted significantly, and outer synchronization is attained after a much shorter transient period than the one in Figure C3.

In order to take a clearer view of the synchronized dynamics of $x_i(t)$ and $y_i(t)$, we also depict their sub-variables in Figure C5, and this further illustrates the effectiveness of the proposed method.

Compared to some existing results, our simulation object is a cross coupled network composed of two different kinds of nodes. Under the proposed control strategy, the synchronization errors converge asymptotically to zero, which implies that the drive network (1) and the response network (2) have achieved outer synchronization. However, the selection of the true values of the unknown system parameters will lead to different rates towards outer synchronization. And this may be due to the cross couplings of the network model.

Numerical simulations for Theorem 2

This subsection considers the system (1)-(B1) with the parameter update laws (3) and the adaptive controllers (B3). Analogously to Figure C1-C2, the validity of the adaptive controllers (B3) and the asymptotic stability of the synchronization errors can be verified. The remainder is to show the effectiveness of the topology update laws. For clarity, the elements of the connection matrix C are set as follows, $c_{ii} = 2$ or $c_{ii} = 3$, $c_{i,i+1} = 1$, $c_{i+1,i} = 4$, $c_{ij} = 0$, where $1 \leqslant i, i+1 \leqslant 6$, $|j - i| \geqslant 2$. Time evolutions of $\bar{c}_{ij}(t)$ are shown in Figure C6, which illustrates the effectiveness of the topology update laws.

References

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