SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

September 2020, Vol. 63 192501:1–192501:12
https://doi.org/10.1007/s11432-019-2706-9

Quantum algorithms of state estimators in classical control systems

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Received 29 June 2019/Revised 14 September 2019/Accepted 24 October 2019/Published online 13 August 2020

Abstract In this paper, quantum algorithms are applied to the design of state estimators in classical control systems under the condition that quantum algorithms can be physically implemented. We demonstrate that the design of state estimators can be solved by quantum algorithms, which may achieve significant acceleration in comparison to traditional classical algorithms. The time complexity can be reduced from $O(n^6)$ to $O(qn)$ when the system matrix is sparse and the condition number κ and the reciprocal of precision ϵ are small in size $O(\text{poly}\log(n))$, where n is the dimension of state $x(t)$ and q is the dimension of input $u(t)$. Our research will provide an entire quantum scheme of constructing state estimators and can be regarded as an attempt to widen application scope of quantum computation.

Keywords quantum computation, estimators for linear systems, optimization algorithms, quantum acceleration, time complexity

Citation Li K, Dai H Y, Zhang M. Quantum algorithms of state estimators in classical control systems. Sci China Inf Sci, 2020, 63(9): 192501, https://doi.org/10.1007/s11432-019-2706-9

1 Introduction

Quantum computation [1] is a new computing model which obeys the rules of quantum mechanics to control the quantum information unit. Quantum computation is distinguished from binary classic calculation based on the transistor. The general digital computing needs to be encoded into binary data, namely the minimum data unit is always one of two state (0 or 1). However, the quantum computation uses qubits that are in quantum superposition. The main research goal of quantum computation is to break the limit of the traditional computer calculation. Quantum computing possesses bigger data processing capacity beyond the classic computer based on the characteristics of coherent superposition and parallel processing [2]. In 1990s, two most famous quantum algorithms, Shor's polynomial-time quantum algorithm [3, 4] for factoring and Grover's quantum algorithm [5] for searching a database in time square-root its size, have been proposed successively and show the power of quantum computation. Harrow, Hassidim, and Lloyd [6] proposed the quantum algorithm for solving linear equations (HHL algorithm) in 2009, which is a remarkable development of quantum algorithms. The HHL algorithm is used to get the solution to a given linear system of equations and can achieve exponential acceleration in comparison to traditional classical algorithms.

Quantum computation has advantages for solving the problems with heavy calculation burden, such as large-scale computational problems [7]. Previous quantum algorithms [6–8] focus more on solving a few single mathematical problems. Sun and Zhang [8] proposed a quantum algorithm for solving Lyapunov

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equation based on [6]. Sun and other coauthors [9] have also demonstrated that the optimal control protocols can be exponentially accelerated by quantum algorithms. Shao [10] proposed the quantum algorithms of matrix multiplication in 2018, which is of great significance as the basis of matrices computing.

With the rapid progress in quantum computing, we should explore more new potential application of quantum algorithms. In this research, we intend to develop an entire quantum scheme of constructing state estimators based on the major building blocks of quantum algorithms.

In the classical control system [11–13], many control problems often involve state variables, for example, using state feedback to realize the pole assignment of the closed-loop system [14]. However, some state variables of classical systems are not easily measured by physical methods, and some intermediate variables do not necessarily have conventional physical meaning that they cannot be measured. In this case, it is necessary to construct state estimator, also known as state observer [15]. State estimator is a kind of dynamic systems based on the measured values of external variables (input variables and output variables). In the early 1960s, Luenberger [16] put forward the concept and construction method of the state estimator in order to realize the state feedback control and other needs. The presence of the state estimator provides a real possibility for the implementation of the state feedback technique, and also has many practical applications in control engineering, such as suppressing the disturbance. When states of an object cannot be measured directly and the state feedback design [17, 18] is to be performed, we need to construct state estimators. It should be underlined that state estimators can be designed by Lyapunov equation methods [12].

However, how to apply quantum algorithms to actual classical control problems has not been fully studied so far. Many classical problems cannot be quantified, or these problems require several quantum algorithms to work together. However, it is difficult to connect different quantum algorithms because these algorithms are independently proposed in general. In this paper, we aim to make full use of the advantage of quantum algorithms to solve classical control problems. In classical control system, the design of state estimator is reduced to solving Sylvester or Lyapunov equations for matrices. Recognizing that Lyapunov equations in [12] can be regarded as a special case of linear equations, and the linear equations can be solved by quantum algorithms [6], we propose an alternative method for classical control problem [8] based on quantum algorithms and complete the design of the state estimators. Our research may provide a useful method for solving the classical control problem by quantum algorithms.

The rest of our paper are organized as follows. Section 2 focuses on how to use these quantum acceleration algorithms to construct state estimators. We propose an integral operation flow of constructing the state estimator based on quantum algorithms. In Section 3, we give a numerical example of constructing state estimators by quantum algorithms. Section 4 summarizes the work of this paper.

2 Construct state estimators by quantum acceleration algorithms

2.1 Classical control problem

If we want to design a state feedback, and states of the object cannot be measured, we need to estimate the state variables. Consider the following nth-order linear time-invariant system Σ_1:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{1}$$

where A is an $n \times n$ square matrix, B is an $n \times q$ matrix and C is a $p \times n$ matrix.

