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Finite-time control for a class of hybrid systems via quantized intermittent control

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Abstract This paper considers the finite-time drive-response synchronization of stochastic nonlinear systems consisting of continuous-time and discrete-time subsystems. To save communication resources and reduce control cost, quantized controllers, which only work on continuous-time intervals, are designed. Owing to the hybrid characteristics of continuous- and discrete-time subsystems, existing finite-time stability theorems are not applicable. By developing novel analytical techniques, three criteria are derived to guarantee the finite-time synchronization. Moreover, the settling time is explicitly estimated. It is shown that the settling time is dependent not only on the control gains and systems' initial conditions, but also on the control width and uncontrolled width. Numerical examples demonstrate the effectiveness of the theoretical analysis.

Keywords hybrid system, finite-time synchronization, quantized intermittent control, stochastic perturbation

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1 Introduction

In recent decades, much attention has been devoted to synchronization owing to its wide applications in engineering such as secure communication, biological systems, and information science [1–5]. Thus far, the synchronization problem for various systems has been considered, including pure continuous-time systems and pure discrete-time systems [6, 7], impulsive systems [8, 9], systems with discontinuous state on the right-hand side [10, 11], and switched systems [12, 13]. In the literature, synchronization can be classified into two categories: asymptotic synchronization and finite-time synchronization. Asymptotic synchronization means that the states of coupled systems can achieve synchronization only when time goes to infinity. Compared with asymptotic synchronization, finite-time synchronization exhibits many advantages because it can not only synchronize coupled systems within a settling time, but also has the properties of better robustness and disturbance rejection [2, 8, 11, 14–17]. Therefore, this paper considers finite-time synchronization for a class of switched systems.

Switched systems are a special kind of hybrid systems. In the previous literature, many researchers studied the synchronization or stabilization problem for switched systems that are composed of either continuous-time subsystems [14–21] or discrete-time ones [22,23]. For instance, authors in [21] investigated

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asymptotic synchronization of continuous-time switched complex networks with impulsive effects, while the authors in [23] considered asymptotic synchronization of discrete-time neural networks with Markovian jump topologies. Specially, authors in [20] investigated finite-time stabilization of continuous-time switched dynamical networks. Recently, another kind of switched system composed of hybrid continuous-time and discrete-time subsystems has been proposed [24–28]. Switched systems that are composed of both continuous-time and discrete-time dynamical subsystems can be found in many applications. A typical example is a continuous-time plant controlled either by a physically implemented regulator or by a digitally implemented one together with a switching rule between them [24]. However, studying the stability or stabilization of such switched systems is not easy work. It was shown in [24–26] that, when all continuous-time subsystems are Hurwitz stable and all discrete-time subsystems are Schur stable, the switched systems can reach exponential stability under arbitrary switching. Since then, the stability and consensus (a special kind of synchronization) of hybrid continuous-time and discrete-time systems has been extended to control theory [27, 28], where exponential consensus and finite-time consensus were considered in [27, 28], respectively. Unfortunately, to the best of our knowledge, all the results concerning stabilization and consensus of this kind of switched systems require that both continuous-time subsystems and discrete-time subsystems are stable or both of them have to be controlled. It seems that this strict condition cannot be removed. Moreover, synchronization of switched systems composed of hybrid continuous-time and discrete-time subsystems has not been reported in the literature, let alone finite-time synchronization of this kind of switched systems. This inspires us to investigate finite-time synchronization of nonlinear systems consisting of continuous-time and discrete-time subsystems, where only continuous-time subsystems are controlled.

From control theory and a practical point of view, controlling a system all the time is costly and is unnecessary. Therefore, intermittent control, which is more effective than continuous-time state feedback control, is proposed because the transmitted signals are inevitably affected by external perturbations, which weaken or intermittently interrupt the signals. Moreover, intermittent control can also greatly reduce control cost and the amount of the transmitted information. In the literature, there are many results concerning synchronization and stabilization by using intermittent control [6, 29–34]. However, to the best of our knowledge, most previous studies achieve only asymptotic synchronization or stabilization when an intermittent control scheme is concerned. Recently, some authors have paid much attention to finite-time synchronization of continuous-time nonlinear systems by using intermittent control. For example, the authors in [35] considered finite-time synchronization of coupled continuous-time nonlinear systems by using a kind of intermittent control. However, their controller is only semi-intermittent; that is, a part of the controller must work all the time. It is extremely difficult to realize finite-time synchronization via strict intermittent control even for continuous-time linear systems because existing control schemes and analysis techniques have difficulty dealing with the increment of the Lyapunov function on uncontrolled time-intervals. As for hybrid continuous-time and discrete-time systems, signals for the discrete-time subsystems are usually weak, hence the controllers may be inactive on discrete-time intervals. In this case, the results in [28] were not applicable anymore. This motivates us to consider finite-time synchronization of hybrid continuous-time and discrete-time systems via intermittent control, where only the continuous-time dynamical system is controlled. This is a challenging problem because classical finite-time stability theorem and analytical techniques cannot be utilized, and hence novel methods must be developed.

