SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

September 2020, Vol. 63 192204:1–192204:13
https://doi.org/10.1007/s11432-019-1473-3

A partial information linear-quadratic optimal control problem of backward stochastic differential equation with its applications

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Received 30 May 2019/Accepted 10 July 2019/Published online 28 July 2020

Abstract In this paper, we investigate a kind of partial information linear-quadratic optimal control problem driven by a backward stochastic differential equation, where the state equation and the cost functional contain diffusion terms. Using maximum principle, we derive the corresponding Hamiltonian system, which is a conditional mean-field forward-backward stochastic differential equation. By the backward separation approach and the filtering technique, we get two Riccati equations, and a backward and a forward optimal filtering equations. Then a feedback form of optimal control is obtained. We also extend the control problem to the case of mean-field backward stochastic differential equation under partial information. A corresponding feedback form of optimal control is also obtained.

Keywords backward stochastic differential equation, feedback representation, linear-quadratic optimal control, mean-field, optimal filter

Citation Huang P Y, Wang G C, Zhang H J. A partial information linear-quadratic optimal control problem of backward stochastic differential equation with its applications. Sci China Inf Sci, 2020, 63(9): 192204, https://doi.org/10.1007/s11432-019-1473-3

1 Introduction

Linear-quadratic (LQ) optimal control problem is an important kind of control problem, whose solution plays a vital role in engineering areas. Searching a feedback form of optimal control is a fundamental issue in studying LQ optimal control problem. In 1999, Dokuchaev and Zhou [1] attempted to consider a special LQ control problem of backward stochastic differential equation (BSDE), where the running cost does not contain states. Kohlmann and Zhou [2] discussed the relationship between a stochastic control problem and a BSDE. Based on [2], Lim and Zhou [3] firstly solved a general LQ optimal control problem of BSDE and gave an explicit form of optimal control. Yu [4] investigated a control problem for forward-backward stochastic differential equation (FBSDE), and gave a representation of optimal control. Wu and Wang [5] researched an LQ control problem of time-delayed BSDE under partial information, and obtained a state feedback of optimal control. Zhang et al. [6] studied a partially observed optimal control problem for FBSDE with Markovian regime switching.

In 1956, inspired by a stochastic toy model for Vlasov kinetic equation of plasma, Kac [7] firstly introduced the mean-field stochastic differential equation (MF-SDE). Mean-field optimal control problem can be regarded as a large population limit of a cooperative game, where players work together to achieve a joint goal. Thus the optimal control problem of the mean-field stochastic system has an important

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significance in reality. Yong [8] dealed with an optimal control problem of MF-SDE. By decoupling the optimality system, a feedback form of optimal control was obtained. Later, Li et al. [9] solved an LQ control problem of mean-field backward stochastic differential equation (MF-BSDE), and an optimal control was represented by two Riccati equations and an MF-SDE. Recently, Douissi et al. [10] considered a mean-field anticipated BSDE driven by fractional Brownian motion, and applied the results to solve an optimal control problem. Ma [11] investigated a control problem of an infinite horizon mean-field FBSDE (MF-FBSDE) with delay and Poisson jump, and gave an Arrow sufficient condition for such problem.

This paper aims at studying two kinds of partial information LQ control problems driven by backward stochastic systems. This work is different from the existing literature in two aspects.

(1) In Section 2, Problem (LQC) is a general LQ control problem of BSDE under partial information, which is different from [3, 12, 13]. Due to the partial information setup, some difficulties arise from the study of Problem (LQC). (i) Filtering equation (22) of Hamiltonian system (4) is a conditional MF-FBSDE. In general, it is difficult to prove the existence and uniqueness of (22). By the virtue of the backward separation approach and the stochastic filtering, we overcome this difficulty. (ii) In (13), if $h(t)$ takes the form of $\mathrm{d}h(t) = \gamma(t)\mathrm{d}t + \eta(t)\mathrm{d}W(t) + \bar{\eta}(t)\mathrm{d}\overline{W}(t)$, it is uncertain whether the filtering equation of $h(t)$ admits a unique solution. Whereas, if $h(t)$ takes the form of (14), we prove that the filtering equation of (14) is uniquely solvable.

(2) In Section 3, Problem (MFC) is not a trivial extension of Problem (LQC). Then, Problem (MFC) cannot be solved by using results obtained in Problem (LQC). See Subsection 3.1 for details. Problems (LQC) and (MFC) have applications in mathematical finance and economics, for example, stochastic pension fund optimization problems [12] and portfolio selection problems [14].

The rest of this paper is organized as follows. In Section 2, we give some notations and formulate an LQ optimal control problem of BSDE with partial information, and represent optimal control by two Riccati equations, two filtering equations of a BSDE and an SDE, respectively. In Section 3, we investigate an optimal control problem of MF-BSDE with partial information, and an explicit feedback of optimal control is obtained. In Section 4, we end this paper with some concluding remarks.

2 An LQ optimal control of BSDE with partial information

2.1 Notations

Throughout this paper, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leqslant t\leqslant T}, P)$ be a complete filtered probability space with a natural filtration $\{\mathcal{F}_t, 0 \leqslant t \leqslant T\}$ generated by an $\mathcal{F}_t$-adapted, 2-dimensional standard Brownian motion $\{W(s), \overline{W}(s), 0 \leqslant s \leqslant t\}$. $\mathcal{F}_t^W = \sigma\{W(s) : 0 \leqslant s \leqslant t\}$ is a sub-filtration of $\mathcal{F}_t$, and $\hat{g}(\cdot) = \mathbb{E}[g(\cdot)|\mathcal{F}_t^W]$ is the optimal filter of $g(\cdot)$ with respect to $\mathcal{F}_t^W$. A^τ denotes the transpose of matrix A. For simplicity, we introduce the notations: $\mathcal{L}^2_{\mathcal{F}_t^W}(0, T; \mathbb{R}) = \{x(\cdot) : \Omega \times [0, T] \to \mathbb{R}|x(\cdot)$ is an $\mathcal{F}_t^W$-adapted process, and satisfies $\mathbb{E}\int_0^T |x(t)|^2\mathrm{d}t < +\infty\}$, $\mathcal{L}^2_{\mathcal{F}}(\Omega, \mathcal{F}_T, P; \mathbb{R}) = \{\xi : \Omega \to \mathbb{R}|\xi$ is an $\mathcal{F}_T$-measurable random variable, and satisfies $\mathbb{E}|\xi|^2 < +\infty\}$.

2.2 Problem formulation

Consider a linear controlled BSDE

$$\begin{cases} \mathrm{d}y^u(t) = [A(t)y^u(t) + B(t)u(t) + C(t)z^u(t) + \bar{C}(t)\bar{z}^u(t)]\mathrm{d}t + z^u(t)\mathrm{d}W(t) + \bar{z}^u(t)\mathrm{d}\overline{W}(t), \\ y^u(T) = \xi, \end{cases} \tag{1}$$

and a cost functional

$$J(u(\cdot)) = \frac{1}{2}\mathbb{E}\left\{\int_0^T \Big[Q(t)(y^u(t))^2 + S(t)(z^u(t))^2 + \bar{S}(t)(\bar{z}^u(t))^2 + R(t)u^2(t)\Big]\mathrm{d}t + \Phi(y^u(0))^2\right\}, \tag{2}$$

where $u(\cdot)$ is the control process; $A(\cdot), B(\cdot), C(\cdot), \bar{C}(\cdot), Q(\cdot), S(\cdot), \bar{S}(\cdot), R(\cdot)$ are uniformly bounded and deterministic functions; moreover, $Q(\cdot), S(\cdot)$, $\bar{S}(\cdot), \Phi \geqslant 0$; $R(\cdot) > 0$; $\xi \in \mathcal{L}^2_{\mathcal{F}}(\Omega, \mathcal{F}_T, P; \mathbb{R})$.

The admissible control set is

$$\mathcal{U}=\mathcal{L}^2_{\mathcal{F}_t^W}(0,T;\mathbb{R}).$$

Our backward stochastic LQ optimal control problem can be stated as follows.