The state estimator Σ_2 is designed as follows:

$$\begin{cases} \dot{z}(t) = Fz(t) + Lu(t) + Gy(t), \\ \hat{x}(t) = T^{-1}z(t), \end{cases} \tag{2}$$

where F is an $n \times n$ matrix, L is an $n \times p$ matrix, G is an $n \times q$ matrix and T is an $n \times n$ matrix. Figure 1 shows the system structure diagram. $\hat{x}(t)$ is the estimate of the Σ_1's state variable $x(t)$. These matrices F, L, G, and T are the design objects of the state estimator.

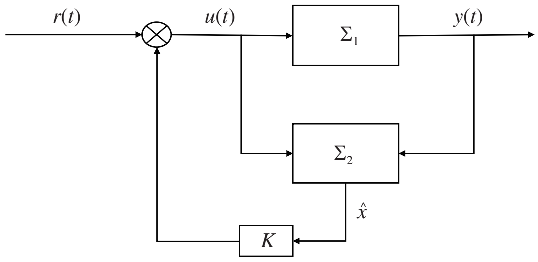

Figure 1 The system diagram about state estimator. Σ_1 is the original linear system, Σ_2 is the state estimator to be designed, and state feedback K is the coefficient matrix.

The design goal of the state estimator is to make the state estimator's output $\hat{x}(t)$ as close as possible to $x(t)$ to achieve the state estimation, so we need to find the minimum value of $\hat{x}(t) - x(t)$.

$$\dot{e}(t) = \dot{z}(t) - T\dot{x}(t) = Fe(t) + (FT + GC - TA)x(t) + (L - TB)u(t). \tag{3}$$

In order to make the steady-state error $e(t)$ close to zero, it is necessary to satisfy $FT + GC - TA = 0, L - TB = 0$ and the eigenvalues of F are all negative. The state estimator design is divided into the following steps.

(1) Choose an $n \times n$ matrix F to satisfy $\mathrm{Re}\lambda(F) < 0$, which is a necessary condition for that the steady state error of the state estimator approaches zero and the system is stable. If $\lambda(F) \cap \lambda(A) = \emptyset$, $FT - TA = -GC$ has a unique solution.

(2) Select an arbitrary $n \times q$ G so that (F, G) is controllable.

(3) Solve the unique T in the Lyapunov equation $FT - TA = -GC$. If T is nonsingular, the program goes to step (4), otherwise goes back to step (2).

(4) Calculate the matrix $L = TB$.

The above procedure guarantees that the state estimator Σ_2 can estimate the state variables x in classical control systems, which is a proven conclusion in linear system theory [15–17]. Constructing the state estimator needs to solve matrix equations about unknown matrices F, L, G, and T, which may cost much computing resource by classical algorithms. Many studies have shown that quantum algorithms can achieve significant acceleration in comparison to traditional classical algorithms. Therefore, we propose an entire scheme to construct state estimators based on quantum algorithms.

2.2 Quantum algorithms

Using quantum algorithms to construct state estimators of classical control system is divided into the following steps.

(1) Choose an $n \times n$ matrix F to satisfy $\mathrm{Re}\lambda(F) < 0$ and $\lambda(F) \cap \lambda(A) = \emptyset$. Then select an arbitrary $n \times q$ G so that (F, G) is controllable. We select randomly $\lambda(F)$ that satisfies $\mathrm{Re}\lambda(F) < 0$, then $P(\lambda(F) \cap \lambda(A) = \emptyset) = 1$. And we construct matrices F and G that F is diagonal and G is controllable canonical form, which can ensure (F, G) is controllable. The time complexity of selecting F and G under the above rules is 0.

(2) Solve the unique T in the Lyapunov equation $FT - TA = -GC$ by HHL algorithms. We know that T is nonsingular in a high probability. Therefore, we think it is rational to suppose that T is nonsingular in our scheme.

(3) Calculate the $L = TB$ by the quantum algorithms of matrix multiplication [10].

We presente an entire scheme (Figure 2) of constructing the state estimator based on some existing quantum algorithms, including the quantum algorithms of linear equation [6], Lyapunov equation [8] and matrix multiplication [10]. In our scheme, the step of solving $H|r\rangle = |v\rangle$ adopts quantum algorithms in [6, 8], and the step of calculating $|L\rangle = |TB\rangle$ adopts the algorithm of matrix multiplication in [10]. The detailed process is shown as follows.

The Lyapunov equation is given as follows:

$$FT - TA = -GC, \tag{4}$$

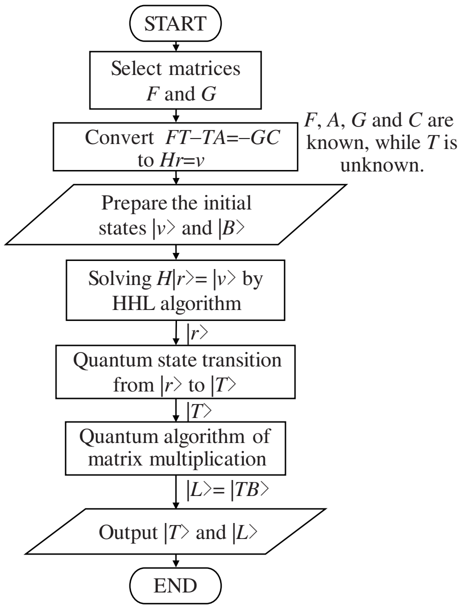

Figure 2 The flow chart of quantum algorithms of constructing state estimators. Information of the matrices T and L is stored in quantum registers.

where $A_{n\times n}$, $F_{n\times n}$, $G_{n\times m}$ and $C_{m\times n}$ are known matrices, while $T_{n\times n}$ is an unknown matrix.