Quantized control has received considerable attention in recent years because it can save communication channels and bandwidth [19, 20, 35–39]. Asymptotic synchronization of a continuous system was realized in [38] by quantized sampled-data control. The authors in [20] investigated finite-time stabilization of switched continuous-time dynamical networks via quantized control, where the subsystems are all continuous and controlled. Yet, authors seldom consider finite-time synchronization of nonlinear systems via strict quantized intermittent control. Obviously, finite-time synchronization of hybrid continuous-time and discrete-time systems by using strict quantized intermittent control has many advantages such as reducing control cost, saving communication channels and bandwidth, as well as enhancing the properties of robustness and disturbance rejection. By designing a new quantized intermittent controller and developing novel analytical methods, this issue is studied in the present paper.

On the other hand, stochastic perturbations caused by environmental uncertainties are unavoidable in the process of information transmission. It is well known that small perturbations will lead to drastic changes to the state of chaotic systems, making synchronization difficult to accomplish. In the literature, stabilization and synchronization of coupled systems with stochastic perturbations have been extensively studied. However, most of the existing literature considers stochastic perturbations in continuous-time [18, 29, 31, 40–43] and in discrete-time [37, 44–47], separately. Specially, finite-time synchronization of continuous-time complex networks with stochastic perturbations was studied in [48]. However, the results in [48] cannot be applied to systems with hybrid continuous-time and discrete-time stochastic perturbations, because the analytical methods for continuous-time and discrete-time stochastic perturbations are completely different. Therefore, it is necessary to consider nonlinear systems consisting of continuous-time and discrete-time subsystems with both continuous-time and discrete-time stochastic perturbations.

Motivated by the above discussion, this paper considers finite-time drive-response synchronization of hybrid continuous-time and discrete-time systems with stochastic perturbations by designing a quantized intermittent controller (QIC). The main contributions are listed as follows:

(i) A new hybrid stochastic continuous-time and stochastic discrete-time system is given by systematically discretizing a continuous-time system. Compared with existing results concerning synchronization of pure continuous-time systems or pure discrete-time systems, results in this paper are more practical.

(ii) Considering the fact that, in many cases, the signals transmitted from a discrete-time drive system are weak or even unavailable, only the QIC with a logarithmic quantizer is added to continuous-time subsystem, while the discrete-time system is not controlled. Moreover, different from existing results which are obtained by using periodic intermittent control [6, 29–35], our control scheme does not need to be periodic, which makes our results more general and practical.

(iii) It is the first time that finite-time synchronization is realized by using a strict QIC, which takes full advantages of both finite-time control and quantized control.

(iv) Without using the classical finite-time stability theorem proposed in [49], a set of novel analytical methods are established that can deal with not only the increment of the Lyapunov function on uncontrolled time intervals and the uncertainties induced by quantization but also the hybrid stochastic perturbations.

(v) For the convenience of engineering technicians in practical applications, three finite-time synchronization criteria are provided under different conditions. Moreover, optimization algorithms are given to estimate the settling time. It is interesting to find that both control width and rest width (the time-interval width without control) of the drive-response systems have an important influence on the settling time. Most of existing results including those obtained in [28, 35, 48] are essentially extended.

The rest of this paper is organized as follows. Section 2 proposes a nonlinear system consisting of continuous-time and discrete-time dynamics and introduces some necessary definitions and assumptions. Section 3 develops three new criteria for the finite-time synchronization. Section 4 provides numerical examples to illustrate the effectiveness of the theoretical results. Finally, Section 5 is the conclusion.

Notations. Sometimes, the arguments of a function or a matrix will be omitted in the analysis when there is no confusion.

$\mathbb{R}$: the set of real numbers.