Problem (LQC). Find a $u^*(\cdot)\in\mathcal{U}$, such that

$$J(u^*(\cdot))=\min_{u(\cdot)\in\mathcal{U}} J(u(\cdot)) \tag{3}$$

subject to (1) and (2). If we can find such an admissible control $u^*(\cdot)\in\mathcal{U}$ satisfying (3), then we call it an optimal control of Problem (LQC), and the corresponding optimal state process is denoted by $(y,z,\bar{z})$. Similar convention is taken for the subsequent optimal state processes. For simplicity of notation, in what follows we shall often suppress the time variable t if no confusion can arise.

Note that since cost functional (2) contains $(z^u)^2$ and $(\bar{z}^u)^2$, Problem (LQC) covers [12, 13, 15] as special cases. Within the setup of Problem (LQC), in general it is difficult to obtain a state feedback representation of optimal control. Here in this paper, two Riccati equations and two filtering equations are introduced to overcome this difficulty.

2.3 Main results

The following proposition can be obtained from the studies in [12, 16]. It is very useful for us to make further discussion.

Proposition 1. u^* is an optimal control of Problem (LQC) if and only if u^* can be represented by

$$u^*(t)=-R^{-1}(t)B(t)\hat{p}(t),$$

where $(y,z,\bar{z},p)$ is the unique solution of Hamiltonian system:

$$\begin{cases} \mathrm{d}y=(Ay+Bu^*+Cz+\bar{C}\bar{z})\mathrm{d}t+z\mathrm{d}W+\bar{z}\mathrm{d}\overline{W},\\ \mathrm{d}p=-(Qy+Ap)\mathrm{d}t-(Sz+Cp)\mathrm{d}W-(\bar{S}\bar{z}+\bar{C}p)\mathrm{d}\overline{W},\\ y(T)=\xi,\quad p(0)=-\Phi y(0). \end{cases} \tag{4}$$

Note that Eq. (4) is an initial coupled FBSDE and contains the conditional expectation $\hat{p}$ through u^*, so that it is difficult to prove its existence and uniqueness.

Lemma 1. Suppose that Eq. (4) admits a unique solution $(y,z,\bar{z},p)$. Then, the following relations are satisfied:

(a)

$$\hat{y}(t)=\Sigma(t)\hat{p}(t)-\hat{h}(t) \tag{5}$$

and

$$\hat{\bar{z}}=-(1+\Sigma\bar{S})^{-1}\Sigma\bar{C}\hat{p}, \tag{6}$$

where

$$\begin{cases} \dot{\Sigma}-2A\Sigma-Q\Sigma^2+\tilde{C}(I+\Sigma\tilde{S})^{-1}\tilde{C}^{\tau}\Sigma+R^{-1}B^2=0,\\ \Sigma(T)=0, \end{cases} \tag{7}$$

$$\begin{cases} \mathrm{d}\hat{h}=[(A+\Sigma Q)\hat{h}+(1+\Sigma S)^{-1}C\hat{\eta}]\mathrm{d}t+\hat{\eta}\mathrm{d}W,\\ \hat{h}(T)=-\hat{\xi}, \end{cases} \tag{8}$$

$$\tilde{C}=(C,\bar{C}),\quad \tilde{S}=\begin{pmatrix} S & 0\\ 0 & \bar{S}\end{pmatrix}.$$

(b)

$$\hat{p}(t)=-[\Pi(t)\hat{y}(t)+\hat{q}(t)], \tag{9}$$

where

$$\begin{cases} \dot{\Pi} + 2A\Pi + [R^{-1}B^2 + \tilde{C}(I + \Sigma\tilde{S})^{-1}\tilde{C}^\tau\Sigma]\Pi^2 - Q = 0, \\ \Pi(0) = \Phi, \end{cases} \tag{10}$$

$$\begin{cases} \mathrm{d}\hat{q} = \{C(1 + \Sigma S)^{-1}\Pi\hat{\eta} - [A + (R^{-1}B^2 + \tilde{C}(I + \Sigma\tilde{S})^{-1}\tilde{C}^\tau\Sigma)\Pi]\hat{q}\}\mathrm{d}t \\ \qquad + [(\Pi - S)(1 + \Sigma S)^{-1}\hat{\eta} + C(1 + \Sigma S)^{-1}(\Pi\hat{h} - \hat{q})]\mathrm{d}W, \\ \hat{q}(0) = 0. \end{cases} \tag{11}$$

In addition,

$$\hat{p}(t) = (1 + \Pi\Sigma)^{-1}(\Pi\hat{h} - \hat{q}). \tag{12}$$

Proof. (a) Noting the terminal condition of (4), we set

$$y(t) = \Sigma(t)p(t) - h(t), \quad \Sigma(T) = 0, \tag{13}$$

where $h(\cdot)$ satisfies

$$\begin{cases} \mathrm{d}h(t) = \gamma(t)\mathrm{d}t + \eta(t)\mathrm{d}W(t), \\ h(T) = -\xi, \end{cases} \tag{14}$$

Σ is a deterministic and differentiable function, and γ and η are $\mathcal{F}_t$-adapted processes.

Applying Itô's formula to (13), we have

$$\begin{aligned} 0 &= \mathrm{d}(y - \Sigma p + h) \\ &= [(-\dot{\Sigma} + 2A\Sigma + Q\Sigma^2)p - R^{-1}B^2\hat{p} - (A + \Sigma Q)h + Cz + \bar{C}\bar{z} + \gamma]\mathrm{d}t \\ &\quad + [(1 + \Sigma S)z + \Sigma Cp + \eta]\mathrm{d}W + [(1 + \Sigma\bar{S})\bar{z} + \Sigma\bar{C}p]\mathrm{d}\overline{W}. \end{aligned}$$

Assuming the existence of $(1 + \Sigma S)^{-1}$ and $(1 + \bar{\Sigma} S)^{-1}$, we obtain

$$\begin{cases} z = -(1 + \Sigma S)^{-1}(\Sigma Cp + \eta), \\ \bar{z} = -(1 + \Sigma\bar{S})^{-1}\Sigma\bar{C}p, \end{cases} \tag{15}$$

and

$$(\dot{\Sigma} - 2A\Sigma - Q\Sigma^2)p + R^{-1}B^2\hat{p} + (A + \Sigma Q)h - Cz - \bar{C}\bar{z} - \gamma = 0. \tag{16}$$

Taking $\mathbb{E}[\cdot|\mathcal{F}_t^W]$ on both sides of (13) and the second equality of (15), we derive (5) and (6). Substituting (15) into (16), and taking $\mathbb{E}[\cdot|\mathcal{F}_t^W]$, we get

$$[\dot{\Sigma} - 2A\Sigma - Q\Sigma^2 + \tilde{C}(I + \Sigma\tilde{S})^{-1}\tilde{C}^\tau\Sigma + R^{-1}B^2]\hat{p} + (A + \Sigma Q)\hat{h} + (1 + \Sigma S)^{-1}C\hat{\eta} - \hat{\gamma} = 0.$$

If ordinary differential equation (7) admits a unique differentiable solution Σ, then

$$\hat{\gamma} = (A + \Sigma Q)\hat{h} + (1 + \Sigma S)^{-1}C\hat{\eta}.$$

The filtering equation of (14) takes the form of (8).

(b) According to (15) and Proposition 1, Hamiltonian system (4) turns into

$$\begin{cases} \mathrm{d}y = [Ay - R^{-1}B^2\hat{p} - C(1 + \Sigma S)^{-1}(\Sigma Cp + \eta) - (1 + \Sigma\bar{S})^{-1}\Sigma\bar{C}^2p]\mathrm{d}t \\ \qquad - (1 + \Sigma S)^{-1}(\Sigma Cp + \eta)\mathrm{d}W - (1 + \Sigma\bar{S})^{-1}\Sigma\bar{C}p\mathrm{d}\overline{W}, \\ \mathrm{d}p = -(Qy + Ap)\mathrm{d}t + [S(1 + \Sigma S)^{-1}(\Sigma Cp + \eta) - Cp]\mathrm{d}W + [\bar{S}(1 + \Sigma\bar{S})^{-1}\Sigma - 1]\bar{C}p\mathrm{d}\overline{W}, \\ y(T) = \xi, \quad p(0) = -\Phi y(0). \end{cases}$$

Now we conjecture that p and y are related by

$$p(t) = -[\Pi(t)y(t) + q(t)], \quad \Pi(0) = \Phi, \tag{17}$$

and

$$\begin{cases} \mathrm{d}q(t) = \alpha(t)\mathrm{d}t + \beta(t)\mathrm{d}W(t), \\ q(0) = 0, \end{cases}$$

where Π is deterministic and differentiable, and α and β are $\mathcal{F}_t$-adapted.