It should be underlined that Eq. (4) for matrix T can be transformed into the form of a linear equation for vector r:

$$Hr = v, \tag{5}$$

where $H = A^{\mathrm{T}} \otimes I_n - I_n \otimes F$ (the selection of F can guarantee the non-singularity of H), $r = \mathrm{vec}(T)$ and $v = \mathrm{vec}(GC)$.

$W \otimes S$ denotes the Kronecker product of matrices W and S, which is expressed as

$$W \otimes S = \begin{bmatrix} w_{1,1}S & w_{1,2}S & \cdots & w_{1,n}S \\ w_{2,1}S & w_{2,2}S & \cdots & w_{2,n}S \\ \vdots & \vdots & \vdots & \vdots \\ w_{n,1}S & w_{n,2}S & \cdots & w_{n,n}S \end{bmatrix}, \tag{6}$$

and define $\mathrm{vec}(S) = [\, s_1^{\mathrm{T}} \; s_2^{\mathrm{T}} \; \cdots \; s_n^{\mathrm{T}} \,]^{\mathrm{T}}$, where s_i are the ith column vectors of the matrix S.

When the matrix H is sparse, we can solve the linear equation (5) by using HHL algorithm:

$$\begin{aligned} |r\rangle &= H^{-1}|v\rangle \\ &= H^{-1}\sum_{i=1}^{n^2}\beta_i|\mu_i\rangle \\ &= \sum_{i=1}^{n^2}\beta_i H^{-1}|\mu_i\rangle \\ &= \sum_{i=1}^{n^2}\beta_i \frac{1}{\lambda_i}|\mu_i\rangle, \end{aligned} \tag{7}$$

where λ_i is the eigenvalue of the matrix H, μ_i is its eigenvector.

Note: Ket $|\cdot\rangle$ and bra $\langle\cdot|$ are the Dirac notations, and represent the state vector and its conjugate vector, respectively. Readers can further refer to [1].

The complete algorithm of constructing the state estimator is as follows.

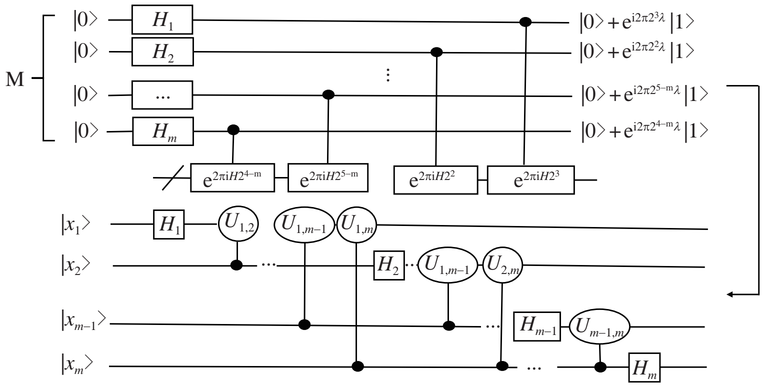

Figure 3 Quantum circuit for phase estimation. There are two registers in the circuit, one for storing state $|v\rangle$ and the other named M for storing the eigenvalues. Initialize M to $|0\rangle^{\otimes m}$. And the bottom half of the picture is a Fourier transform. The x_i is the representation of qubits of eigenvalue. Accuracy is reserved to four decimal places.

(1) Construct the quantum state $|v\rangle = \frac{1}{||v||_2}\sum_{i=1}^{n^2} v_i|i\rangle$, where v_i is the ith element of the vector v. The state of the quantum system at this moment is

$$|v\rangle = \sum_{i=1}^{n^2} \beta_i|\mu_i\rangle, \tag{8}$$

where $|\mu_i\rangle$ is the eigenvector basis of the matrix H.

(2) The operation of quantum phase estimation [19] is performed to obtain the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n^2}$ of the matrix H. The quantum circuit [1] for phase estimation is given in Figure 3. After performing the operation of phase estimation, the state of the quantum system is translated into

$$\sum_{i=1}^{n^2} \beta_i|\lambda_i\rangle|\mu_i\rangle, \tag{9}$$

where H_i is a Hadamard gate acting on the ith qubit in quantum circuit of Figure 3, $H_i = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. And $U_{i,j}$ is a controlled gate,

$$U_{i,j} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\theta_{i,j}} \end{bmatrix},$$

where the phase-shift $\theta_{i,j} = 2\pi x_j/2^{j-i+1}$. Only when the jth quantum bit is $|1\rangle$, $U_{i,j}$ will act on the ith quantum bit. The state $|\lambda_i\rangle = |x_1\rangle_1|x_2\rangle_2\cdots|\lambda_m\rangle_m$, and the eigenvalue $\lambda = x_1 \times 2^{m-5} + x_2 \times 2^{m-6} + \cdots + x_m \times 2^{-4}$, because the accuracy of the eigenvalue is reserved to four decimal places.

