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A parameter formula connecting PID and ADRC

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Abstract This paper presents a parameter formula connecting the well-known proportional-integral-derivative (PID) control and the active disturbance rejection control (ADRC). On the one hand, this formula gives a quantitative lower bound to the bandwidth of the extended state observer (ESO) used in ADRC, implying that the ESO is not necessarily of high gain. On the other hand, enlightened by the design of ADRC, a new PID tuning rule is provided, which can guarantee both strong robustness and nice tracking performance of the closed-loop systems under the PID control. Moreover, it is proved that the ESO can be rewritten as a suitable linear combination of the three terms in PID, which can give a better estimate for the system uncertainty than the single integral term in the PID controller. Theoretical results are verified also by simulations in the paper.

Keywords nonlinear uncertain systems, proportional-integral-derivative, PID, active disturbance rejection control, ADRC, extended state observer, ESO

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1 Introduction

Despite of the remarkable progress of modern control theory over the past sixty years, it is widely recognized that the classical proportional-integral-derivative (PID) control is by far the most widely and successfully used controller in engineering systems [1]. However, it has also been pointed out that most of the practical PID loops are poorly tuned, and there is strong evidence that PID controllers remain poorly understood [2]. Therefore, as mentioned in [3], better understanding of the PID control may considerably improve its widespread practice, and so contribute to better product quality. Recently, some theoretical investigations on the global convergence of the PID controller for a basic class of nonlinear uncertain systems are given [4, 5], where some necessary and sufficient conditions for the selection of the PID parameters are provided. These results have rigorously demonstrated in theory that the PID controller does have large-scale robustness with respect to both the uncertain nonlinear structure of the plant and the selection of the controller parameters.

On the other hand, the active disturbance rejection control (ADRC), which was originally proposed by Han in 1998 [6], has attracted more and more attention in both theory and applications [7–11]. This is largely because of its unique ideas and superior performance, which were readily translated into something valuable in engineering practice: the ability in dealing with a vast range of uncertainties and great transient response [12]. Thus, the high level of robustness and the superior transient performance turn out to be the most valuable characteristics of ADRC to make it an appealing solution in dealing with

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real world control problems. However, the research on the theoretical analysis for ADRC was progressing haltingly, especially, on how to tune the ADRC parameters to achieve satisfactory performance of the closed-loop system under practical restrictions.

In this paper, we provide a new parameter formula for the design of PID controller, which is derived from the inherent but rarely noticed relationship between PID and ADRC. This formula is found to be beneficial for the design of both PID and ADRC. On the one hand, this formula gives a quantitative lower bound for the bandwidth of the extended state observer (ESO) used in ADRC, implying that the ESO is not necessarily of high gain, thanks to the parameter manifold provided recently in [4, 5] for the selection of PID parameters for nonlinear uncertain systems. On the other hand, enlightened by the structure of the reduced-order ESO in ADRC, a new and concrete tuning rule for PID parameters is found from the unbounded parameter manifold given in [4, 5], which can guarantee global convergence, strong robustness and nice tracking performance, both for the transient phase and the steady state. Moreover, we show that the ESO actually corresponds to a suitable linear combination of the proportional-integral-derivative terms in PID, and also demonstrate that the ESO can give better estimates for the system uncertainty than the single integral term of the PID controller. Our theoretical results are also verified by some numerical simulations.

The rest of the paper is organized as follows. The detailed problem description is presented in Section 2. Section 3 introduces the main results of this paper. Some simulation verifications of the theoretical analysis are given in Section 4. Finally, the conclusion is presented in Section 5.

2 Problem description

Consider the following second-order nonlinear uncertain system:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = f(x_1, x_2, t) + u(t), \end{cases} \tag{1}$$

where $(x_1, x_2) \in \mathbb{R}^2$ is the system state vector and can be measured, $u(t)$ is the control input, $f(x_1, x_2, t) \in \mathbb{R}$ is an unknown nonlinear function of the state (x_1, x_2) and time t.

The control objective is to make the controlled variable x_1 track a given bounded reference signal $y^*(t)$, which satisfies

$$\lim_{t\to\infty} y^*(t) = y^{**}, \quad \lim_{t\to\infty} \dot{y}^*(t) = 0, \quad \lim_{t\to\infty} \ddot{y}^*(t) = 0,$$

where $\dot{y}^*(t), \ddot{y}^*(t)$ are the first and second derivatives of $y^*(t)$, respectively, and y^{**} is a constant.

To have a nice transient control performance, we introduce the following desired transient process to be tracked by $x_1(t)$, which is shaped from $y^*(t)$ by a stable linear filter:

$$\ddot{r} = -2c_r\dot{r} - c_r^2(r - y^*(t)), \quad r(0) = x_1(0), \quad \dot{r}(0) = x_2(0), \tag{2}$$

where c_r is a parameter for tuning the speed of the transient process.

In this paper, the classical PID controller for the system (1) is described as follows:

$$u_{\text{pid}} = -k_p(x_1 - r) - k_d(x_2 - \dot{r}) - k_i \int_0^t (x_1(\tau) - r(\tau))\mathrm{d}\tau + \ddot{r}, \tag{3}$$

where k_p, k_d, k_i are the controller parameters to be discussed in this paper.

On the other hand, according to the idea of ADRC, f can be viewed as the total disturbance of the system and treated as an extended state of the system to be estimated by an ESO so that it can be compensated for in time.

Because the state x_2 is measurable, the following reduced-order ESO can be designed [12]:

$$\begin{cases} \dot{\xi} = -\omega_o\xi - \omega_o^2 x_2 - \omega_o u, \quad \xi(0) = -\omega_o x_2(0), \\ \hat{f} = \xi + \omega_o x_2, \end{cases} \tag{4}$$

where $\hat{f}$ is the estimation of the total disturbance $f(x_1, x_2, t)$, $\hat{f}(0) = 0$, and ω_o is the only parameter of the ESO (4) to be tuned.

Remark 1. Actually, ω_o of the ESO (4) works as the bandwidth for filtering the total disturbance f. In this paper, ω_o is named as the bandwidth of the ESO (4).

Then, the corresponding ADRC law for tracking the transient process $r(t)$ can be designed as [12]

$$u = -k_{ap}(x_1 - r) - k_{ad}(x_2 - \dot{r}) - \hat{f} + \ddot{r}, \quad k_{ap} > 0,\ k_{ad} > 0, \tag{5}$$

where k_{ap}, k_{ad} are two controller parameters to be tuned. In the ADRC law (5), the term $-\hat{f}$, which is an estimate of f, tries to compensate for the total disturbance, and $\ddot{r}$ is a feedforward term.

In the ADRC (5), the PD feedback parameters k_{ap} and k_{ad} can assign the closed-loop poles of a second-order integrator system such that the system has a desired dynamic response. Hence, the ADRC law (5) aims to achieve the result that the transient performance of the closed-loop system is kept near the one of a second-order linear system whose poles are at the desired positions designed by PD parameters k_{ap} and k_{ad}, even if f is nonlinear and unknown. This is the reason why ADRC has high level of robustness and superior transient performance. Moreover, the physical meaning of ADRC parameters is clear. k_{ap} and k_{ad} are used to assign the poles of an ideal second-order system, which correspond to a desired transient performance. ω_o represents the ability compensating for the unknown f such that the performance of the closed-loop system is close to that of the ideal second-order linear system with the given closed-loop poles determined by k_{ap} and k_{ad}.