Applying Itô's formula to (17), we have

$$\begin{aligned} 0 = \{&[\dot{\Pi} + 2A\Pi + C^2(1+\Sigma S)^{-1}\Sigma\Pi^2 + \bar{C}^2(1+\Sigma\bar{S})^{-1}\Sigma\Pi^2 - Q]y + R^{-1}B^2\Pi^2\hat{y} \\ &+ [A + C^2(1+\Sigma S)^{-1}\Sigma\Pi + \bar{C}^2(1+\Sigma\bar{S})^{-1}\Sigma\Pi]q + R^{-1}B^2\Pi\hat{q} - C(1+\Sigma S)^{-1}\Pi\eta + \alpha\}\mathrm{d}t \\ &+ [(S-\Pi)(1+\Sigma S)^{-1}(\Sigma C p + \eta) - Cp + \beta]\mathrm{d}W + [(\bar{S}-\Pi)(1+\Sigma\bar{S})^{-1}\Sigma - 1]\bar{C}p\mathrm{d}\overline{W}. \end{aligned}$$

Then we arrive at

$$\begin{aligned} \{&\dot{\Pi} + 2A\Pi + [C^2(1+\Sigma S)^{-1}\Sigma + \bar{C}^2(1+\Sigma\bar{S})^{-1}\Sigma + R^{-1}B^2]\Pi^2 - Q\}\hat{y} + [A + C^2(1+\Sigma S)^{-1}\Sigma\Pi \\ &+\bar{C}^2(1+\Sigma\bar{S})^{-1}\Sigma\Pi + R^{-1}B^2\Pi]\hat{q} - C(1+\Sigma S)^{-1}\Pi\hat{\eta} + \hat{\alpha} = 0, \end{aligned}$$

and

$$(S-\Pi)(1+\Sigma S)^{-1}(\Sigma C\hat{p} + \hat{\eta}) - C\hat{p} + \hat{\beta} = 0.$$

We see that Π should satisfy (10), and

$$\begin{cases} \hat{\alpha} = C(1+\Sigma S)^{-1}\Pi\hat{\eta} - [A + (R^{-1}B^2 + \tilde{C}(I+\Sigma\tilde{S})^{-1}\tilde{C}^\tau\Sigma)\Pi]\hat{q}, \\ \hat{\beta} = (\Pi - S)(1+\Sigma S)^{-1}(\Sigma C\hat{p} + \hat{\eta}) + C\hat{p}. \end{cases} \tag{18}$$

Besides, Eq. (9) can be easily obtained by taking $\mathbb{E}[\cdot|\mathcal{F}_t^W]$ on both sides of (17).

At last, from (5) and (9), we derive (12). Combining (18) and (12), we get that $\hat{q}$ is represented by (11). What's more, according to [3], Eqs. (7) and (10) admit unique solutions, and $\Sigma \geqslant 0, \Pi \geqslant 0$.

2.4 Representation of optimal control

Theorem 1. The optimal control of Problem (LQC) is represented by

$$u^*(t) = R^{-1}(t)B(t)[\Pi(t)\hat{y}(t) + \hat{q}(t)], \tag{19}$$

where $\hat{g}(\cdot) = \mathbb{E}[g(\cdot)|\mathcal{F}_t^W]$, and $\Pi, \hat{q}, \hat{y}$ are given by (10), (11) and (24), respectively.

Proof. The proof can be divided into four steps.

Step 1. The open-loop form of optimal control. According to Proposition 1, if u^* is an optimal control of Problem (LQC), then it is necessary to satisfy

$$u^*(t) = -R^{-1}(t)B(t)\hat{p}(t), \tag{20}$$

where $(y, z, \bar{z}, p)$ is the unique solution to

$$\begin{cases} \mathrm{d}y = (Ay + Bu^* + Cz + \bar{C}\bar{z})\mathrm{d}t + z\mathrm{d}W + \bar{z}\mathrm{d}\overline{W}, \\ \mathrm{d}p = -(Qy + Ap)\mathrm{d}t - (Sz + Cp)\mathrm{d}W - (\bar{S}\bar{z} + \bar{C}p)\mathrm{d}\overline{W}, \\ y(T) = \xi, \quad p(0) = -\Phi y(0). \end{cases} \tag{21}$$

Applying Lemma 5.4 in [17] to (21), we get the optimal filtering $(\hat{y}, \hat{z}, \hat{\bar{z}}, \hat{p})$ of $(y, z, \bar{z}, p)$ governed by

$$\begin{cases} \mathrm{d}\hat{y} = (A\hat{y} + Bu^* + C\hat{z} + \bar{C}\hat{\bar{z}})\mathrm{d}t + \hat{z}\mathrm{d}W, \\ \mathrm{d}\hat{p} = -(Q\hat{y} + A\hat{p})\mathrm{d}t - (S\hat{z} + C\hat{p})\mathrm{d}W, \\ \hat{y}(T) = \hat{\xi}, \quad \hat{p}(0) = -\Phi\hat{y}(0). \end{cases} \tag{22}$$

Note that Eq. (22) is not a standard FBSDE because an additional filtering estimate $\hat{\bar{z}}$ is contained. Therefore, its existence and uniqueness is not an immediate result of Theorem 2.3 in [18].

Step 2. Existence and uniqueness of (23)**.** Introduce a Riccati equation:

$$\begin{cases} \dot{\Sigma} - 2A\Sigma - Q\Sigma^2 + \tilde{C}(I + \Sigma\tilde{S})^{-1}\tilde{C}^{\tau}\Sigma + R^{-1}B^2 = 0, \\ \Sigma(T) = 0, \end{cases}$$

which is uniquely solvable and the solution $\Sigma \geqslant 0$. Next, we introduce a new FBSDE:

$$\begin{cases} \mathrm{d}\hat{y} = \{A\hat{y} - [R^{-1}B^2 + (1 + \Sigma\bar{S})^{-1}\Sigma\bar{C}^2]\hat{p} + C\hat{z}\}\mathrm{d}t + \hat{z}\mathrm{d}W, \\ \mathrm{d}\hat{p} = -(Q\hat{y} + A\hat{p})\mathrm{d}t - (S\hat{z} + C\hat{p})\mathrm{d}W, \\ \hat{y}(T) = \hat{\xi}, \quad \hat{p}(0) = -\Phi\hat{y}(0). \end{cases} \tag{23}$$

It follows from Theorem 2.3 in [18] that Eq. (23) admits a unique solution $(\hat{y}, \hat{z}, \hat{p})$.

Step 3. Equivalence between (22) **and** (23) **with** (20)**.** Firstly, we prove that the solution $(\hat{y}, \hat{z}, \hat{p})$ of (23) is a solution of (22). Set $u^* = -R^{-1}B\hat{p}$, and then Eq. (21) is uniquely solvable. According to (6) in Lemma 1, it is easy to see that the solution $(\hat{y}, \hat{z}, \hat{p})$ of (23) solves (22).

Secondly, we prove that for fixed u^*, the solution $(\hat{y}, \hat{z}, \hat{\bar{z}}, \hat{p})$ of (22) is a solution of (23). Take $u^* = -R^{-1}B\hat{p}$. Then $(y, z, \bar{z}, p)$ is the unique solution of (21). Similar to the above analysis, we obtain $\hat{\bar{z}} = -(1 + \Sigma\bar{S})^{-1}\Sigma\bar{C}\hat{p}$. Putting $\hat{\bar{z}} = -(1 + \Sigma\bar{S})^{-1}\Sigma\bar{C}\hat{p}$ and $u^* = -R^{-1}B\hat{p}$ into (22), we arrive at (23), which implies that $(\hat{y}, \hat{z}, \hat{p})$ is a solution of (23).