(3) We perform controlled rotation on an auxiliary qubit $|0\rangle$ based on the eigenvalue information λ_i obtained through (9).

$$\sum_{i=1}^{n^2} \beta_i \left(\sqrt{1 - C^2/\lambda_i^2}|0\rangle + C/\lambda_i|1\rangle\right)|\lambda_i\rangle|\mu_i\rangle. \tag{10}$$

(4) We perform the inverse operation of step (2) in the quantum system, the eigenvalue information $|\lambda_i\rangle$ is converted to $|0\rangle^{\otimes m}$. Then the system state is transformed into

$$\sum_{i=1}^{n^2} \beta_i \left(\sqrt{1 - C^2/\lambda_i^2}|0\rangle + C/\lambda_i|1\rangle\right)|\mu_i\rangle. \tag{11}$$

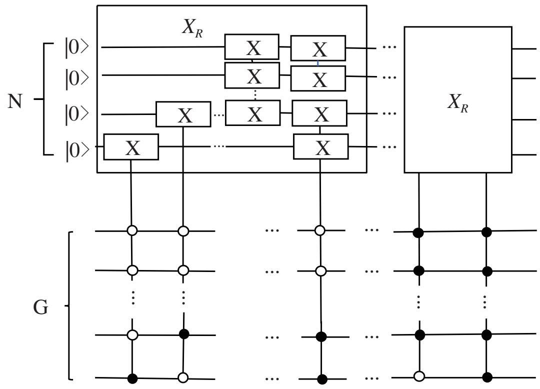

Figure 4 Quantum circuit for the quantum state transition. Convert the output of the HHL algorithm to the input of the quantum algorithms of matrix multiplication. The position information of the quantum state $|x\rangle$ is stored in the register G, which controls the number of rows of the quantum state $|T\rangle$ in the register N. Where X is the NOT gate.

(5) Then we measure the ancillary qubit. When the measurement result is $|0\rangle$, it is regarded as a failure. Repeat the above steps until we get auxiliary qubit $|1\rangle$. And when the measurement result is $|1\rangle$, the quantum system state will collapse to

$$|r\rangle = \sum_{i=1}^{n^2} \beta_i \frac{1}{\lambda_i} |\mu_i\rangle = \frac{1}{\|T\|_F} \sum_{i=1}^{n^2} r_i |i\rangle. \tag{12}$$

Eq. (12) is the solution of the linear equation (5).

(6) Obtain the state $|T\rangle$ by performing controlled rotation on the auxiliary qubits of the register N as shown in Figure 4.

$$|r\rangle = \frac{1}{\|T\|_F} \sum_{i=1}^{n^2} r_i |i\rangle \mapsto |T\rangle = \frac{1}{\|T\|_F} \sum_{i=1}^{n} \sum_{j=1}^{n} t_{i,j} |i,j\rangle = \frac{1}{\|T\|_F} \sum_{i=1}^{n} \|T_{i\cdot}\|_2 |i\rangle |T_{i\cdot}\rangle. \tag{13}$$

The quantum state of the register N is

$$|T_{F\cdot}\rangle = \frac{1}{\|T\|_F} \sum_{i=1}^{n} \|T_{i\cdot}\|_2 |i\rangle. \tag{14}$$

Note: $L_{ij} = T_{i\cdot}^{\mathrm{T}} B_{\cdot j} = \|T_{i\cdot}\|_2 \|B_{\cdot j}\|_2 \langle T_{i\cdot}|B_{\cdot j}\rangle$, where $|T_{i\cdot}\rangle = \frac{1}{\|T_{i\cdot}\|_2} \sum_{j=1}^{n} t_{ij}|j\rangle|$, $|B_{\cdot j}\rangle = \frac{1}{\|B_{\cdot j}\|_2} \sum_{i=1}^{n} b_{ij}|i\rangle$. By quantum algorithms of matrix multiplication, we can estimate $\langle T_{i\cdot}|B_{\cdot j}\rangle$ efficiently.

(7) Prepare the state $|B\rangle$:

$$\begin{aligned} |B\rangle &= \frac{1}{\|B\|_F} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{i,j} |i,j\rangle \\ &= \frac{1}{\|B\|_F} \sum_{j=1}^{n} \|B_{\cdot j}\|_2 |j\rangle |B_{\cdot j}\rangle, \\ |B_{\cdot F}\rangle &= \frac{1}{\|B\|_F} \sum_{j=1}^{n} \|B_{\cdot j}\|_2 |j\rangle, \end{aligned} \tag{15}$$

where $\|T\|_F = \sqrt{\Sigma_{ij} |t_{ij}|^2}$ is the Frobenius norm of T. Therefore, we have

$$|T_{F\cdot}\rangle \otimes |B_{\cdot F}\rangle \otimes |0,0\rangle = \frac{1}{\|T\|_F \|B\|_F} \sum_{i,j=1}^{n} \|T_{i\cdot}\|_2 \|B_{\cdot j}\|_2 |i,j\rangle |0,0\rangle. \tag{16}$$

(8) By control transformation, we can prepare $|T_{i\cdot}\rangle$ and $|B_{\cdot j}\rangle$ in the last register:

$$\frac{1}{\|T\|_F \|B\|_F} \sum_{i,j=1}^{n} \|T_{i\cdot}\|_2 \|B_{\cdot j}\|_2 |i,j\rangle \otimes \frac{1}{\sqrt{2}} (|0\rangle |T_{i\cdot}\rangle + |1\rangle |B_{\cdot j}\rangle). \tag{17}$$

Table 1 The time complexity comparison between classical algorithms and our scheme