How to tune the bandwidth ω_o of ESO such that the unknown f can be timely estimated is a crucial problem for ADRC applications. The existing analysis results on this problem are mostly qualitative and ω_o is usually taken to be large enough. In Section 3, a quantitative lower bound to ω_o will be given by employing the result provided in [4, 5] on the selection of PID parameters.

To this intention, we explore the relationship between the ADRC (5) and the PID (3). Substituting (5) into (4) gives

$$\hat{f} = \omega_o k_{ad}(x_1 - r) + \omega_o(x_2 - \dot{r}) + \omega_o k_{ap} \int_0^t (x_1(\tau) - r(\tau))\mathrm{d}\tau. \tag{6}$$

Therefore, the ADRC law (5) can be rewritten as

$$u = -(k_{ap} + \omega_o k_{ad})(x_1 - r) - (k_{ad} + \omega_o)(x_2 - \dot{r}) - \omega_o k_{ap} \int_0^t (x_1(\tau) - r(\tau))\mathrm{d}\tau + \ddot{r}. \tag{7}$$

Comparing (7) with (3), the ADRC (5) suggests a new tuning law for PID as follows:

$$k_p = k_{ap} + \omega_o k_{ad}, \quad k_d = k_{ad} + \omega_o, \quad k_i = \omega_o k_{ap}. \tag{8}$$

The above simple parameter formula (8), which connects PID and ADRC, is quite meaningful. It suggests the following.

(1) The main results provided in [4, 5] on the selection of PID parameters for guaranteeing the global asymptotic stability of the closed-loop system, may be used to find quantitative lower bounds for the parameters $(k_{ap}, k_{ad}, \omega_o)$ of ADRC (4) and (5). In Section 3, this quantitative lower bound for ADRC will be firstly given. This result will show that the parameters of ADRC are not necessarily of high gain. Moreover, theoretical analysis will demonstrate that the performance of the closed-loop system may be improved by tuning the bandwidth ω_o of ESO.

(2) The formula (8) provides a new and concrete tuning rule for PID parameters rather than taken arbitrarily from a given unbounded parameter manifold as in [4, 5]. Specifically, when the parameters (k_p, k_d, k_i) of PID are tuned by the formula (8), PID law can be divided into two parts:

$$u_{pid} = \underbrace{-k_{ap}(x_1 - r) - k_{ad}(x_2 - \dot{r}) + \ddot{r}}_{u_{pd_0}} \underbrace{-\omega_o k_{ad}(x_1 - r) - \omega_o(x_2 - \dot{r}) - \omega_o k_{ap} \int_0^t (x_1(\tau) - r(\tau))\mathrm{d}\tau}_{u_{pid_f}}. \tag{9}$$

u_{pid_f} is the suitable linear combination of the P part $\omega_o k_{ad}(x_1 - r)$, the D part $\omega_o(x_2 - \dot{r})$ and the I part $\omega_o k_{ap} \int_0^t (x_1(\tau) - r(\tau))\mathrm{d}\tau$, which has the same function of the ESO (4). Then, from the above discussion on the ADRC (5), u_{pid_f} may have a good function of timely estimating and compensating for the unknown disturbance f. Another part u_{pd_0}, consisting of the PD error feedback and the feedforward, will assign the given closed-loop poles to the ideal system. Therefore, the ADRC (5) educes a new PID tuning method, which makes the PID (3) have the excellent capability similar to the advantages of the ADRC (5). In the classical PID controller, only the integral part is designed to estimate f at least in the steady state. However, the new design method (8), obtained from the ADRC (5), implies that u_{pid_f}, the combination of the three terms of P-I-D, can contribute to a better capability of estimating f than the pure integral part. In Section 3, it will be proved that the output $\hat{f}$ of the ESO (4), which corresponds to the combination of the P-I-D three terms (6), has a better capability for estimating the dynamic process of an unknown function f.

3 Main results

Before presenting the main results, we introduce a definition for a class of unknown nonlinear functions f. Define the following function space:

$$\mathcal{F} = \Bigg\{ f \in C^1(\mathbb{R}^2 \times \mathbb{R}^+) \Big| f(x_1, x_2, t) = h(x_1, x_2) + w(t), \left|\frac{\partial h}{\partial x_1}\right| \leqslant L_1, \left|\frac{\partial h}{\partial x_2}\right| \leqslant L_2, \Big|w(t)\Big| \leqslant L_3,$$
$$\Big|\dot{w}(t)\Big| \leqslant L_3, \lim_{t\to\infty} w(t) \text{ exists}, \forall x_1, x_2 \in \mathbb{R}, \forall t \in \mathbb{R}^+ \Bigg\}, \tag{10}$$

where L_1, L_2, L_3 are positive constants, and $C^1(\mathbb{R}^2 \times \mathbb{R}^+)$ denotes the space of all functions from $\mathbb{R}^2 \times \mathbb{R}^+$ to $\mathbb{R}$ which are continuous in (x_1, x_2) and t, with continuous partial derivatives with respect to (x_1, x_2).

3.1 A lower bound to the bandwidth of the ESO with guaranteed performance

As suggested by the formula (8) and the manifold provided in [4, 5] for the selection of PID parameters, a quantitative lower bound to the parameter ω_o of the ESO (4) may be obtained. To this end, we let

$$\Omega = \big\{\omega \in \mathbb{R} \big| n_0\omega^4 + n_1\omega^3 + n_2\omega^2 + n_3\omega + n_4 = 0\big\},$$

where $n_0 = k_{ad}^2$ and

$$\begin{aligned} n_1 &= 2k_{ad}[k_{ad}(k_{ad} - L_2) - L_1], \\ n_2 &= 2k_{ad}(k_{ap} - L_1)(k_{ad} - L_2) + [k_{ad}(k_{ad} - L_2) - L_1]^2 - L_2^2 k_{ap}, \\ n_3 &= 2[k_{ad}(k_{ad} - L_2) - L_1](k_{ap} - L_1)(k_{ad} - L_2) - L_2^2 k_{ap}(k_{ad} - L_2), \\ n_4 &= (k_{ap} - L_1)^2(k_{ad} - L_2)^2. \end{aligned} \tag{11}$$

We can now obtain a lower bound ω_o^* to the ESO parameter ω_o as follows:

$$\begin{aligned} \bar{\omega}_o &= \begin{cases} 0, & \Omega = \emptyset \quad \text{or} \quad \max\{\Omega\} \leqslant 0, \\ \max\{\Omega\}, & \max\{\Omega\} > 0, \end{cases} \\ \omega_o^* &= \max\left\{0, \frac{L_1 - k_{ap}}{k_{ad}}, L_2 - k_{ad}, \bar{\omega}_o\right\}, \end{aligned} \tag{12}$$

where $\emptyset$ represents the empty set.

Remark 2. From (12), it can be seen that ω_o^* only depends on the constants L_1, L_2, k_{ap}, k_{ad} and is irrelevant to the disturbance $w(t)$, initial values and the reference signal $y^*(t)$.