Step 4. The feedback representation of optimal control. Inserting (9) into (20) leads to (19).

With the help of (9), the first equation of (23) is written as

$$\begin{cases} \mathrm{d}\hat{y}(t) = [f_1(t)\hat{y}(t) + C(t)\hat{z}(t) + f_2(t)]\mathrm{d}t + \hat{z}(t)\mathrm{d}W(t), \\ \hat{y}(T) = \hat{\xi}, \end{cases}$$

where $f_1 = A + [R^{-1}B^2 + (1 + \Sigma\bar{S})^{-1}\bar{C}^2\Sigma]\Pi$, and $f_2 = [R^{-1}B^2 + (1 + \Sigma\bar{S})^{-1}\bar{C}^2\Sigma]\hat{q}$. By solving it, we have

$$\hat{y}(t) = \mathbb{E}\left[\hat{\xi}X(T) - \int_t^T f_2(s)X(s)\mathrm{d}s\middle|\mathcal{F}_t^W\right] \tag{24}$$

with

$$X(s) = \exp\left\{-\int_t^s \left[f_1(r) + \frac{1}{2}C^2(r)\right]\mathrm{d}r - \int_t^s C(r)\mathrm{d}W(r)\right\}.$$

Note that since $\hat{q}(t)$ given by (11) is a stochastic process, Eq. (19) covers [12, 13, 15] as special cases.

3 An LQ optimal control of MF-BSDE with partial information

3.1 Problem formulation

Consider the following Problem (MFC): $\min_{u(\cdot)\in\mathcal{U}} J(u(\cdot))$ with a cost functional

$$\begin{aligned} J(u(\cdot)) =& \frac{1}{2}\mathbb{E}\Bigg\{\int_0^T [Q(t)(y^u(t))^2 + \tilde{Q}(t)(\mathbb{E}y^u(t))^2 + R(t)u^2(t) + \tilde{R}(t)(\mathbb{E}u(t))^2 + S(t)(z^u(t))^2 \\ &+ \bar{S}(t)(\bar{z}^u(t))^2]\mathrm{d}t + \Phi(y^u(0))^2\Bigg\} \end{aligned} \tag{25}$$

subject to an MF-BSDE

$$\begin{cases} \mathrm{d}y^u(t) = [A(t)y^u(t) + \tilde{A}(t)\mathbb{E}y^u(t) + B(t)u(t) + \tilde{B}(t)\mathbb{E}u(t) + C(t)z^u(t) + \bar{C}(t)\bar{z}^u(t)]\mathrm{d}t \\ \qquad\qquad + z^u(t)\mathrm{d}W(t) + \bar{z}^u(t)\mathrm{d}\overline{W}(t), \\ y^u(T) = \xi, \end{cases} \tag{26}$$

where $u(\cdot)$ is the control process; $A(\cdot),\tilde{A}(\cdot),B(\cdot),\tilde{B}(\cdot),C(\cdot),\bar{C}(\cdot),Q(\cdot),\tilde{Q}(\cdot),R(\cdot),\tilde{R}(\cdot),S(\cdot),\bar{S}(\cdot)$ are uniformly bounded and deterministic, and $\Phi,Q(\cdot),Q(\cdot)+\tilde{Q}(\cdot),S(\cdot),\bar{S}(\cdot)\geqslant 0$. Besides, there exists a constant $\rho>0$, such that $R(\cdot),R(\cdot)+\tilde{R}(\cdot)\geqslant\rho$; $\xi\in\mathcal{L}^2_{\mathcal{F}}(\Omega,\mathcal{F}_T,P;\mathbb{R})$; $\mathcal{U}=\mathcal{L}^2_{\mathcal{F}^W_t}(0,T;\mathbb{R})$. Under such conditions, MF-BSDE (26) admits a unique solution, and Problem (MFC) has a unique optimal control (see Theorem 3.1 in [19] and Theorem 2.2 in [9] for details).

We emphasize that Problem (MFC) cannot be solved by transforming it into Problem (LQC). A main reason is as follows.

Taking $\mathbb{E}[\cdot]$ on both sides of (26), we get

$$\begin{cases}\mathrm{d}\mathbb{E}y^u=[(A+\tilde{A})\mathbb{E}y^u+(B+\tilde{B})\mathbb{E}u+C\mathbb{E}z^u+\bar{C}\mathbb{E}\bar{z}^u]\mathrm{d}t,\\ \mathbb{E}y^u(T)=\mathbb{E}\xi,\end{cases}$$

and then

$$\begin{cases}\mathrm{d}(y^u-\mathbb{E}y^u)=[A(y^u-\mathbb{E}y^u)+B(u-\mathbb{E}u)+C(z^u-\mathbb{E}z^u)+\bar{C}(\bar{z}^u-\mathbb{E}\bar{z}^u)]\mathrm{d}t+z^u\mathrm{d}W+\bar{z}^u\mathrm{d}\overline{W},\\ y^u(T)-\mathbb{E}y^u(T)=\xi-\mathbb{E}\xi.\end{cases}$$

Set

$$Y^u=\begin{pmatrix}y^u-\mathbb{E}y^u\\ \mathbb{E}y^u\end{pmatrix},\quad Z^u=\begin{pmatrix}z^u-\mathbb{E}z^u\\ \mathbb{E}z^u\end{pmatrix},\quad \bar{Z}^u=\begin{pmatrix}\bar{z}^u-\mathbb{E}\bar{z}^u\\ \mathbb{E}\bar{z}^u\end{pmatrix},\quad U=\begin{pmatrix}u-\mathbb{E}u\\ \mathbb{E}u\end{pmatrix},$$

and

$$\mathcal{A}=\begin{pmatrix}A&0\\0&A+\tilde{A}\end{pmatrix},\quad \mathcal{B}=\begin{pmatrix}B&0\\0&B+\tilde{B}\end{pmatrix},\quad \mathcal{C}=\begin{pmatrix}C&0\\0&C\end{pmatrix},\quad \bar{\mathcal{C}}=\begin{pmatrix}\bar{C}&0\\0&\bar{C}\end{pmatrix},$$

$$\mathcal{D}=\begin{pmatrix}1&1\\0&0\end{pmatrix},\quad \tilde{\xi}=\begin{pmatrix}\xi-\mathbb{E}\xi\\ \mathbb{E}\xi\end{pmatrix},\quad \mathcal{Q}=\begin{pmatrix}Q&0\\0&Q+\tilde{Q}\end{pmatrix},\quad \mathcal{R}=\begin{pmatrix}R&0\\0&R+\tilde{R}\end{pmatrix},$$

$$\mathcal{S}=\begin{pmatrix}S&0\\0&S\end{pmatrix},\quad \bar{\mathcal{S}}=\begin{pmatrix}\bar{S}&0\\0&\bar{S}\end{pmatrix},\quad \tilde{\Phi}=\begin{pmatrix}\Phi&0\\0&\Phi\end{pmatrix}.$$

Then state equation (26) is written as

$$\begin{cases}\mathrm{d}Y^u=(\mathcal{A}Y^u+\mathcal{B}U+\mathcal{C}Z^u+\bar{\mathcal{C}}\bar{Z}^u)\mathrm{d}t+\mathcal{D}Z^u\mathrm{d}W+\mathcal{D}\bar{Z}^u\mathrm{d}\overline{W},\\ Y^u(T)=\tilde{\xi},\end{cases}\tag{27}$$

and cost functional (25) becomes

$$J(u(\cdot))=\frac{1}{2}\mathbb{E}\left\{\int_0^T[(Y^u)^\tau\mathcal{Q}Y^u+U^\tau\mathcal{R}U+(Z^u)^\tau\mathcal{S}Z^u+(\bar{Z}^u)^\tau\bar{\mathcal{S}}\bar{Z}^u]\mathrm{d}t+(Y^u(0))^\tau\tilde{\Phi}Y^u(0)\right\}.$$

Note that since $\mathcal{D}$ is not reversible, Eq. (27) is not a standard BSDE. Besides, the control U has the form of

$$U=\begin{pmatrix}u-\mathbb{E}u\\ \mathbb{E}u\end{pmatrix}.$$

The collection of all such processes is not $\mathcal{U}\times\mathcal{U}$. This tells us that Problem (MFC) cannot be solved by using the results of Problem (LQC).