Step	Time complexity of classical algorithms	Our scheme when quantum states can be efficiently prepared	Our scheme when quantum states cannot be efficiently prepared
Select F and G	0	0	0
Preparing states $\|\nu\rangle$ and $\|B\rangle$	/	$O(\log(n)/\epsilon^2)$ [10]	$O(n^2)$ [21]
$H\|r\rangle = \|\nu\rangle$	$O(n^6)$	$O(\kappa_1^2 \log(n)/\epsilon)$ [6,8]	$O(\kappa_1^2 \log(n)/\epsilon)$ [6,8]
$\|L\rangle = \|TB\rangle$	$O(n^2q)$	$O(\kappa_2\sqrt{n}/\epsilon + nq)$ [10]	$O(\kappa_2\sqrt{n}/\epsilon + nq)$ [10]
Sum	$O(n^6)$	$O(\kappa_2\sqrt{n}/\epsilon + nq)$	$O(\kappa_2\sqrt{n}/\epsilon + n^2)$

(9) Run program of matrix multiplication in [10], we can get the desired state:

$$|L\rangle = \frac{1}{\|T\|_F\|B\|_F}\sum_{i,j=1}^{n} \|T_{i\cdot}\|_2\|B_{\cdot j}\|_2\langle T_{i\cdot}|B_{\cdot j}\rangle|i,j\rangle|0\rangle + |0\rangle^{\perp}. \tag{18}$$

(10) By measuring quantum states $|T\rangle$ and $|L\rangle$, we may get matrices T and L. Therefore, we get the state estimator $\sum_2$.

2.3 Algorithm complexity analysis

Then we can estimate the final complexity of our scheme, which consists of three parts: the prepared of quantum states, HHL algorithm and the quantum algorithms of matrix multiplication.

(1) The time complexity of preparing quantum states $|v\rangle$ and $|B\rangle$. From [10], we know, for any vector, its quantum state can be prepared in time $O(\log(n)/\epsilon^2)$, where ϵ is the precision.

(2) The time complexity of HHL algorithm. When the Hermitian matrix H of (5) is sparse, the quantum algorithm can achieve exponential acceleration compared with traditional classical algorithms [6]. When the matrices A and F are sparse, $H = A^{\mathrm{T}} \otimes I_n - I_n \otimes F$ is sparse, and the HHL algorithm can be efficiently simulated in time $O({\kappa_1}^2\log(n)/\epsilon)$ [6, 8, 20], where $\kappa_1 = \lambda_{\max}/\lambda_{\min}$ is the condition number of the matrix H.

(3) The time complexity of matrix multiplication. The complexity of quantum algorithms of matrix multiplication is $O(\|T\|_F\|B\|_F/\epsilon\|TB\|_F) = O(\sqrt{n}\kappa_2/\epsilon)$, where κ_2 is the condition number of T. Besides, the norms of $|B_{\cdot j}\rangle$ can be evaluated by the classical method $(1 \leqslant j \leqslant q)$, which costs $O(nq)$ [10].

Remark: In fact, HHL algorithm requires the matrix H to be symmetric and sparse. Nevertheless, the symmetrical condition can be satisfied by transforming the linear equation (5) to (19) [6].

$$\begin{bmatrix} 0 & H \\ H^{\dagger} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ r \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix}. \tag{19}$$

where $H^{\dagger}$ is the conjugate transpose of the matrix H. Besides, the precision ϵ means that if s is the exact result and s' is the result getting from quantum algorithms, then $|s - s'| \leqslant \epsilon$.

The paper makes the holistic complexity comparison between our scheme and the classical algorithms. In the process of constructing the state estimator, solving the Lyapunov equation is the key step, and the existing numerical algorithms must run in $O(n^6)$ time generally, which can be reduced to $O({\kappa_1}^2\log(n)/\epsilon)$ time by quantum algorithms. From Table 1, we know that the holistic complexity of classical algorithms is $O(n^6)$, while our scheme just needs $O(qn)$ when κ and $1/\epsilon$ are small in size $O(\text{poly}\log(n))$.

Ref. [21] proposed that when quantum states cannot be efficiently prepared, quantum states may be prepared in $O(n^2)$ time, in which the holistic time complexity can be reduced from $O(n^6)$ to $O(n^2)$ by our scheme.

3 An illustrative example

An illustrative example is given to clarify the process of our algorithms.

Consider a second-order linear time-invariant system Σ_1:

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \\ y(t) = [0 \quad 1]x(t). \end{cases} \tag{20}$$

Choose an 2×2 matrix F to satisfy $\mathrm{Re}\lambda(F) < 0$ and $\lambda(F) \cap \lambda(A) = \emptyset$. Then select an arbitrary 2×1 G so that (F, G) is controllable. We select randomly $F = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$ and $G = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The equations that need to be solved are as follows:

$$\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} T - T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \tag{21}$$

$$L = T \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{22}$$

Eq. (21) for matrix T can be transformed into the form of a linear equation $Hr = v$, where

$$H = A^{\mathrm{T}} \otimes I_n - I_n \otimes F = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix}, \quad v = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

The complete algorithm of constructing the state estimator is as follows.