The following theorem shows that ω_o^* can indeed serve as a lower bound to the ESO parameter ω_o.

Theorem 1 (Stability)**.** Consider the ADRC-controlled nonlinear uncertain system (1), (4) and (5), where the nonlinear unknown function $f \in \mathcal{F}$. Then, for any given L_1, L_2, k_{ap}, k_{ad}, the closed-loop system signals will be bounded and satisfy

$$\lim_{t\to\infty} x_1(t) = y^{**}, \quad \lim_{t\to\infty} x_2(t) = 0,$$

for any initial value $(x_1(0), x_2(0)) \in \mathbb{R}^2$ and any y^{**}, as long as the ESO parameter $\omega_o > \omega_o^*$.

The proof of Theorem 1 is given in Appendix A.

Theorem 1 gives a tuning method of ADRC which makes the closed-loop system globally bounded with vanishing tracking error. It can be seen from Theorem 1 that the lower bound to the parameter of the ESO (4), i.e., ω_o^*, can be calculated through (12). This result indicates that the parameter of ESO is not necessarily of high gain.

The next theorem will further show that the tracking performance may be improved by tuning the parameter $\omega_o > \omega_o^*$.

Denote the tracking error as $e(t) = r(t) - x_1(t)$ and the estimation error as $e_f = \hat{f} - f(x_1, x_2, t)$. Note that $\ddot{e}(t) + k_{ad}\dot{e}(t) + k_{ap}e(t) = 0$ is the ideal performance for the tracking of the pole-placement control. Theorem 2 describes the distance between the real closed-loop performance and the ideal one.

Theorem 2 (Pole-placement performance)**.** Consider the ADRC-controlled nonlinear uncertain system (1), (4) and (5), where the nonlinear unknown function $f \in \mathcal{F}$. Then, there exist positive constants $\eta_1 = |e_f(0)|$ and η_2, which depend on $(e_f(0), k_{ap},\ k_{ad}, L_1, L_2, L_3, \dot{r}, \ddot{r})$, such that for all $\omega_o > \omega_o^*$, the closed-loop equation has the following property:

$$|\ddot{e}(t) + k_{ad}\dot{e}(t) + k_{ap}e(t)| = |e_f(t)| \leqslant \eta_1 \mathrm{e}^{-\omega_o t} + \frac{\eta_2}{\omega_o}, \quad t \geqslant 0. \tag{13}$$

The proof of Theorem 2 is given in Appendix A.

From (13), the dynamic response can be divided into two parts. The first part $\eta_1 \mathrm{e}^{-\omega_o t}$, which is related to the initial value of the estimation error e_f, can be rapidly tuned to zero by the parameter ω_o. The second part $\frac{\eta_2}{\omega_o}$ decreases with ω_o. Thus, by increasing the parameter ω_o, the dynamic response of the closed-loop system can be made close to the ideal trajectory.

3.2 A new tuning rule for PID

In this subsection, a new tuning rule for PID controller is proposed, which can guarantee the robustness as well as nice tracking performance of the closed-loop system.

According to the parameter formula (8) and Theorem 2, a new tuning rule for the PID law (3) is given as follows:

$$\begin{aligned} &k_p = k_{ap} + \omega_o k_{ad}, \quad k_d = k_{ad} + \omega_o, \quad k_i = \omega_o k_{ap}, \\ &k_{ap} > 0, \quad k_{ad} > 0, \quad \omega_o > \omega_o^*. \end{aligned} \tag{14}$$

Under the new tuning rule (14), the PID controlled closed-loop system (1) and (3) can share the similar properties as that of the ADRC-based closed-loop system (1), (4) and (5). Thus, according to the advantages of ADRC, the PID controller defined by (3) and (14) also has the ability to timely estimate and compensate for the disturbances and uncertainties, so that the closed-loop system has guaranteed strong robustness and superior tracking performance.

The following corollary can be directly obtained based on Theorem 2.

Corollary 1. Consider the PID-controlled nonlinear uncertain system (1), (3) and (14), where the nonlinear unknown function $f \in \mathcal{F}$. Then, there exist positive constants η_1, η_2, which are the same as in Theorem 2, such that for any given $L_1, L_2, k_{ap}, k_{ad} > 0$, any initial value $(x_1(0), x_2(0)) \in \mathbb{R}^2$ and any setpoint y^{**}, the closed-loop system has the following properties whenever $\omega_o > \omega_o^*$:

(1) $\lim_{t\to\infty} x_1(t) = y^{**}, \lim_{t\to\infty} x_2(t) = 0$;

(2) $|\ddot{e}(t) + k_{ad}\dot{e}(t) + k_{ap}e(t)| \leqslant \eta_1 \mathrm{e}^{-\omega_o t} + \frac{\eta_2}{\omega_o}, t \geqslant 0$.

To further elaborate on the nice performance of the ESO, we note that the integral term $\hat{f}_{\rm I} = k_i \int_0^t (x_1(\tau) - r(\tau)){\rm d}\tau$ of PID controller (3) is usually regarded to have the ability to eliminate the constant disturbance, while in the ADRC frame, the ESO (4) is known to have the ability to timely estimate the dynamic disturbance. The following theorem compares the estimation property of the integral term of PID controller (3) with that of the ESO (4) in the frequency domain.

Define $e_{f_{\rm I}}(t) = \hat{f}_{\rm I} - f(x_1, x_2, t)$. Let $E_f(s), E_{f_{\rm I}}(s), \hat{F}_{\rm I}(s)$ and $\hat{F}(s)$ denote the Laplace transforms of $e_f(t), e_{f_{\rm I}}(t), \hat{f}_{\rm I}(t)$ and $\hat{f}(t)$, respectively. It can be obtained from Theorem 1 that the unknown f on the system trajectories is bounded, and thus the Laplace transform of f exists. Let $F(s)$ denote the Laplace transform of the unknown f, and let $G_{e_f}(s), G_{e_{f_{\rm I}}}(s)$ be the transfer functions from $F(s)$ to $E_f(s)$ and $E_{f_{\rm I}}(s)$, respectively.

Theorem 3. Consider the system (1), (4) and (5) and the system (1), (3), which are connected by the formula (14). For any $f \in \mathcal{F}$, the integral term of PID controller (3) and the ESO (4) have the following properties when $\omega_o > \omega_o^*$:

(1) For any ω, $\frac{|G_{e_f}(i\omega)|}{|G_{e_{f_{\rm I}}}(i\omega)|} < 1$. Moreover, $\lim_{t\to\infty} \frac{e_f(t)}{e_{f_{\rm I}}(t)} = \frac{k_{ap}}{k_{ap}+\omega_o k_{ad}}$.

(2) $\hat{F}_{\rm I}(s) = \frac{k_{ap}}{s^2+k_{ad}s+k_{ap}} \hat{F}(s)$.

The proof of Theorem 3 is given in Appendix A.

From (1) of Theorem 3, it can be seen that the steady estimation error of the ESO (4) is smaller than that of the integral term for the total disturbances f at any frequency. Moreover, it can be seen that the ratio of the steady estimation error is $\frac{k_{ap}}{k_{ap}+\omega_o k_{ad}}$, which decreases with the increase of either ω_o or k_{ad}. The result (2) of Theorem 3 shows that phase-lag of the response of the ESO (4) is smaller than that of the integral term, particularly for rapidly varying disturbances.