We remark that the above analysis is inspired by Yong [8] and Wang et al. [20].

3.2 Main results

Lemma 2. Let $(y,z,\bar{z},u^*)$ be the optimal solution of Problem (MFC). Then p is the solution to

$$\begin{cases} \mathrm{d}p=-(Ap+\tilde{A}\mathbb{E}p+Qy+\tilde{Q}\mathbb{E}y)\mathrm{d}t-(Sz+Cp)\mathrm{d}W-(\bar{S}\bar{z}+\bar{C}p)\mathrm{d}\overline{W}, \\ p(0)=-\Phi y(0), \end{cases}$$

and the following condition holds:

$$Ru^*+\tilde{R}\mathbb{E}u^*+B\mathbb{E}[p|\mathcal{F}_t^W]+\tilde{B}\mathbb{E}p=0,\quad t\in[0,T].$$

Proof. By the variational method, the result can be obtained. Here we omit the proof for simplicity.

From Lemma 2, we know that if u^* is an optimal control of Problem (MFC), then it satisfies

$$\begin{cases} \mathrm{d}y=(Ay+\tilde{A}\mathbb{E}y+Bu^*+\tilde{B}\mathbb{E}u^*+Cz+\bar{C}\bar{z})\mathrm{d}t+z\mathrm{d}W+\bar{z}\mathrm{d}\overline{W}, \\ \mathrm{d}p=-(Ap+\tilde{A}\mathbb{E}p+Qy+\tilde{Q}\mathbb{E}y)\mathrm{d}t-(Sz+Cp)\mathrm{d}W-(\bar{S}\bar{z}+\bar{C}p)\mathrm{d}\overline{W}, \\ y(T)=\xi,\quad p(0)=-\Phi y(0), \end{cases} \tag{28}$$

and

$$Ru^*+\tilde{R}\mathbb{E}u^*+B\hat{p}+\tilde{B}\mathbb{E}p=0, \tag{29}$$

where $\hat{p}=\mathbb{E}[p|\mathcal{F}_t^W]$.

Since Eq. (28) is an initial coupled MF-FBSDE, it is uncertain whether it has a unique solution.

Lemma 3. Assume that (28) has a unique solution. Then, we have the relations as follows:

(a)

$$\hat{y}(t)=\sigma(t)[\hat{p}(t)-\mathbb{E}p(t)]+\delta(t)\mathbb{E}p(t)+\hat{\varphi}(t), \tag{30}$$

$$\hat{\bar{z}}=-(1+\sigma\bar{S})^{-1}\sigma\bar{C}\hat{p}, \tag{31}$$

where

$$\begin{cases} \dot{\sigma}-2A\sigma-Q\sigma^2+\tilde{C}(I+\sigma\tilde{S})^{-1}\tilde{C}^{\tau}\sigma+R^{-1}B^2=0, \\ \sigma(T)=0, \end{cases} \tag{32}$$

$$\begin{cases} \dot{\delta}-2(A+\tilde{A})\delta-(Q+\tilde{Q})\delta^2+\tilde{C}(I+\sigma\tilde{S})^{-1}\tilde{C}^{\tau}\sigma+(R+\tilde{R})^{-1}(B+\tilde{B})^2=0, \\ \delta(T)=0, \end{cases} \tag{33}$$

$$\begin{cases} \mathrm{d}\hat{\varphi}=\{(A+\sigma Q)\hat{\varphi}+[\tilde{A}+(Q+\tilde{Q})\delta-\sigma Q]\mathbb{E}\varphi+C(1+\sigma S)^{-1}\hat{\gamma}_2\}\mathrm{d}t+\hat{\gamma}_2\mathrm{d}W, \\ \hat{\varphi}(T)=\hat{\xi}, \end{cases} \tag{34}$$

$$\begin{cases} \mathrm{d}\mathbb{E}\varphi=\{[A+\tilde{A}+(Q+\tilde{Q})\delta]\mathbb{E}\varphi+C(1+\sigma S)^{-1}\mathbb{E}\gamma_2\}\mathrm{d}t, \\ \mathbb{E}\varphi(T)=\mathbb{E}\xi, \end{cases} \tag{35}$$

and

$$\tilde{C}=(C,\bar{C}),\quad \tilde{S}=\begin{pmatrix} S & 0 \\ 0 & \bar{S} \end{pmatrix}.$$

(b)

$$\hat{p}(t)=-[\bar{\sigma}(t)(\hat{y}(t)-\mathbb{E}y(t))+\bar{\delta}(t)\mathbb{E}y(t)+\hat{\bar{\varphi}}(t)], \tag{36}$$

where

$$\begin{cases} \dot{\bar{\sigma}}+2A\bar{\sigma}+[R^{-1}B^2+\tilde{C}(I+\sigma\tilde{S})^{-1}\tilde{C}^{\tau}\sigma]\bar{\sigma}^2-Q=0, \\ \bar{\sigma}(0)=\Phi, \end{cases} \tag{37}$$

$$\begin{cases} \dot{\bar{\delta}}+2(A+\tilde{A})\bar{\delta}+[(R+\tilde{R})^{-1}(B+\tilde{B})^2+\tilde{C}(I+\sigma\tilde{S})^{-1}\tilde{C}^{\tau}\sigma]\bar{\delta}^2-(Q+\tilde{Q})=0, \\ \bar{\delta}(0)=\Phi, \end{cases} \tag{38}$$

$$\begin{cases} \mathrm{d}\hat{\bar{\varphi}} = (K_1\hat{\bar{\varphi}} + K_5\mathbb{E}\bar{\varphi} + K_6)\mathrm{d}t \\ \qquad + [(S-\sigma)(1+\sigma S)^{-1}\hat{\gamma}_2 + K_2K_4 + K_3K_4\mathbb{E}\bar{\varphi} - K_4(1+\bar{\sigma}\sigma)^{-1}\hat{\bar{\varphi}}]\mathrm{d}W, \\ \hat{\bar{\varphi}}(0) = 0, \end{cases} \tag{39}$$

$$\begin{cases} \mathrm{d}\mathbb{E}\bar{\varphi} = [(K_1+K_5)\mathbb{E}\bar{\varphi} + K_7]\mathrm{d}t, \\ \mathbb{E}\bar{\varphi}(0) = 0, \end{cases} \tag{40}$$

and K_i $(i = 1, \ldots, 7)$ are given by

$$\begin{cases} K_1 = -[A + R^{-1}B^2\bar{\sigma} + \tilde{C}(I+\sigma\tilde{S})^{-1}\tilde{C}^{\tau}\sigma\bar{\sigma}], \\ K_2 = (1+\bar{\sigma}\sigma)^{-1}[(1+\bar{\delta}\delta)^{-1}(\bar{\delta}\delta - \bar{\sigma}\sigma)\bar{\delta}\mathbb{E}\varphi - \bar{\sigma}(\hat{\varphi} - \mathbb{E}\varphi) - \bar{\delta}\mathbb{E}\varphi], \\ K_3 = (1+\bar{\sigma}\sigma)^{-1}(1+\bar{\delta}\delta)^{-1}(\bar{\delta}\delta - \bar{\sigma}\sigma), \\ K_4 = C[1 + \sigma(1+\sigma S)^{-1}(\bar{\sigma} - S)], \\ K_5 = R^{-1}B^2\bar{\sigma} + \tilde{C}(I+\sigma\tilde{S})^{-1}\tilde{C}^{\tau}\sigma(\bar{\sigma} - \bar{\delta}) - (R+\tilde{R})^{-1}(B+\tilde{B})^2\bar{\delta} - \tilde{A}, \\ K_6 = -\bar{\sigma}C(1+\sigma S)^{-1}\hat{\gamma}_2 + (\bar{\sigma} - \bar{\delta})C(1+\sigma S)^{-1}\mathbb{E}\gamma_2, \\ K_7 = -\bar{\delta}C(1+\sigma S)^{-1}\mathbb{E}\gamma_2. \end{cases}$$

Proof. (a) Set

$$y(t) = \sigma(t)[p(t) - \mathbb{E}p(t)] + \delta(t)\mathbb{E}p(t) + \varphi(t), \quad \sigma(T) = \delta(T) = 0, \tag{41}$$

and

$$\begin{cases} \mathrm{d}\varphi(t) = \gamma_1(t)\mathrm{d}t + \gamma_2(t)\mathrm{d}W(t), \\ \varphi(T) = \xi, \end{cases}$$

where σ, δ are deterministic and differentiable, and γ_1, γ_2 are $\mathcal{F}_t$-adapted.