(1) Construct the quantum state $|v\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |11\rangle)$. The state of the quantum system at this moment is

$$|v\rangle = \sum_{i=1}^{4} \beta_i |\mu_i\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |11\rangle). \tag{23}$$

(2) The operation of quantum phase estimation [19] is performed to obtain the eigenvalues $1, 2, 3, 4$ of matrix H. After performing the operation of phase estimation in Figure 3, the state of the quantum system is translated into

$$\sum_{i=1}^{4} \beta_i |\lambda_i\rangle |\mu_i\rangle. \tag{24}$$

(3) We use the eigenvalue information λ_i of the matrix H to perform controlled rotation on an auxiliary qubit $|0\rangle$.

$$\sum_{i=1}^{4} \beta_i \left(\sqrt{1 - 1/\lambda_i^2} |0\rangle + 1/\lambda_i |1\rangle \right) |\lambda_i\rangle |\mu_i\rangle. \tag{25}$$

(4) We perform the inverse operation of step (2) in the quantum system, the eigenvalue information $|\lambda_i\rangle$ is converted to $|0\rangle^{\otimes m}$. Then the system state is transformed into

$$\sum_{i=1}^{4} \beta_i \left(\sqrt{1 - 1/\lambda_i^2} |0\rangle + 1/\lambda_i |1\rangle \right) |\mu_i\rangle. \tag{26}$$

(5) Then we measure the ancillary qubit. Repeat the above steps until we get auxiliary qubit $|1\rangle$. Then the quantum system state will collapse to

$$|r\rangle = H^{-1}|v\rangle = \sum_{i=1}^{4} \beta_i \frac{1}{\lambda_i} |\mu_i\rangle = \frac{1}{\sqrt{410}}(-8|00\rangle - 3|01\rangle + 16|10\rangle + 9|11\rangle). \tag{27}$$

Eq. (27) is the solution of the linear equation.

(6) Obtain the state $|T\rangle$ by performing controlled rotation on the auxiliary qubits of the register N as shown in Figure 4.

$$|r\rangle = \frac{1}{\sqrt{410}}(-8|00\rangle - 3|01\rangle + 16|10\rangle + 9|11\rangle) \mapsto |T\rangle = \frac{1}{\sqrt{410}}(-8|0,0\rangle + 16|0,1\rangle - 3|1,0\rangle + 9|1,1\rangle). \tag{28}$$

The quantum state of the register N is

$$|T_{F\cdot}\rangle = \frac{1}{\sqrt{410}}(8\sqrt{5}|0\rangle + 3\sqrt{10}|1\rangle). \tag{29}$$

(7) Prepare the state $|B\rangle$:

$$|B\rangle = |B_{\cdot F}\rangle = |0\rangle. \tag{30}$$

Therefore, we have

$$|T_{F\cdot}\rangle \otimes |B_{\cdot F}\rangle \otimes |0,0\rangle = \frac{1}{\sqrt{410}}(8\sqrt{5}|0,0\rangle + 3\sqrt{10}|1,0\rangle)|0,0\rangle. \tag{31}$$

(8) By control transformation, we can prepare $|T_{i\cdot}\rangle$ and $|B\rangle$ in the last register,

$$|T_{F\cdot}\rangle \otimes |B_{\cdot F}\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle|T_{i\cdot}\rangle + |1\rangle|B\rangle). \tag{32}$$

(9) Run program of matrix multiplication in [10], we can get the desired state

$$|L\rangle = \frac{1}{\sqrt{73}}(8|0\rangle + 3|1\rangle)|0\rangle + |0\rangle^{\perp}. \tag{33}$$

(10) By measuring quantum states $|T\rangle$ and $|L\rangle$, we may get matrices

$$T = C_1 \begin{bmatrix} -\frac{8}{\sqrt{410}} & \frac{16}{\sqrt{410}} \\ -\frac{3}{\sqrt{410}} & \frac{9}{\sqrt{410}} \end{bmatrix}, \quad L = C_2 \begin{bmatrix} \frac{8}{\sqrt{73}} \\ \frac{3}{\sqrt{73}} \end{bmatrix}.$$

Substituting T and L into (21) and (22), respectively, we get $C_1 = \frac{\sqrt{410}}{24}$, $C_2 = -\frac{\sqrt{73}}{24}$. So far, the state estimator Σ_2 in (2) is designed as follows:

$$\begin{cases} \dot{z}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} z(t) - \begin{bmatrix} \frac{1}{3} \\ \frac{1}{8} \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} y(t), \\ \hat{x}(t) = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{8} & \frac{3}{8} \end{bmatrix}^{-1} z(t). \end{cases} \tag{34}$$

Remark: The above procedure of our quantum scheme is given based on the assumption that accurate quantum measurement probabilities are available. In the actual experiments, we have to perform quantum operations on a batch of initial quantum states to obtain a set of target quantum states, and get the estimated probabilities. Then there are errors between estimated and real probabilities. Therefore, the impact of the inaccuracy of the quantum measurement result on the state estimator is discussed in Appendix A.

4 Conclusion

In summary, we propose an entire quantum scheme to construct state estimators in classical control systems. Our scheme can reduce the time complexity of constructing state estimator with dimension n

to $O(qn)$ in comparison to the time complexity $O(n^6)$ in classical algorithms when quantum states can be efficiently prepared, the system matrix is sparse, and both the condition number κ and the reciprocal of precision ϵ are small in size $O(\text{poly}\ \log(n))$. And the time complexity of our scheme is $O(n^2)$ when the quantum states cannot be efficiently prepared. Our scheme can get the desired acceleration when the system matrix of the classical control system is sparse. When the system matrix is not sparse, in our opinion, another entire quantum scheme can also be constructed based on [22] in which a quantum algorithm of linear equation for dense matrices has been investigated.