Remark 3. From the formula (6), (14) and Theorem 3, it can be concluded that the P-term and the D-term of PID controller also contribute to the estimation and compensation for the disturbances, rather than the single I-term. This seems to be a somewhat striking property that has not been revealed in the investigation of the classical PID before.

4 Simulations

In this section, some simulations are presented to verify the main results of the paper.

In the simulations, the unknown nonlinear f can be one of following cases:

$$\begin{aligned} &{\rm C1}: f(t) = 2x_1 + 6{\rm sin}x_2 + 1, && {\rm C2}: f(t) = 5{\rm cos}x_1 + 2x_2 - 2,\\ &{\rm C3}: f(t) = 3x_1 + 2x_2 - w_1(t), && {\rm C4}: f(t) = 6{\rm cos}x_1{\rm sin}x_2 - w_2(t), \end{aligned}$$

where

$$w_1(t) = \begin{cases} \sin(t), & \text{if } t < 4 \text{ s},\\ \sin(4), & \text{else}, \end{cases} \qquad w_2(t) = \begin{cases} \cos(t), & \text{if } t < 4 \text{ s},\\ \cos(4), & \text{else}. \end{cases}$$

The initial values of the state are $x_1(0) = 0, x_2(0) = 0$, and the reference signal is $y^*(t) = 2$. Then, the desired transient process $r(t)$ is designed as follows:

$$\ddot{r} = -2c_r\dot{r} - c_r^2(r-2),\ c_r = 5,\ r(0) = 0,\ \dot{r}(0) = 0. \tag{15}$$

According to Theorem 1, it can be calculated that $\omega_o^* = 6$. In the simulations, the parameters in the formula (14) are chosen as $k_{ap} = 4, k_{ad} = 4, \omega_o = 10$.

The simulation results in the C1 case are shown in Figures 1 and 2.

Figure 1 is the response curves of the state (x_1, x_2) based on the ADRC (4) and (5) (the blue line), and the PID controller (3) and (14) (the red dash line). It is shown that under the parameter formula (14), the closed-loop system (1) and (3) and the closed-loop system (1), (4) and (5) have the same dynamic responses.

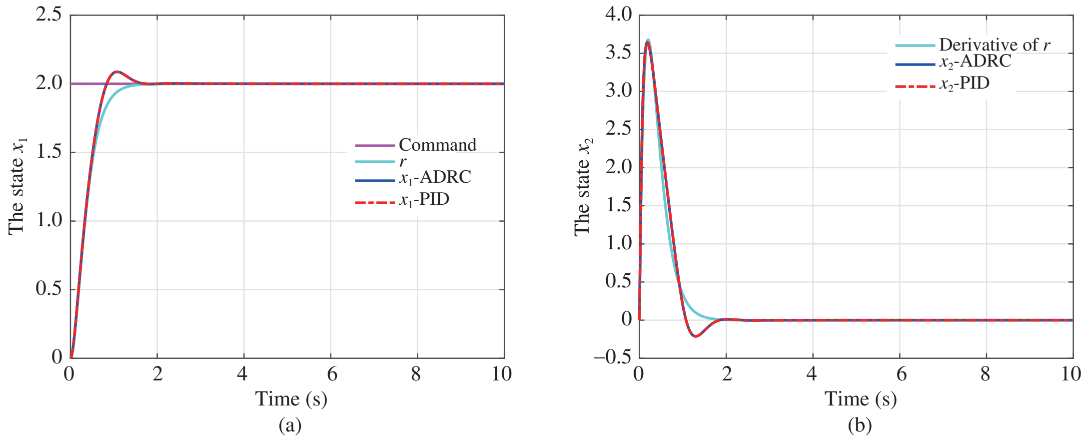

Figure 1 The response curves of the state (a) x_1 and (b) x_2.

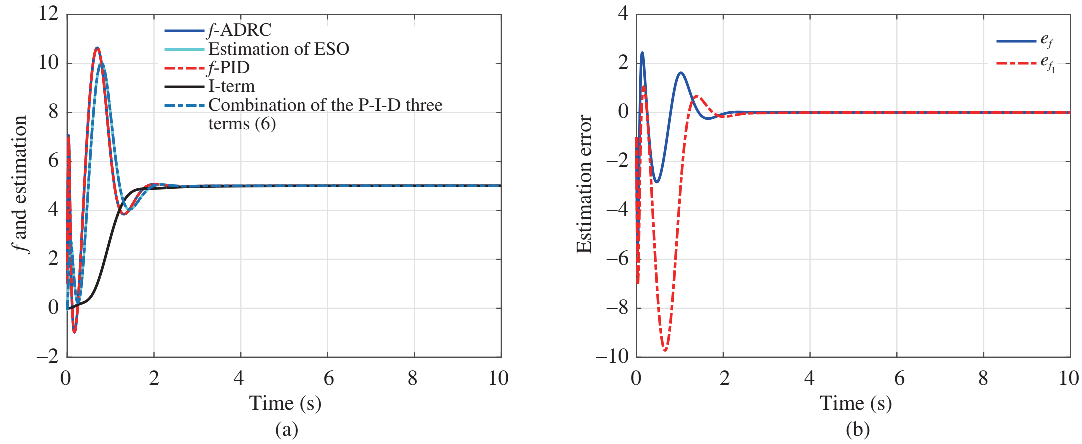

Figure 2 The estimations (a) and the estimation errors (b) of the disturbance f.

Figure 2(a) are the estimations of the unknown f based respectively on the ESO (4), the I-term of PID controller (3) and (14), and the combination of the P-I-D three terms (6) in the PID controller (9). In Figure 2(a), the blue line represents f of the closed-loop system (1), (4) and (5), and the red dash line represents f of the closed-loop system (1), (3) and (14). Figure 2(b) is the estimation errors of f based on the ESO (4) (the blue line) and the I-term of the PID controller (9) (the red dash line), respectively.

Figure 2(a) shows that compared to the I-term, the ESO (4) can track the unknown disturbance more quickly. Moreover, it also verifies that the combination of the P-I-D three terms (6) has the same capability for estimating the unknown f as that of the ESO (4). From Figure 2(b), it can be seen that the estimation error of the ESO (4) is smaller than that of the I-term, although both gradually tend to zero.

Figure 3 shows the response curves of the state x_1, based on the ADRC (4) and (5), and the PID controller (3) and (14) under difference situations C1–C4, respectively. It can be seen that both the PID (3) and ADRC (4) and (5), which are connected by the formula (14), can deal with a vast range of uncertainties in the sense that the closed-loop system has strong robustness and great tracking performance.

Figure 4 shows the curves of the unknown f and the combination of the P-I-D three terms (6) in the cases C1–C4, respectively. It indicates that the combination of the P-I-D three terms (6), which has the same function of the ESO (4), has the ability to timely estimate a large range of the unknown dynamic function f. This is the reason why both the PID (3) and ADRC (4) and (5), tuned according to the formula (14), have the capability to keep the tracking performance close to the ideal one $r(t)$.