Taking $\mathbb{E}[\cdot]$ on both sides of (28) and (29), we obtain

$$\begin{cases} \mathrm{d}\mathbb{E}y = [(A+\tilde{A})\mathbb{E}y + (B+\tilde{B})\mathbb{E}u^* + C\mathbb{E}z + \bar{C}\mathbb{E}\bar{z}]\mathrm{d}t, \\ \mathrm{d}\mathbb{E}p = -[(A+\tilde{A})\mathbb{E}p + (Q+\tilde{Q})\mathbb{E}y]\mathrm{d}t, \\ \mathbb{E}y(T) = \mathbb{E}\xi, \quad \mathbb{E}p(0) = -\Phi\mathbb{E}y(0), \end{cases} \tag{42}$$

$$(R+\tilde{R})\mathbb{E}u^* + (B+\tilde{B})\mathbb{E}p = 0, \tag{43}$$

and then

$$\begin{cases} \mathrm{d}(y - \mathbb{E}y) = [A(y-\mathbb{E}y) + B(u^* - \mathbb{E}u^*) + C(z - \mathbb{E}z) + \bar{C}(\bar{z} - \mathbb{E}\bar{z})]\mathrm{d}t + z\mathrm{d}W + \bar{z}\mathrm{d}\overline{W}, \\ \mathrm{d}(p - \mathbb{E}p) = -[A(p-\mathbb{E}p) + Q(y-\mathbb{E}y)]\mathrm{d}t - (Sz + Cp)\mathrm{d}W - (\bar{S}\bar{z} + \bar{C}p)\mathrm{d}\overline{W}, \\ y(T) - \mathbb{E}y(T) = \xi - \mathbb{E}\xi, \quad p(0) - \mathbb{E}p(0) = 0, \end{cases} \tag{44}$$

$$R(u^* - \mathbb{E}u^*) + B(\hat{p} - \mathbb{E}p) = 0.$$

From (41), we get

$$y - \mathbb{E}y = \sigma(p - \mathbb{E}p) + (\varphi - \mathbb{E}\varphi), \tag{45}$$

and

$$\mathbb{E}y = \delta\mathbb{E}p + \mathbb{E}\varphi. \tag{46}$$

Taking $\mathbb{E}[\cdot|\mathcal{F}_t^W]$ on both sides of (41), we obtain (30).

(i) Applying Itô's formula to (45), we have

$$\begin{aligned} 0 = {} & [(-\dot{\sigma} + 2A\sigma + Q\sigma^2)(p - \mathbb{E}p) - R^{-1}B^2(\hat{p} - \mathbb{E}p) + C(z - \mathbb{E}z) + \bar{C}(\bar{z} - \mathbb{E}\bar{z}) + (A + \sigma Q)(\varphi - \mathbb{E}\varphi) \\ & - (\gamma_1 - \mathbb{E}\gamma_1)]\mathrm{d}t + [(1+\sigma S)z + \sigma Cp - \gamma_2]\mathrm{d}W + [(1+\sigma\bar{S})\bar{z} + \sigma\bar{C}p]\mathrm{d}\overline{W}, \end{aligned}$$

which yields

$$(-\dot{\sigma} + 2A\sigma + Q\sigma^2)(p - \mathbb{E}p) - R^{-1}B^2(\hat{p} - \mathbb{E}p) + C(z - \mathbb{E}z) + \bar{C}(\bar{z} - \mathbb{E}\bar{z}) + (A+\sigma Q)(\varphi - \mathbb{E}\varphi) - (\gamma_1 - \mathbb{E}\gamma_1) = 0, \tag{47}$$

and

$$\begin{cases}(1+\sigma S)z+\sigma Cp-\gamma_2=0,\\(1+\sigma\bar{S})\bar{z}+\sigma\bar{C}p=0.\end{cases}\tag{48}$$

Assuming the existence of $(1+\sigma S)^{-1}$ and $(1+\sigma\bar{S})^{-1}$, we obtain

$$\begin{cases}z=-(1+\sigma S)^{-1}(\sigma Cp-\gamma_2),\\\bar{z}=-(1+\sigma\bar{S})^{-1}\sigma\bar{C}p,\end{cases}\tag{49}$$

$$\begin{cases}\mathbb{E}z=-(1+\sigma S)^{-1}(\sigma C\mathbb{E}p-\mathbb{E}\gamma_2),\\\mathbb{E}\bar{z}=-(1+\sigma\bar{S})^{-1}\sigma\bar{C}\mathbb{E}p.\end{cases}\tag{50}$$

From the second equality of (49), we get (31). Putting (49) and (50) into (47), and taking $\mathbb{E}[\cdot|\mathcal{F}_t^W]$ on both sides of it, we get

$$\begin{aligned}&[\dot{\sigma}-2A\sigma-Q\sigma^2+C^2(1+\sigma S)^{-1}\sigma+\bar{C}^2(1+\sigma\bar{S})^{-1}\sigma+R^{-1}B^2](\hat{p}-\mathbb{E}p)-(A+\sigma Q)(\hat{\varphi}-\mathbb{E}\varphi)\\&\quad+(\hat{\gamma}_1-\mathbb{E}\gamma_1)-C(1+\sigma S)^{-1}(\hat{\gamma}_2-\mathbb{E}\gamma_2)=0,\end{aligned}$$

which yields (32), and

$$(\hat{\gamma}_1-\mathbb{E}\gamma_1)-(A+\sigma Q)(\hat{\varphi}-\mathbb{E}\varphi)-C(1+\sigma S)^{-1}(\hat{\gamma}_2-\mathbb{E}\gamma_2)=0.\tag{51}$$

(ii) Differentiating to (46), we have

$$\begin{aligned}0=&[-\dot{\delta}+2(A+\tilde{A})\delta+(Q+\tilde{Q})\delta^2-C^2(1+\sigma S)^{-1}\sigma-\bar{C}^2(1+\sigma\bar{S})^{-1}\sigma-(R+\tilde{R})^{-1}(B+\tilde{B})^2]\mathbb{E}p\\&+[(A+\tilde{A})+\delta(Q+\tilde{Q})]\mathbb{E}\varphi+C(1+\sigma S)^{-1}\mathbb{E}\gamma_2-\mathbb{E}\gamma_1,\end{aligned}$$

which implies (33) and

$$[(A+\tilde{A})+(Q+\tilde{Q})\delta]\mathbb{E}\varphi+C(1+\sigma S)^{-1}\mathbb{E}\gamma_2-\mathbb{E}\gamma_1=0.\tag{52}$$

With the help of (51) and (52), $\hat{\varphi}$ and $\mathbb{E}\varphi$ have the forms of (34) and (35), respectively.