The method of the paper can be applied to classical control systems, such as large-scale control systems that are a class of general classical control systems with high dimension. In general, the system matrix of large-scale control systems is more likely to be sparse. For example, Ref. [23] has studied the problem of linear-quadratic regulator (LQR) with high dimension, in which the system matrix is sparse, and both the state estimator and the state feedback control can be constructed based on the Lyapunov equation. This implies that our quantum scheme is justified to design the state estimator and the state feedback control in above the large-scale control systems. Moreover, large-scale control systems with high dimension can fully demonstrate the significant acceleration of quantum algorithms in comparison to traditional classical algorithms. Therefore, our research proposed an entire quantum scheme for solving classical control problems, and can enrich the application area of quantum computation.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 61673389, 61273202).

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Appendix A The impact of the inaccuracy of quantum measurement on the state estimator

The design goal of the state estimator is to make the state estimator's output $\hat{x}(t)$ as close as possible to the system state $x(t)$. Therefore, we make the steady state error $J = \lim_{t\to+\infty} \|\hat{T}(\hat{x}(t) - x(t))\|$ as a measure of the quality of the state observer constructed by matrices $\hat{T}$ and $\hat{L}$. Suppose there are N measured quantum states, then the measurement error $\delta = O(\frac{1}{N})$.

After measuring quantum states $|T\rangle$ and $|L\rangle$, we get the normalized matrices $\hat{T_0}$ and $\hat{L_0}$. Because of the inaccuracy of quantum measurement, there is a deviation between the real values T_0, L_0 and measured values $\hat{T_0}, \hat{L_0}$. Therefore, the coefficients $\hat{C_1}$ and $\hat{C_2}$ obtained based on $\hat{T_0}, \hat{L_0}$ also deviate from the true values C_1 and C_2,

$$T = C_1T_0,\ \hat{T} = \hat{C_1}\hat{T_0} = \hat{C_1}\left(T_0 + \Delta\left(\frac{1}{N}\right)\right), \tag{A1}$$

$$L = C_2L_0,\ \hat{L} = \hat{C_2}\hat{L_0} = \hat{C_2}\left(L_0 + \Delta\left(\frac{1}{N}\right)\right), \tag{A2}$$

where T, L are the expected target matrices, $\hat{T}, \hat{L}$ are the corresponding matrices with errors and $\Delta(\frac{1}{N})$ is a set of matrices whose element δ_{ij} satisfies $\delta_{ij} = O(\frac{1}{N})$. The matrices T_0 and L_0 are normalized, so $C_1 = \|T\|$ and $C_2 = \|L\|$. Substituting $\hat{T}$ into (4), we get

$$\begin{aligned}
&\hat{C_1}\left(FT_0 + F\Delta\left(\frac{1}{N}\right) - T_0A - \Delta\left(\frac{1}{N}\right)A\right)_{11} = (GC)_{11}\\
&\mapsto \hat{C_1}\left(\frac{1}{C_1}GC + F\Delta\left(\frac{1}{N}\right) - \Delta\left(\frac{1}{N}\right)A\right)_{11} = (GC)_{11}\\
&\mapsto \hat{C_1}\left(\frac{1}{C_1}V_{11} + \lambda(f_1)O\left(\frac{1}{N}\right) - \sum_i a_{i1}O\left(\frac{1}{N}\right)\right) = V_{11},
\end{aligned} \tag{A3}$$

where $V_{11} = (GC)_{11}$ is the first element of matrix GC, a_{i1} is the non-zero element of the first column of A, the matrix F is a diagonal matrix constructed based on eigenvalue $\lambda(f_i)(\lambda(f_i) < 0)$ and A is a sparse matrix. According to (A3), there is

$$\hat{C_1} = \frac{V_{11}}{V_{11} + (\lambda(f_1) - \sum_i a_{i1})C_1O(\frac{1}{N})}C_1. \tag{A4}$$

The value of N is big enough, so $\hat{C_1} \approx C_1 = \|T\|$. Based on the equation $\hat{L} = \hat{T}B$, we can get

$$\begin{aligned}
&\hat{C_2}\|\hat{L_0}\| = \|C_1\left(T_0 + \Delta\left(\frac{1}{N}\right)\right)B\|,\\
&\hat{C_2} = \|L + C_1\Delta\left(\frac{1}{N}\right)B\| \approx \|L\|.
\end{aligned} \tag{A5}$$

Let $\delta T = \hat{T} - T, \delta L = \hat{L} - L$, then we may know $\delta T \subseteq \Delta(\frac{1}{N})\|T\|, \delta L \subseteq \Delta(\frac{1}{N})\|L\|$. Let $e(t) = \hat{T}(\hat{x}(t) - x(t))$ and we obtain

$$\begin{aligned}
\dot{e}(t) &= \dot{z}(t) - \hat{T}\dot{x}(t) = Fe(t) + (F\hat{T} + GC - \hat{T}A)x(t) + (\hat{L} - \hat{T}B)u(t)\\
&= Fe(t) + (F\delta T - \delta TA)x(t) + (\delta L - \delta TB)u(t).
\end{aligned} \tag{A6}$$

Set $\delta U = F\delta T - \delta TA \subseteq \|U\|\Delta(\frac{1}{N})$ and $\delta V = \delta L - \delta TB \subseteq \|V\|\Delta(\frac{1}{N})$, then we get

$$\|U\| \leqslant (\lambda(f) + a)\|T\|,\ \|V\| \leqslant (\|L\| + nb\|T\|), \tag{A7}$$

where $\lambda(f) = \max(|\lambda(f_i)|)$, $a = \max(|a_{ij}|)$ and $b = \max(|b_{ij}|)$.