To verify the results of Corollary 1, Figure 5 shows the curves of $\ddot{e}(t) + k_{ad}\dot{e}(t) + k_{ap}e(t)$ based on the PID controller (3), (14), when the parameter ω_o varies. It shows that the increase of the parameter ω_o

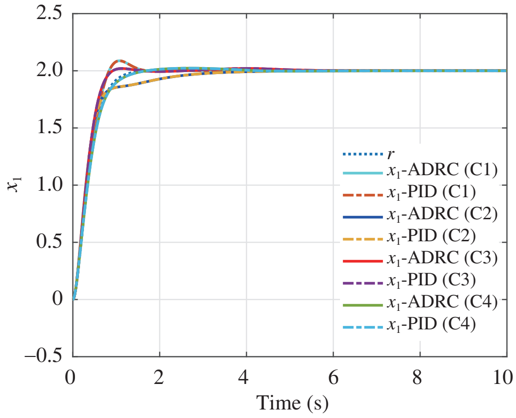

Figure 3 The response curves of the state x_1 under C1–C4.

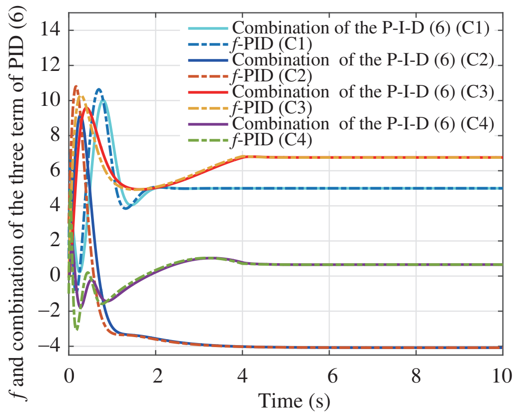

Figure 4 The curves of disturbances f and combination of the P-I-D three terms (6), under C1–C4.

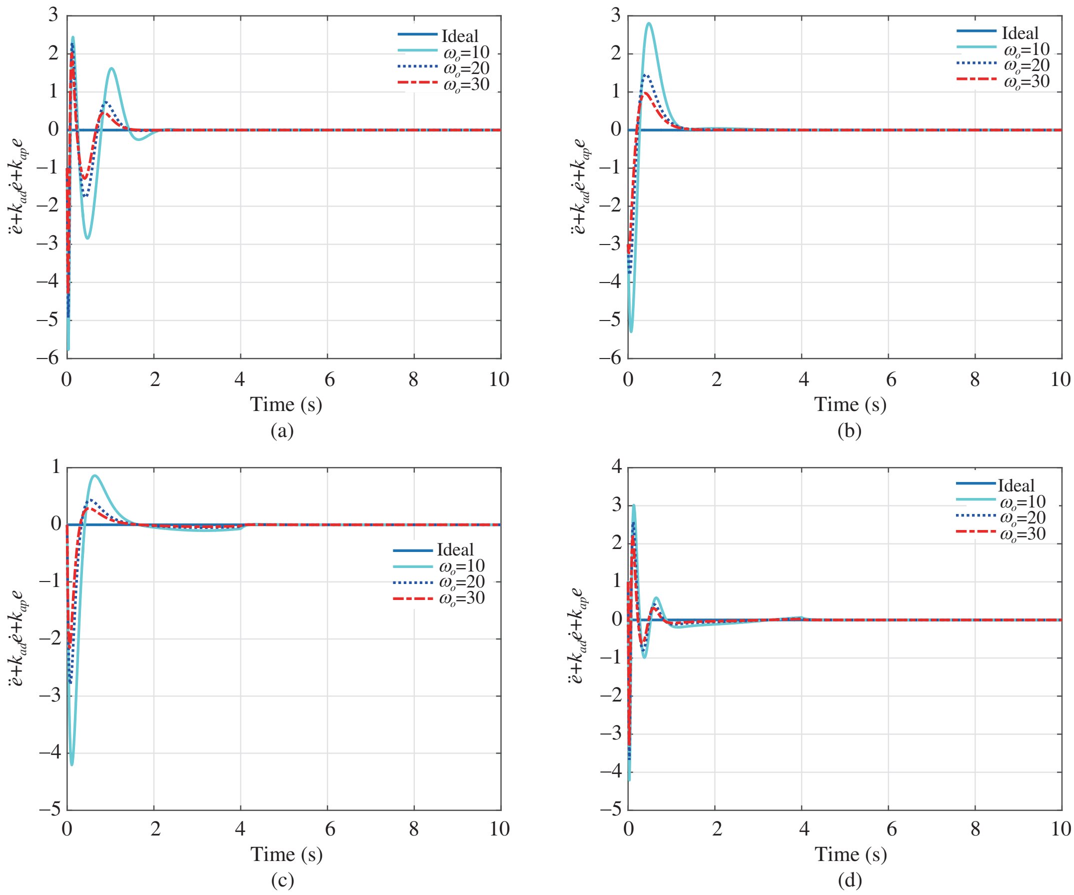

Figure 5 The curves of $\ddot{e}(t) + k_{ad}\dot{e}(t) + k_{ap}e(t)$ under (a) C1, (b) C2, (c) C3, and (d) C4.

will lead to the decrease of $|\ddot{e}(t) + k_{ad}\dot{e}(t) + k_{ap}e(t)|$, i.e., the dynamic response of the tracking error e can be tuned close to the ideal trajectory by only increasing ω_o.

5 Conclusion

In this paper, a new and simple parameter formula connecting PID and ADRC is discovered and is shown to be able to improve the design of both controllers significantly. Firstly, a quantitative lower bound ω_o^*

to ω_o, the bandwidth of the ESO (4), is provided for guaranteeing the global boundness and asymptotic tracking of the ADRC. This result shows that the design parameters of the ADRC are not necessarily of high gain. It is further proved that the tracking performance of the closed-loop system can be improved by increasing $\omega_o > \omega_o^*$. Then, a novel PID controller tuning rule, suggested by the design of ADRC, is provided. Because the PID controller (3) has the same function of the ADRC (5) by this tuning rule, the robustness and excellent tracking performance of the closed-loop system can be guaranteed. Finally, it is demonstrated that the steady estimation error of the ESO (4) is less and phase-lag of the response of the ESO (4) is smaller than that of the single integral term of PID controller (3). We believe that the tuning formula provided in this paper has wide applicability in practical control systems.

The formula proposed in this paper is not only limited to the second-order system. Similar parameter formulas can also be deduced for the multi-input multi-output systems and higher order systems. This belongs to future investigation.

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Appendix A

Proof of Theorem 1

Substituting (5) into (4), by Laplace transform, we obtain

$$\hat{F}(s) = \omega_o k_{ad}(X_1(s) - R(s)) + \omega_o s(X_1(s) - R(s)) + \frac{\omega_o k_{ap}(X_1(s) - R(s))}{s}, \tag{A1}$$

where $X_1(s), R(s)$ are the Laplace transforms of the state $x_1(t)$ and the transient process $r(t)$, respectively. Taking the inverse Laplace transform for (A1), we have

$$\hat{f} = \omega_o k_{ad}(x_1 - r) + \omega_o(x_2 - \dot{r}) + \omega_o k_{ap} \int_0^t (x_1(\tau) - r(\tau))\mathrm{d}\tau. \tag{A2}$$

Hence, the control law (5) can be rewritten as

$$u = -k_p(x_1 - r) - k_d(x_2 - \dot{r}) - k_i \int_0^t (x_1(\tau) - r(\tau))\mathrm{d}\tau + \ddot{r}, \tag{A3}$$

where $k_p = k_{ap} + \omega_o k_{ad}, k_d = k_{ad} + \omega_o, k_i = \omega_o k_{ap}$.