(b) Substituting (49) and (50) into (42) and (44), we have

$$\begin{cases}\mathrm{d}\mathbb{E}y=[(A+\tilde{A})\mathbb{E}y+(B+\tilde{B})\mathbb{E}u^*-C(1+\sigma S)^{-1}(\sigma C\mathbb{E}p-\mathbb{E}\gamma_2)-(1+\sigma\bar{S})^{-1}\sigma\bar{C}^2\mathbb{E}p]\mathrm{d}t,\\\mathrm{d}\mathbb{E}p=-[(A+\tilde{A})\mathbb{E}p+(Q+\tilde{Q})\mathbb{E}y]\mathrm{d}t,\\\mathbb{E}y(T)=\mathbb{E}\xi,\quad \mathbb{E}p(0)=-\Phi\mathbb{E}y(0),\end{cases}$$

and

$$\begin{cases}\mathrm{d}(y-\mathbb{E}y)=[A(y-\mathbb{E}y)+B(u^*-\mathbb{E}u^*)-C^2\sigma(1+\sigma S)^{-1}(p-\mathbb{E}p)+C(1+\sigma S)^{-1}(\gamma_2-\mathbb{E}\gamma_2)\\\qquad\qquad-\bar{C}^2\sigma(1+\sigma\bar{S})^{-1}(p-\mathbb{E}p)]\mathrm{d}t-(1+\sigma S)^{-1}(\sigma Cp-\gamma_2)\mathrm{d}W-(1+\sigma\bar{S})^{-1}\sigma\bar{C}p\mathrm{d}\overline{W},\\\mathrm{d}(p-\mathbb{E}p)=-[A(p-\mathbb{E}p)+Q(y-\mathbb{E}y)]\mathrm{d}t-[(1-(1+\sigma S)^{-1}\sigma S)Cp+(1+\sigma S)^{-1}S\gamma_2]\mathrm{d}W\\\qquad\qquad-[1-(1+\sigma\bar{S})^{-1}\sigma\bar{S}]\bar{C}p\mathrm{d}\overline{W},\\y(T)-\mathbb{E}y(T)=\xi-\mathbb{E}\xi,\quad p(0)-\mathbb{E}p(0)=0.\end{cases}$$

Now we conjecture that

$$p(t)=-[\bar{\sigma}(t)(y(t)-\mathbb{E}y(t))+\bar{\delta}(t)\mathbb{E}y(t)+\bar{\varphi}(t)],\quad \bar{\sigma}(0)=\Phi,\quad \bar{\delta}(0)=\Phi,\tag{53}$$

and

$$\mathrm{d}\bar{\varphi}(t)=\bar{\gamma}_1(t)\mathrm{d}t+\bar{\gamma}_2(t)\mathrm{d}W(t),\quad \bar{\varphi}(0)=0,$$

where $\bar{\sigma}$, $\bar{\delta}$ are deterministic and differentiable, and $\bar{\gamma}_1$, $\bar{\gamma}_2$ are $\mathcal{F}_t$-adapted. It is easy to see that

$$\begin{cases}p-\mathbb{E}p=-[\bar{\sigma}(y-\mathbb{E}y)+(\bar{\varphi}-\mathbb{E}\bar{\varphi})],\\\mathbb{E}p=-(\bar{\delta}\mathbb{E}y+\mathbb{E}\bar{\varphi}).\end{cases}\tag{54}$$

Taking $\mathbb{E}[\cdot|\mathcal{F}_t^W]$ on both sides of (53), we get (36).

Similar to the above deduction, from (54), we derive that Eqs. (37) and (38) hold, and

$$\begin{cases} \hat{\bar{\gamma}}_1 - \mathbb{E}\bar{\gamma}_1 = -[A + R^{-1}B^2\bar{\sigma} + \tilde{C}(I+\sigma\tilde{S})^{-1}\tilde{C}^\tau\sigma\bar{\sigma}](\hat{\bar{\varphi}} - \mathbb{E}\bar{\varphi}) - \bar{\sigma}C(1+\sigma S)^{-1}(\hat{\gamma}_2 - \mathbb{E}\gamma_2), \\ \hat{\bar{\gamma}}_2 = (S-\bar{\sigma})(1+\sigma S)^{-1}\hat{\gamma}_2 + [1 + (1+\sigma S)^{-1}\sigma(\bar{\sigma} - S)]C\hat{p}, \\ \mathbb{E}\bar{\gamma}_1 = -\{(A+\tilde{A}) + [(R+\tilde{R})^{-1}(B+\tilde{B})^2 + \tilde{C}(I+\sigma\tilde{S})^{-1}\tilde{C}^\tau\sigma]\bar{\delta}\}\mathbb{E}\bar{\varphi} - \bar{\delta}C(1+\sigma S)^{-1}\mathbb{E}\gamma_2. \end{cases}$$

Obviously, Eqs. (32), (33), (37), and (38) are uniquely solvable, and $\sigma \geqslant 0, \delta \geqslant 0, \bar{\sigma} \geqslant 0, \bar{\delta} \geqslant 0$.

Putting (30) into (36), we get

$$\begin{aligned} \hat{p} = &(1+\bar{\sigma}\sigma)^{-1}[(1+\bar{\delta}\delta)^{-1}(\bar{\delta}\delta - \bar{\sigma}\sigma)\bar{\delta}\mathbb{E}\varphi - \bar{\sigma}(\hat{\varphi} - \mathbb{E}\varphi) - \bar{\delta}\mathbb{E}\varphi] \\ &+ (1+\bar{\sigma}\sigma)^{-1}(1+\bar{\delta}\delta)^{-1}(\bar{\delta}\delta - \bar{\sigma}\sigma)\mathbb{E}\bar{\varphi} - (1+\bar{\sigma}\sigma)^{-1}\hat{\bar{\varphi}}. \end{aligned}$$

Then, $\hat{\bar{\varphi}}$ and $\mathbb{E}\bar{\varphi}$ are the solutions to (39) and (40), respectively. It is easy to get that Eqs. (34), (35), (39), and (40) are uniquely solvable.

Now we give a hypothesis as follows.

Hypothesis 1. $R^{-1}(R+\tilde{R})^{-1}\tilde{R}B(B+\tilde{B}) - (R+\tilde{R})^{-1}\tilde{B}(B+\tilde{B}) - R^{-1}B\tilde{B} \leqslant 0$, which guarantees the existence and uniqueness of (60) below.

3.3 The feedback form of optimal control

Theorem 2. Suppose that Hypothesis 1 holds. Then the optimal control of Problem (MFC) is

$$u^* = -R^{-1}[\tilde{R}(R+\tilde{R})^{-1}(B+\tilde{B}) - \tilde{B}](\bar{\delta}\mathbb{E}y + \mathbb{E}\bar{\varphi}) + R^{-1}B[\bar{\sigma}(\hat{y} - \mathbb{E}y) + \bar{\delta}\mathbb{E}y + \hat{\bar{\varphi}}], \tag{55}$$

where $\bar{\sigma}, \bar{\delta}, \hat{\bar{\varphi}}, \mathbb{E}\bar{\varphi}, \mathbb{E}y$ and $\hat{y}$ are given by (37)–(40), (61), and (62), respectively.

Proof. We will use four steps to prove this result.

Step 1. The unexplicit form of optimal control. According to Lemma 2, if u^* is optimal, then

$$Ru^* + \tilde{R}\mathbb{E}u^* + B\hat{p} + \tilde{B}\mathbb{E}p = 0, \tag{56}$$

where $(y, z, \bar{z}, p)$ satisfies

$$\begin{cases} \mathrm{d}y = (Ay + \tilde{A}\mathbb{E}y + Bu^* + \tilde{B}\mathbb{E}u^* + Cz + \bar{C}\bar{z})\mathrm{d}t + z\mathrm{d}W + \bar{z}\mathrm{d}\overline{W}, \\ \mathrm{d}p = -(Ap + \tilde{A}\mathbb{E}p + Qy + \tilde{Q}\mathbb{E}y)\mathrm{d}t - (Sz + Cp)\mathrm{d}W - (\bar{S}\bar{z} + \bar{C}p)\mathrm{d}\overline{W}, \\ y(T) = \xi, \quad p(0) = -\Phi y(0). \end{cases} \tag{57}$$

Applying Lemma 5.4 in [17] to (57), we get

$$\begin{cases} \mathrm{d}\hat{y} = (A\hat{y} + \tilde{A}\mathbb{E}y + Bu^* + \tilde{B}\mathbb{E}u^* + C\hat{z} + \bar{C}\hat{\bar{z}})\mathrm{d}t + \hat{z}\mathrm{d}W, \\ \mathrm{d}\hat{p} = -(A\hat{p} + \tilde{A}\mathbb{E}p + Q\hat{y} + \tilde{Q}\mathbb{E}y)\mathrm{d}t - (S\hat{z} + C\hat{p})\mathrm{d}W, \\ \hat{y}(T) = \hat{\xi}, \quad \hat{p}(0) = -\Phi\hat{y}(0). \end{cases} \tag{58}$$

Step 2. The unique solvability of (60)**.** Introduce a Riccati equation:

$$\begin{cases} \dot{\sigma} - 2A\sigma - Q\sigma^2 + \tilde{C}(I+\sigma\tilde{S})^{-1}\tilde{C}^\tau\sigma + R^{-1}B^2 = 0, \\ \sigma(T) = 0. \end{cases} \tag{59}$$

Based on Lemma 3, we know that Eq. (59) admits a unique solution $\sigma \geqslant 0$.

