SCIENCE CHINA
Information Sciences

• LETTER •

September 2020, Vol. 63 199204:1–199204:3
https://doi.org/10.1007/s11432-018-9888-0

Robust variable normalization least mean *p*-power algorithm

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Received 12 November 2018/Revised 25 January 2019/Accepted 29 April 2019/Published online 26 March 2020

Citation Xiong K, Zhang Y, Wang S Y. Robust variable normalization least mean *p*-power algorithm. Sci China Inf Sci, 2020, 63(9): 199204, https://doi.org/10.1007/s11432-018-9888-0

Dear editor,

Adaptive filtering algorithms have been widely applied in system identification, channel equalization, and echo cancellation over the past decades [1–3]. Generally, adaptive filtering algorithms can be generalized by the least mean square (LMS)-based algorithms. According to the shape of the error nonlinearities in the weight update, the LMS-based algorithms are therefore divided into the V-shaped, Λ-shaped, and M-shaped algorithms [4]. However, none of these three types of algorithms can significantly improve filtering performance for both Gaussian and non-Gaussian noises simultaneously.

To this end, we present a novel robust variable normalization least mean *p*-power (VNLMP) algorithm using the variable normalization related to the norm of the input and the power of the error for the least mean *p*-power (LMP) algorithm [5], which can adaptively change between the V-shaped and M-shaped algorithms. Actually, VNLMP uses high-order and low-order moment information of the estimation error simultaneously thanks to its variable normalization, thus leading to the filtering performance improvement in both Gaussian and non-Gaussian noises.

Problem formulation. For the adaptive filtering, we consider the following system identification model [1, 3]:

$$d(i) = \boldsymbol{w}_o^{\mathrm{T}} \boldsymbol{u}(i) + v(i), \tag{1}$$

where $d(i)$ is the desired response; $\boldsymbol{w}_o = [w_{o,1}, w_{o,2}, \ldots, w_{o,N}]^{\mathrm{T}}$ is the unknown optimal weight vector in the finite impulse response (FIR) system of length N with $(\cdot)^{\mathrm{T}}$ being the transpose; $\boldsymbol{u}(i) = [u_i, u_{i-1}, \ldots, u_{i-N+1}]^{\mathrm{T}}$ is the input vector at discrete time i; $v(i)$ is the disturbance noise. According to (1), the estimation error is therefore defined by

$$e(i) = d(i) - \boldsymbol{w}^{\mathrm{T}}(i)\boldsymbol{u}(i), \tag{2}$$

where $\boldsymbol{w}(i) = [w_{i,1}, w_{i,2}, \ldots, w_{i,N}]^{\mathrm{T}}$ is the weight vector of the adaptive filter.

The weight update in the LMS-based algorithm [4] with error nonlinearity is generalized by

$$\boldsymbol{w}(i+1) = \boldsymbol{w}(i) + \mu f(e(i))e(i)\boldsymbol{u}(i), \tag{3}$$

where $\mu > 0$ is the step-size and $f(e(i))$ is a nonlinear even function of $e(i)$.

According to the shape of $f(e(i))$, the existing LMS-based algorithms can be generalized by the V-shaped, Λ-shaped, and M-shaped algorithms [4]. Because the V-shaped algorithms cannot combat impulsive noise, and the Λ-shaped and M-shaped algorithms generate slow convergence rate in the absence of impulsive noise, all these three types of algorithms cannot significantly improve filtering performance for both Gaussian and non-Gaussian noises simultaneously. Thus, we propose a novel VNLMP algorithm to address these issues in the following.