Therefore, we can obtain

$$\dot{e}(t) = Fe(t) + \delta Ux(t) + \delta Vu(t). \tag{A8}$$

Therefore, Eq. (A8) can be transformed into

$$\dot{e_i}(t) = \lambda(f_i)e_i(t) + \sum_{j=1}^{n}\delta U_{ij}x_j(t) + \sum_{j=1}^{q}\delta V_{ij}u_j(t), \tag{A9}$$

and

$$e_i(t) = E_i\mathrm{e}^{\lambda(f_i)t} + \mathrm{e}^{\lambda(f_i)t}\int \mathrm{e}^{-\lambda(f_i)t}\left(\sum_{j=1}^{n}\delta U_{ij}x_j(t) + \sum_{j=1}^{q}\delta V_{ij}u_j(t)\right)\mathrm{d}t. \tag{A10}$$

Let $y_i(t) = \sum_{j=1}^{n}\delta U_{ij}x_j(t) + \sum_{j=1}^{q}\delta V_{ij}u_j(t)$, then we may infer

$$\begin{aligned}
|y_i(t)| &= \left|\sum_{j=1}^{n}\delta U_{ij}x_j(t) + \sum_{j=1}^{q}\delta V_{ij}u_j(t)\right|\\
&\leqslant \left|\sum_{j=1}^{n}\delta U_{ij}x_j(t)\right| + \left|\sum_{j=1}^{q}\delta V_{ij}u_j(t)\right|
\end{aligned}$$

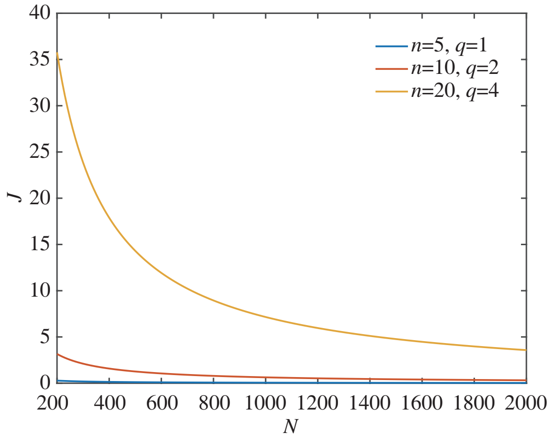

Figure A1 (Color online) The impact of the quantum measurement accuracy on the steady state error J of state estimator.

$$\leqslant n x_m \|U\| O\left(\frac{1}{N}\right) + q u_m \|V\| O\left(\frac{1}{N}\right), \tag{A11}$$

where $x_m = \max(|x_j(t)|)$ and $u_m = \max(|u_j(t)|)$.

According to (A10), it follows:

$$\begin{aligned}\lim_{t\to+\infty} |e_i(t)| &\leqslant \lim_{t\to+\infty} |\mathrm{e}^{\lambda(f_i)t} \int \mathrm{e}^{-\lambda(f_i)t} y_i(t)\mathrm{d}t| \\ &\leqslant -\frac{n x_m}{\lambda(f_i)} \|U\| O\left(\frac{1}{N}\right) - \frac{q u_m}{\lambda(f_i)} \|V\| O\left(\frac{1}{N}\right) \\ &\leqslant -\frac{n x_m(\lambda(f)+a)}{\lambda(f_i)} \|T\| O\left(\frac{1}{N}\right) - \frac{q u_m}{\lambda(f_i)} (\|L\| + nb\|T\|) O\left(\frac{1}{N}\right),\end{aligned} \tag{A12}$$

and the steady state error of state estimator $J = \lim_{t\to+\infty} \|\hat{T}(\hat{x(t)} - x(t))\|$ satisfies

$$\begin{aligned}J &= \lim_{t\to+\infty} \|e(t)\| \\ &\leqslant \sqrt{\sum_{i=1}^{n} \left(\frac{n x_m(\lambda(f)+a)}{\lambda(f_i)} \|T\| O\left(\frac{1}{N}\right) + \frac{q u_m}{\lambda(f_i)} (\|L\| + nb\|T\|) O\left(\frac{1}{N}\right)\right)^2}.\end{aligned} \tag{A13}$$

When all eigenvalues of the system matrix A are negative, the original linear system Σ_1 is stable, $\lim_{t\to+\infty} x(t) = x$ and

$$\begin{aligned}J &\leqslant \sqrt{\sum_{i=1}^{n} \left(\frac{n x_m(\lambda(f)+a)}{\lambda(f_i)} \|T\| O\left(\frac{1}{N}\right) + \frac{q u_m}{\lambda(f_i)} (\|L\| + nb\|T\|) O\left(\frac{1}{N}\right)\right)^2} \\ &\approx O\left(\frac{\sqrt{n} n^2 q}{N}\right).\end{aligned} \tag{A14}$$

With the increase of the number N of measured quantum states, the steady state error J of state estimator decreases. We can simulate the error in Figure A1.

In addition, when the system matrix A has positive eigenvalues, the Σ_1 is an unstable system. With the increase of $x(t)$, the steady state error J increases. For the unstable system Σ_1, the inaccuracy of quantum measurement will seriously affect the state estimator.