Now we introduce an auxiliary MF-FBSDE:

$$\begin{cases} \mathrm{d}\hat{y} = \{A\hat{y} + \tilde{A}\mathbb{E}y + C\hat{z} - [\bar{C}^2(1+\sigma\bar{S})^{-1}\sigma + R^{-1}B^2]\hat{p} \\ \qquad + [R^{-1}(R+\tilde{R})^{-1}\tilde{R}B(B+\tilde{B}) - (R+\tilde{R})^{-1}\tilde{B}(B+\tilde{B}) - R^{-1}B\tilde{B}]\mathbb{E}p\}\mathrm{d}t + \hat{z}\mathrm{d}W, \\ \mathrm{d}\hat{p} = -(A\hat{p} + \tilde{A}\mathbb{E}p + Q\hat{y} + \tilde{Q}\mathbb{E}y)\mathrm{d}t - (S\hat{z} + C\hat{p})\mathrm{d}W, \\ \hat{y}(T) = \hat{\xi}, \quad \hat{p}(0) = -\Phi\hat{y}(0). \end{cases} \tag{60}$$

According to Theorem 4.1 in [21], Eq. (60) has a unique solution $(\hat{y},\hat{z},\hat{p})$ under Hypothesis 1.

Step 3. Eq. (60) **is equivalent to** (58) **with** (56)**.** On the one hand, we prove that the solution $(\hat{y},\hat{z},\hat{p})$ of (60) solves (58). Set

$$Ru^*+\tilde{R}\mathbb{E}u^*+B\hat{p}+\tilde{B}\mathbb{E}p=0.$$

Then Eq. (57) admits a unique solution. We use the decoupling technique to solve (56) and (57), which leads to a derivation of our Riccati equations. According to (31) in Lemma 3, it is easy to see that the solution $(\hat{y},\hat{z},\hat{p})$ of (60) solves (58).

On the other hand, we can also prove that the solution of (58) is a solution of (60).

Step 4. Optimal feedback. Eq. (55) can be obtained by (36) and (56).

Putting (43), (50) and the second equality of (54) into the first equation of (42), it turns into

$$\begin{cases}\dfrac{\mathrm{d}\mathbb{E}y(t)}{\mathrm{d}t}+L_1(t)\mathbb{E}y(t)=-L_2(t),\\ \mathbb{E}y(T)=\mathbb{E}\xi.\end{cases}$$

Solving it, we have

$$\mathbb{E}y(t)=\mathrm{e}^{\int_t^T L_1(r)\mathrm{d}r}\mathbb{E}\xi+\int_t^T L_2(s)\mathrm{e}^{\int_t^s L_1(r)\mathrm{d}r}\mathrm{d}s, \tag{61}$$

where

$$\begin{cases}L_1=-\{(A+\tilde{A})+[(R+\tilde{R})^{-1}(B+\tilde{B})^2+\tilde{C}(I+\sigma\tilde{S})^{-1}\tilde{C}^\tau\sigma]\bar{\delta}\},\\ L_2=-[(R+\tilde{R})^{-1}(B+\tilde{B})^2+\tilde{C}(I+\sigma\tilde{S})^{-1}\tilde{C}^\tau\sigma]\mathbb{E}\bar{\varphi}-C(1+\sigma S)^{-1}\mathbb{E}\gamma_2.\end{cases}$$

Substituting (36) and the second equality of (54) into the first equation of (60), $\hat{y}$ is written as

$$\begin{cases}\mathrm{d}\hat{y}=(L_3\hat{y}+C\hat{z}+L_4)\mathrm{d}t+\hat{z}\mathrm{d}W,\\ \hat{y}(T)=\hat{\xi},\end{cases}$$

whose unique solution is

$$\hat{y}(t)=\mathbb{E}\left[\hat{\xi}\Gamma(T)-\int_t^T L_4(s)\Gamma(s)\mathrm{d}s\bigg|\mathcal{F}_t^W\right], \tag{62}$$

where

$$\begin{cases}\Gamma(s)=\exp\left\{-\int_t^s\left[L_3(r)+\dfrac{1}{2}C^2(r)\right]\mathrm{d}r-\int_t^s C(r)\mathrm{d}W(r)\right\},\\ L_3=A+[R^{-1}B^2+\bar{C}^2\sigma(1+\sigma\bar{S})^{-1}]\bar{\sigma},\\ L_4=L_5\mathbb{E}y-[R^{-1}(R+\tilde{R})^{-1}\tilde{R}B(B+\tilde{B})-(R+\tilde{R})^{-1}\tilde{B}(B+\tilde{B})-R^{-1}B\tilde{B}]\mathbb{E}\bar{\varphi}\\ \qquad+[R^{-1}B^2+\bar{C}^2\sigma(1+\sigma\bar{S})^{-1}]\hat{\bar{\varphi}},\\ L_5=\tilde{A}+[\bar{C}^2(1+\sigma\bar{S})^{-1}\sigma+R^{-1}B^2](\bar{\delta}-\bar{\sigma})\\ \qquad-[R^{-1}(R+\tilde{R})^{-1}\tilde{R}B(B+\tilde{B})-(R+\tilde{R})^{-1}\tilde{B}(B+\tilde{B})-R^{-1}B\tilde{B}]\bar{\delta},\end{cases}$$

and $\mathbb{E}y$ is given by (61).

Note that the results obtained in this section extend those of [3,8,9]. Compared with [20], we obtain a state feedback representation of optimal control of Problem (MFC). In addition, Problem (MFC) distinguishes itself from Problem (LQC), due to the following facts. (i) Unlike (17) in Problem (LQC), we set $p(t)=-[\bar{\sigma}(t)(y(t)-\mathbb{E}y(t))+\bar{\delta}(t)\mathbb{E}y(t)+\bar{\varphi}(t)]$, which simplifies the deductions of Problem (MFC). (ii) Since Problem (MFC) contains mean-field, Hamiltonian system (28) is more complicated than Hamiltonian system (4) of Problem (LQC).

4 Conclusion and outlook

In this paper, we study a kind of general LQ optimal control problem of BSDE under partial information. By the filtering technique and the backward separation approach, we obtain an explicit form of optimal control, which is represented by two Riccati equations and two filtering equations. Then we extend Problem (LQC) to Problem (MFC). We emphasize that Problem (MFC) is not a simple extension of Problem (LQC) because of the fact that Problem (MFC) is an LQ problem with a constrained control set.

Note that, the coefficients in Problem (LQC) and Problem (MFC) are deterministic. Otherwise, there is an immediate difficulty to solve the partial information control problems with stochastic coefficients. One main reason is that $\mathbb{E}[A(t)y(t)|\mathcal{F}_t^W] = A(t)\mathbb{E}[y(t)|\mathcal{F}_t^W]$ does not always hold if $A(t)$ is an $\mathcal{F}_t$-adapted stochastic process. We will focus on the stochastic case in the future.

Acknowledgements This work was supported in part by National Natural Science Foundation of China (Grant Nos. 11371228, 61821004, 61633015). The authors would like to thank the anonymous referees for their valuable comments, which led to a much better version of this paper.

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