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Proposed algorithm. Inspired by the LMS-based algorithm of which the shape of the error nonlinearity is related to the power of the estimation error, we design a novel variable normalization for adaptive change of the shape of the error nonlinearity in LMP [5] to combat both Gaussian and non-Gaussian noises efficiently. First, the normalized term of LMP [5] in the weight update is considered using the product of the qth moment of the norm of the input and the $(p-q)$th moment of the error, i.e.,

$$\boldsymbol{w}(i+1)=\boldsymbol{w}(i)+\mu\frac{|e(i)|^{p-1}\mathrm{sign}[e(i)]}{\alpha+\|\boldsymbol{u}(i)\|^{q}|e(i)|^{p-q}}\boldsymbol{u}(i), \tag{4}$$

where $p>0$, $\mu>0$ is the step-size, and α is a small positive constant. It has been proved that the error nonlinear function is optimal regarding the steady-state mean square error when the order of the error in the denominator of the weight update is one order larger than that of the numerator [6]. Therefore, $q\in[0,2]$ is chosen to ensure the stability and filtering accuracy of (4), which is also explained in [7,8]. According to (3), the error nonlinear function of (4) is rewritten as

$$f(e(i))=\frac{|e(i)|^{p-2}}{\alpha+\|\boldsymbol{u}(i)\|^{q}|e(i)|^{p-q}}. \tag{5}$$

The input signal is assumed to be a stationary sequence of independent zero-mean Gaussian random variable with a finite variance σ_u^2 and dimension N. By setting $\sigma_u^2=0.01$, $N=32$, $p=4$, and $\alpha=1$, we plot the error surface of (5) in Figure 1, where $\boldsymbol{u}(i)$ is obtained as an average over 100 independent runs. As can be seen from Figure 1, Eq. (4) is a V-shaped algorithm when $q=2$ and an M-shaped algorithm otherwise.

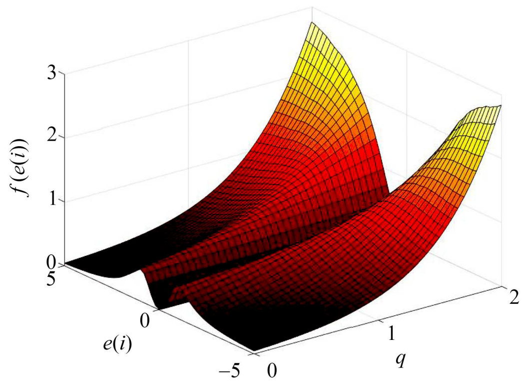

Figure 1 (Color online) Surface of the error nonlinear function (5).

It is worth mentioning that Eq. (4) with a fixed q can only be used in a specific noise environment. Thus, a variable order of error in (4) is used to combine the characteristics of both V-shaped and M-shaped algorithms. Generally, different noises require different shapes of error nonlinear functions. For example, the V-shaped algorithms used in the presence of Gaussian and sub-Gaussian noises cannot combat impulsive noise, and the M-shaped algorithms used in the presence of impulsive noise cannot improve filtering accuracy for Gaussian and sub-Gaussian noises. Therefore, $q\in[0,2]$ in (4) is required for different noises.

In Figure 1, a large error requires a small q for guaranteeing the convergence of the algorithm, and a small error requires a large q to improve the steady-state performance. Thus, to obtain variable $q(i)$, we need to design a monotone decreasing function with respect to $e(i)$ with upper and lower bounds of 2 and 0, respectively. Motivated by the "S" shape function of sigmoid function [9], we define a sigmoid function as follows:

$$\mathrm{sgm}\,[e(i)]=\frac{1}{1+\exp\left(-\beta|e(i)|^{p}\right)}, \tag{6}$$

where parameter $p>0$ is the same as that in (4), and $\beta>0$ is the steepness parameter which controls the steepness of the sigmoid function curve.

The sigmoid function (6) is a symmetric function regarding the origin and has the maximum of 1 and only a global minimum of $\mathrm{sgm}[0]=0.5$. Thus, using the sigmoid function (6), we propose the variable method for $q(i)$ as follows:

$$\begin{aligned}q(i)&=4\left(1-\mathrm{sgm}\,[e(i)]\right)\\&=4-\frac{4}{1+\exp\left(-\beta|e(i)|^{p}\right)}\in[0,2].\end{aligned} \tag{7}$$

In (7), a larger β results in a steeper sigmoid curve, which can be used in impulsive noise, while Eq. (4) requires a smaller q to combat such noise. On the contrary, a smaller β can be used in non-impulsive noise to smooth the steady-state performance. In addition, $q(i)$ with a larger p has a steeper curve at moderate errors and a smoother curve at smaller errors. Otherwise, $q(i)$ has a contrary curve. Therefore, $q(i)$ with different p can be applied in different noise environments.