Because $\lim_{t\to\infty} w(t)$ exists, which can be denoted by a constant c, we have

$$e_i(t) = \int_0^t e(\tau)\mathrm{d}\tau + \frac{h(y^{**}, 0) + c}{k_i}, \quad e_d(t) = \dot{e}(t), \quad g(e, e_d) = -h(y^{**} - e, -e_d) + h(y^{**}, 0).$$

Based on the definition of $\mathcal{F}$, it can be seen that $g \in \mathcal{F}$ and $g(0,0)=0$. Then the closed-loop system (1) and (A3) turns into

$$\begin{cases} \dot{e}_i = e, \\ \dot{e} = e_d, \\ \dot{e}_d = -k_i e_i - k_p e - k_d e_d + g(e, e_d) + \Delta(t), \end{cases} \tag{A4}$$

where $\Delta(t) = g(e + y^{**} - r, e_d - \dot{r}) - g(e, e_d) + c - w(t)$. By the mean value theorem, it can be obtained that, for any $t \in \mathbb{R}^+$, there is

$$g(e + y^{**} - r, e_d - \dot{r}) - g(e, e_d) = \frac{\partial g}{\partial e}\Big|_{(\bar{e}, \bar{e}_d)} (y^{**} - r) - \frac{\partial g}{\partial e_d}\Big|_{(\bar{e}, \bar{e}_d)} \dot{r}, \tag{A5}$$

where $\bar{e} = e + \theta(y^{**} - r), \bar{e}_d = e_d - \theta\dot{r}, \theta \in (0,1)$. Because $f \in \mathcal{F}$, it can be deduced that $|\Delta| \leqslant L_1|y^{**} - r| + L_2|\dot{r}| + c + L_3$. Moreover, $(0,0,0)$ is an equilibrium of (A4), when t approaches infinity.

Following the analysis in [4, 5], we let

$$b(e) = \begin{cases} \dfrac{g(e,0)}{e}, & e \neq 0, \\ \dfrac{\partial g}{\partial e}(0,0), & e = 0, \end{cases} \quad \text{and} \quad a(e, e_d) = \begin{cases} \dfrac{g(e, e_d) - g(e,0)}{e_d}, & e_d \neq 0, \\ \dfrac{\partial g}{\partial e_d}(e, 0), & e_d = 0, \end{cases}$$

and then $g(e, e_d)$ can be expressed as

$$g(e, e_d) = b(e)e + a(e, e_d)e_d.$$

By the mean value theorem again and the definition of $\mathcal{F}$, we have $|b(e)| \leqslant L_1, |a(e, e_d)| \leqslant L_2$ for all e, e_d.

Hence, the closed-loop system (A4) can be rewritten as

$$\begin{cases} \dot{e}_i = e, \\ \dot{e} = e_d, \\ \dot{e}_d = -k_i e_i - \phi(e)e - \psi(e, e_d)e_d + \Delta(t), \end{cases} \tag{A6}$$

where $\phi(e) = k_p - b(e), \psi(e, e_d) = k_d - a(e, e_d)$. By the fact that $\omega_o > \frac{L_1 - k_{ap}}{k_{ad}}, \omega_o > L_2 - k_{ad}$, there exist $\phi(e) \geqslant k_p - L_1 > 0$ and $\psi(e, e_d) \geqslant k_d - L_2 > 0$.

To construct a Lyapunov function, we consider the following matrix P:

$$P = \frac{1}{2}\begin{bmatrix} \mu k_i & k_i & \delta \\ k_i & k_p - L_1 + \mu k_d & \mu \\ \delta & \mu & 1 \end{bmatrix}, \quad \mu = \frac{2((k_p - L_1)(k_d - L_2) + k_i)}{4(k_p - L_1) + L_2^2}, \tag{A7}$$

where δ satisfies $0 < \delta < \frac{2\mu k_i}{(k_p - L_1) + \mu k_d}$. We will first show that the matrix P is positive definite.

Based on the definition of ω_o^* and the assumption $\omega_o > \omega_o^*$, it can be obtained that

$$(k_p - L_1)(k_d - L_2) - k_i > L_2\sqrt{k_i(k_d - L_2)}; \tag{A8}$$

thus,

$$\begin{aligned} &(k_p - L_1)(k_d - L_2) - k_i > 0, \\ &[(k_p - L_1)(k_d - L_2) - k_i]^2 > L_2^2 k_i (k_d - L_2). \end{aligned} \tag{A9}$$

Because

$$\mu - k_d + L_2 = \frac{-2(k_p - L_1)(k_d - L_2) + 2k_i - L_2^2(k_d - L_2)}{4(k_p - L_1) + L_2^2} < 0, \tag{A10}$$

and

$$4(-\mu + k_d - L_2)(-k_i + \mu(k_p - L_1)) - \mu^2 L_2^2 = \frac{4[[(k_p - L_1)(k_d - L_2) - k_i]^2 - L_2^2 k_i (k_d - L_2)]}{4(k_p - L_1) + L_2^2} > 0, \tag{A11}$$

we have

$$-k_i + \mu(k_p - L_1) > 0. \tag{A12}$$

Then, based on (A10)–(A12), the following three inequalities can be verified:

$$\mu k_i > 0, \quad \begin{vmatrix} \mu k_i & k_i \\ k_i & k_p - L_1 + \mu k_d \end{vmatrix} = k_i[\mu(k_p - L_1 + \mu k_d) - k_i] > 0, \tag{A13}$$

and

$$\begin{vmatrix} \mu k_i & k_i & \delta \\ k_i & k_p - L_1 + \mu k_d & \mu \\ \delta & \mu & 1 \end{vmatrix} > k_i(\mu(k_p - L_1) + \mu^2 k_d - k_i - \mu^3) > 0. \tag{A14}$$

Thus, the matrix P is positive definite.

We are now in a position to consider the following Lyapunov function [4,5]:

$$V(e_i,e,e_d)=[e_i,e,e_d]P[e_i,e,e_d]^{\mathrm{T}}+\int_0^e(L_1-b(s))s\mathrm{d}s. \tag{A15}$$

Because $0\leqslant\int_0^e(L_1-b(s))s\mathrm{d}s\leqslant L_1e^2$, from (A15), we have

$$[e_i,e,e_d]P[e_i,e,e_d]^{\mathrm{T}}\leqslant V(e_i,e,e_d)\leqslant[e_i,e,e_d]P_0[e_i,e,e_d]^{\mathrm{T}}, \tag{A16}$$

where

$$P_0=\frac{1}{2}\begin{bmatrix}\mu k_i & k_i & \delta\\ k_i & k_p+L_1+\mu k_d & \mu\\ \delta & \mu & 1\end{bmatrix}.$$

It can be deduced that

$$\lambda_{\min}(P)\,\|[e_i,e,e_d]\|^2\leqslant V\leqslant\lambda_{\max}(P_0)\,\|[e_i,e,e_d]\|^2, \tag{A17}$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ are the minimum and maximum eigenvalues of the corresponding matrix, respectively.