Finally, replacing q in (4) with $q(i)$ in (7) gives the proposed novel robust VNLMP algorithm, i.e.,

$$\boldsymbol{w}(i+1)=\boldsymbol{w}(i)+\mu\frac{|e(i)|^{p-1}\mathrm{sign}[e(i)]}{\alpha+\|\boldsymbol{u}(i)\|^{q(i)}|e(i)|^{p-q(i)}}\boldsymbol{u}(i). \tag{8}$$

Remark 1. According to (7) used in the normalization, VNLMP can automatically switch between the V-shaped and M-shaped algorithms. When $e(i)$ is very large, i.e., a large $|e(i)|$, we have that $q(i) \to 0$ from (7) and VNLMP in (8) can be viewed as an M-shaped algorithm. This means that VNLMP can combat various outliers including impulsive noise efficiently. When $e(i)$ is very small, i.e., $|e(i)| \to 0$, $q(i) \to 2$ is obtained from (7) and VNLMP reduces to a V-shaped algorithm. Therefore, VNLMP can improve the steady-state performance in the presence of both Gaussian and non-Gaussian noises simultaneously.

Remark 2. In the absence of impulsive noise, the error shown in (7) is usually small, and thus a small $q(i)$ is required in (8) to improve the convergence performance. Note that $q(i)$ with $p = 2$ has a smooth curve when the error is not large enough, which leads to the desirable performance. Thus, in the non-impulsive noise environments, we reasonably choose $p = 2$ in (7).

Remark 3. In the VNLMP algorithm shown in (8), there exist four parameters, i.e., error power p, positive constant α, steepness parameter β, and step-size μ. We generally choose p as a positive integer for different applications [4, 7]. As α is used to avoid the normalization of VNLMP being 0, we can set it as a small positive number. According to Figure 1, we see that β can affect the switching rate of VNLMP between the V-shaped and M-shaped algorithms. Specifically, a lager β leads to a more probable M-shaped algorithm, while a smaller β leads to a more probable V-shaped algorithm. Therefore, steepness parameter β together with step-size μ can achieve a trade-off between the transient and steady-state filtering performance in different noise environments. Because the V-shaped algorithm can smooth a small error and the M-shaped algorithm can combat a large error, a relatively small β is set for non-impulsive noise and a relatively large β for impulsive noise generally.

In Appendix A, the steady-state performance of VNLMP in terms of excess mean-square error (EMSE) and mean square deviation (MSD) is derived with a white Gaussian reference for theoretical analysis. In Appendix B, simulations in the context of system identification confirm the theoretical results and illustrate the superiorities of VNLMP over other typical algorithms for both Gaussian and non-Gaussian noises.

Conclusion. VNLMP is presented using the variable product of the norm of the input and the power of the estimation error as the normalization. VNLMP can adaptively switch between the V-shaped and M-shaped algorithms, and thus utilize the higher-order and lower-order information of the estimation error simultaneously. Therefore, VNLMP implements filtering performance improvement for Gaussian and non-Gaussian noises simultaneously. The analytical results for theoretical analysis of VNLMP are supported by simulations. Simulation results in the context of system identification show that VNLMP outperforms the normalized least mean square (NLMS) algorithm and the normalized least mean 2Lth (NLM2L) algorithm for Gaussian noise. For non-Gaussian noise, VNLMP has comparable performance to NLM2L for uniform noise, and outperforms the generalized maximum correntropy criterion (GMCC) algorithm for impulsive noise.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 61671389), Fundamental Research Funds for the Central Universities (Grant No. XDJK2019B011), and Chongqing Industrial Control System Information Security Technology Support Center.

Supporting information Appendixes A and B. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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