The time derivative of $V(e_i,e,e_d)$ along the trajectories of (A6) is

$$\dot V(e_i,e,e_d)=-[e_i,e,e_d]W(e,e_d)[e_i,e,e_d]^{\mathrm{T}}+[\delta,\mu,1][e_i,e,e_d]^{\mathrm{T}}\Delta, \tag{A18}$$

where

$$W(e,e_d)=\begin{bmatrix}\delta k_i & \dfrac{\delta\phi(e)}{2} & \dfrac{\delta\psi(e,e_d)}{2}\\ \dfrac{\delta\phi(e)}{2} & -k_i+\mu\phi(e) & -\dfrac{\mu a(e,e_d)+\delta}{2}\\ \dfrac{\delta\psi(e,e_d)}{2} & -\dfrac{\mu a(e,e_d)+\delta}{2} & -\mu+\psi(e,e_d)\end{bmatrix}.$$

Let

$$Q(e,e_d)=\begin{bmatrix}-k_i+\mu\phi(e) & -\dfrac{\mu a(e,e_d)}{2}\\ -\dfrac{\mu a(e,e_d)}{2} & -\mu+\psi(e,e_d)\end{bmatrix}.$$

From (A10)–(A12), it is easy to verify that $Q(e,e_d)$ is positive definite. Because $\phi(e),\psi(e,e_d)$ are bounded, based on the same analysis as in [5], there exists a constant δ^*, such that the matrix $W(e,e_d)$ is positive definite when $\delta<\delta^*$. Thus δ can be defined by $\delta<\min\{\frac{2\mu k_i}{k_p-L_1+\mu k_d},\delta^*\}$.

From (A18), we have

$$\dot V\leqslant-\lambda_{\min}(W)\|[e_i,e,e_d]\|^2+c_0\|[e_i,e,e_d]\||\Delta|\leqslant-c_1V+c_2\sqrt{V}|\Delta|, \tag{A19}$$

where $c_0=\max\{\delta,\mu,1\}, c_1=\frac{\lambda_{\min}(W)}{\lambda_{\max}(P_0)}, c_2=\frac{c_0}{\sqrt{\lambda_{\min}(P)}}$. Because $|\Delta|$ is bounded, there exists a constant M_0, such that $|\Delta|\leqslant M_0$. Then, it can be obtained that

$$\sqrt{V}\leqslant\mathrm{e}^{\frac{-c_1t}{2}}\sqrt{V(e_i(0),e(0),e_d(0))}+\frac{c_2M_0}{c_1}\left(1-\mathrm{e}^{\frac{-c_1t}{2}}\right)\leqslant M_1, \tag{A20}$$

where M_1 is a constant. Thus, e_i,e and e_d are bounded. Because $\lim_{t\to\infty}r(t)=y^{**}$, $\lim_{t\to\infty}\dot r(t)=0$ and $\lim_{t\to\infty}w(t)=c$, we know that $\lim_{t\to\infty}\Delta=0$. Thus, for any $\varepsilon>0$, there exists $T>0$, such that for any $t>T$, there is $\dot V\leqslant-c_1V+\varepsilon$.

In conclusion, $\lim_{t\to\infty}x_1(t)=y^{**}$ and $\lim_{t\to\infty}x_2(t)=0$. The proof of Theorem 1 is completed.

Proof of Theorem 2

Because the dynamic equations of e and e_d can be written as follows:

$$\begin{cases}\dot e=e_d,\\ \dot e_d=-k_{ap}e-k_{ad}e_d+e_f,\end{cases} \tag{A21}$$

we get $|\ddot e(t)+k_{ad}\dot e(t)+k_{ap}e(t)|=|e_f(t)|$. Based on the proof of Theorem 1, when $\omega_o>\omega_o^*$, e_f,e and e_d are bounded. Let

$$P_2=\begin{bmatrix}\dfrac{1+k_{ap}}{2k_{ad}}+\dfrac{k_{ad}}{2k_{ap}} & \dfrac{1}{2k_{ap}}\\ \dfrac{1}{2k_{ap}} & \dfrac{1+k_{ap}}{2k_{ad}k_{ap}}\end{bmatrix}.$$

Consider the following Lyapunov function:

$$V_2(e,e_d)=\begin{bmatrix}e & e_d\end{bmatrix}P_2\begin{bmatrix}e\\ e_d\end{bmatrix}. \tag{A22}$$

The time derivative of $V_2(e,e_d)$ along the trajectories of (A21) is

$$\dot V_2=-e^2-e_d^2+\begin{bmatrix}e & e_d\end{bmatrix}\begin{bmatrix}\dfrac{1}{k_{ap}}\\ \dfrac{1+k_{ap}}{k_{ad}k_{ap}}\end{bmatrix}e_f\leqslant-\frac{V_2}{\lambda_{\max}(P_2)}+\frac{\max\left(\frac{1}{k_{ap}},\frac{1+k_{ap}}{k_{ad}k_{ap}}\right)\sqrt{V_2}|e_f|}{\sqrt{\lambda_{\min}(P_2)}}. \tag{A23}$$

From (A23), it can be seen that

$$\sqrt{V_2} \leqslant \frac{\max\left(\frac{1}{k_{ap}}, \frac{1+k_{ap}}{k_{ad}k_{ap}}\right)\lambda_{\max}(P_2)\sup|e_f|}{\sqrt{\lambda_{\min}(P_2)}},$$

i.e.,

$$\|[e, e_d]\| \leqslant \frac{\max\left(\frac{1}{k_{ap}}, \frac{1+k_{ap}}{k_{ad}k_{ap}}\right)\lambda_{\max}(P_2)\sup|e_f|}{\lambda_{\min}(P_2)}.$$

If $\sqrt{V_2} > \frac{\max\left(\frac{1}{k_{ap}}, \frac{1+k_{ap}}{k_{ad}k_{ap}}\right)\lambda_{\max}(P_2)\sup|e_f|}{\sqrt{\lambda_{\min}(P_2)}}$, we know that $\dot{V}_2 < 0$.

From (4), the estimation error e_f satisfies the following equation:

$$\dot{e}_f = -\omega_o e_f - \dot{f}, \tag{A24}$$

where

$$\dot{f} = \frac{\partial f}{\partial x_2}k_{ap}e + \left(\frac{\partial f}{\partial x_2}k_{ad} - \frac{\partial f}{\partial x_1}\right)e_d - \frac{\partial f}{\partial x_2}e_f + \frac{\partial f}{\partial x_1}\dot{r} + \frac{\partial f}{\partial x_2}\ddot{r} + \frac{\partial f}{\partial t}.$$

Because $f \in \mathcal{F}$, we have

$$|\dot{f}| \leqslant \gamma_1|e| + \gamma_2|e_d| + L_2|e_f| + \gamma_3, \tag{A25}$$

where $\gamma_1 = L_2k_{ap}, \gamma_2 = |L_2k_{ad} - L_1|, \gamma_3 = L_1\dot{r} + L_2\ddot{r} + L_3$.

Consider the following Lyapunov function:

$$V_1(e_f) = \frac{1}{2}e_f^2, \tag{A26}$$

the time derivative of $V_1(e_f)$ along the trajectories of (A24) is

$$\dot{V}_1 = -\omega_o e_f^2 - e_f\dot{f} \leqslant -\omega_o e_f^2 + |e_f||\dot{f}| \leqslant -(\omega_o - L_2)e_f^2 + |e_f|(\gamma_1|e| + \gamma_2|e_d| + \gamma_3). \tag{A27}$$

Let

$$\gamma_4 = \frac{\max\left(\frac{1}{k_{ap}}, \frac{1+k_{ap}}{k_{ad}k_{ap}}\right)\lambda_{\max}(P_2)}{\lambda_{\min}(P_2)}, \quad \omega_{o1}^* = L_2 + \gamma_4(\gamma_1 + \gamma_2) + 1.$$

Next, it will be proved that when $\omega_o > \omega_{o1}^*, \sup|e_f| \leqslant \max\{e_f(0), \gamma_3\}$. From (A27), it can be seen that

$$\dot{V}_1 < -(\gamma_4(\gamma_1 + \gamma_2) + 1)e_f^2 + |e_f|(\gamma_4(\gamma_1 + \gamma_2)\sup|e_f| + \gamma_3). \tag{A28}$$

When $|e_f| > \max\{e_f(0), \gamma_3\}$, there is $\dot{V}_1 < 0$; thus, $|e_f| \leqslant \max\{e_f(0), \gamma_3\}$. Moreover, there exists a constant γ_5, which does not depend on ω_o, such that $|\dot{f}| \leqslant \gamma_5$.

If $\omega_o^* \leqslant \omega_o \leqslant \omega_{o1}^*$, then by Theorem 1 and (A19), we know that there exists a constant γ_6, such that $\gamma_6 = \sup_{\omega_o}|\dot{f}|$.

Let $M_{\dot{f}} = \max\{\gamma_5, \gamma_6\}$. Based on the above analysis, it can be deduced that

$$\sqrt{V_1} \leqslant \mathrm{e}^{-\omega_o t}\sqrt{V_1(e_f(0))} + \frac{\sqrt{2}M_{\dot{f}}}{2\omega_o}(1 - \mathrm{e}^{-\omega_o t}); \tag{A29}$$

thus

$$|e_f| \leqslant \mathrm{e}^{-\omega_o t}|e_f(0)| + \frac{M_{\dot{f}}}{\omega_o}(1 - \mathrm{e}^{-\omega_o t}) \leqslant \eta_1\mathrm{e}^{-\omega_o t} + \frac{\eta_2}{\omega_o}, \tag{A30}$$

where $\eta_1 = |e_f(0)|, \eta_2 = M_{\dot{f}}$, which are irrelevant to ω_o. Thus, Eq. (13) is obtained and the proof of Theorem 2 is completed.

Proof of Theorem 3

Based on the proofs of Theorems 1 and 2, when $\omega_o > \omega_o^*$, both f and $\dot{f}$ are bounded. Because the PID controller defined by (3) and (14) is equivalent to the ADRC (4) and (5), the total disturbance f of the closed-loop system defined by (1), (3) and (14) is the same as that of the closed-loop system (1), (4) and (5). Based on (A21), we have

$$\ddot{e} + k_{ad}\dot{e} + k_{ap}e = e_f. \tag{A31}$$

Taking the Laplace transform for (A31), we get

$$E(s) = \frac{1}{s^2 + k_{ad}s + k_{ap}}E_f(s), \tag{A32}$$

where $E(s)$ is the Laplace transform of $e(t)$ and $E_f(s)$ is the Laplace transform of $e_f(t)$. Based on (A24), we know that

$$\dot{e}_{f_{\rm I}} = -k_ie + \dot{e}_f + \omega_o e_f. \tag{A33}$$

Taking the Laplace transform for (A33), it can be obtained from (A32) that

$$E_{f_{\rm I}}(s) = \frac{s^2 + k_ds + k_p}{s^2 + k_{ad}s + k_{ap}}E_f(s), \tag{A34}$$

where $E_{f_{\rm I}}(s)$ is the Laplace transform of $e_{f_{\rm I}}(t)$. Taking the Laplace transform for (A24), it follows that

$$E_f(s) = G_{e_f}(s)F(s), \quad G_{e_f}(s) = \frac{s}{s + \omega_o}, \tag{A35}$$

where $F(s)$ is the Laplace transform of f. Thus,

$$E_{f_{\rm I}}(s) = G_{e_{f_{\rm I}}}(s)F(s), \quad G_{e_{f_{\rm I}}}(s) = \frac{s^3 + k_d s^2 + k_p s}{(s+\omega_o)(s^2 + k_{ad}s + k_{ap})}. \tag{A36}$$

From (A35) and (A36), we obtain

$$\frac{|G_{e_f}(i\omega)|^2}{|G_{e_{f_{\rm I}}}(i\omega)|^2} = \frac{(k_{ap} - \omega^2)^2 + k_{ad}^2\omega^2}{(k_p - \omega^2)^2 + k_d^2\omega^2} < 1. \tag{A37}$$

Because $\lim_{t\to\infty} \frac{e_f(t)}{e_{f_{\rm I}}(t)} = \lim_{s\to 0} \frac{sE_f(s)}{sE_{f_{\rm I}}(s)}$, it is easy to see that

$$\lim_{t\to\infty} \frac{e_f(t)}{e_{f_{\rm I}}(t)} = \frac{k_{ap}}{k_{ap} + \omega_o k_{ad}}.$$

Therefore, the property (1) of Theorem 3 is obtained.

Based on (1) and (5), it can be obtained that

$$\dddot{\hat{f}}_{\rm I} = -k_d \ddot{\hat{f}}_{\rm I} - k_p \dot{\hat{f}}_{\rm I} - k_i \hat{f}_{\rm I} + k_i f. \tag{A38}$$

Taking the Laplace transform for (A38), we have

$$\hat{F}_{\rm I}(s) = \frac{\omega_o k_{ap}}{(s+\omega_o)(s^2 + k_{ad}s + k_{ap})} F(s), \tag{A39}$$

where $\hat{F}_{\rm I}(s)$ is the Laplace transform of $\hat{f}_{\rm I}(t)$ and $F(s)$ is the Laplace transform of f.

Based on (4), the dynamical equation of $\hat{f}$ can be written as follows:

$$\dot{\hat{f}} = -\omega_o(\hat{f} - f). \tag{A40}$$

Taking the Laplace transform for (A40), we have

$$\hat{F}(s) = \frac{\omega_o}{s + \omega_o} F(s), \tag{A41}$$

where $\hat{F}(s)$ is the Laplace transform of $\hat{f}(t)$. Thus,

$$\hat{F}_{\rm I}(s) = \frac{k_{ap}}{s^2 + k_{ad}s + k_{ap}} \hat{F}(s),$$

which is the property (2) of Theorem 3. Hence, the proof of Theorem 3 is completed.