$\mathbb{R}^n$: the n-dimensional Euclidean space.

$\mathbb{N}^+$: the set of nonnegative integers.

T: matrix or vector transpose.

I_n: the $n \times n$ identity matrix.

P: probability measure.

$\|\cdot\|$: the Euclidean norm in $\mathbb{R}^n$.

$\lambda_{\max}(A)$: the largest eigenvalue of A, if A is a symmetric matrix.

$\mathrm{sym}(A)$: the symmetric matrix of matrix A, which is $(A + A^{\mathrm{T}})/2$.

Ω: the canonical space generated by $w(t)$.

$\mathcal{F}$: the associated σ-algebra generated by $\{w(t)\}$ with the probability measure $\mathcal{P}$.

E$\{\cdot\}$: mathematical expectation operator with respect to the given probability measure P.

$(\Omega, \mathcal{F}, \mathcal{P})$: a complete probability space with a natural filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ generated by $\{w(s) : 0 \leqslant s \leqslant t\}$.

2 Problem formulation and preliminaries

Generally, a continuous-time nonlinear system is described as

$$\mathrm{d}x(t) = [Ax(t) + Bf(x(t))]\mathrm{d}t, \tag{1}$$

where $x(t) = (x_1(t), \dots, x_n(t))^{\mathrm{T}} \in \mathbb{R}^n$ is the state of the system, $f(x(t)) = (f_1(x(t)), \dots, f_n(x(t)))^{\mathrm{T}}$ is a nonlinear continuous vector function, and $A, B \in \mathbb{R}^{n\times n}$ are real weighted matrices. Assume $x(0) = \varphi(0) = (\varphi_1(0), \dots, \varphi_n(0))^{\mathrm{T}} \in \mathbb{R}^n$ is the initial value of the system (1).

By using a discretization method [47], the discrete-time form of (1) is

$$x(t+1) = \tilde{A}x(t) + \tilde{B}f(x(t)), \tag{2}$$

where $\tilde{A} = I + Ah, \tilde{B} = hB$, h is a known positive constant.

Taking (1) and (2) as a whole to be the drive system, correspondingly, we consider the response hybrid system as

$$\mathrm{d}y(t) =[Ay(t) + Bf(y(t)) + U(t)]\mathrm{d}t + \sigma(t, e(t))\mathrm{d}\omega(t), \tag{3}$$

and

$$y(t+1) = \tilde{A}y(t) + \tilde{B}f(y(t)) + \tilde{\sigma}(t, e(t))w(t), \tag{4}$$

where $y(t) = (y_1(t), \dots . y_n(t))^{\mathrm{T}} \in \mathbb{R}^n$, $e(t) = y(t) - x(t)$, $U(t) \in \mathbb{R}^n$ is the controller to be designed, $\sigma(t, e(t)) \in \mathbb{R}^{n\times n}$ and $\tilde{\sigma}(t, e(t)) \in \mathbb{R}^{n\times n}$ are noise intensity function matrices, and $\omega(t) = (\omega_1(t), \dots, \omega_n(t))^{\mathrm{T}} \in \mathbb{R}^n$ and $w(t) = (w_1(t), \dots, w_n(t))^{\mathrm{T}} \in \mathbb{R}^n$ are n-dimensional continuous-time and discrete-time Brownian motions defined on $(\Omega, \mathcal{F}, \mathcal{P})$ and $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{P}})$, respectively. The initial condition of system (3) is $y(0) = \phi(0) = (\phi_1(0), \dots, \phi_n(0))^{\mathrm{T}} \in \mathbb{R}^n$.

To study the synchronization of the hybrid system consisting of (3) and (4), the following assumption condition is given, which is inspired by [28]. One can refer to [24–28] for more details for hybrid continuous-time and discrete-time subsystems.

Assumption 1. There exists a time sequence $\{t_k, k \in \mathbb{N}^+\}$ satisfying $0 = t_0 < t_1 < \cdots < t_k < \cdots$, $\lim_{k\to+\infty} t_k = +\infty, k \in \mathbb{N}^+$. Moreover, the continuous-time subsystems (1) and (3) are activated when $t \in (t_{2k}, t_{2k+1}]$, and the discrete-time subsystems (2) and (4) are activated when $t \in (t_{2k+1}, t_{2k+2}]$.

Remark 1. Assumption 1 means that the subsystems (1) and (2) work alternately. Systems satisfying Assumption 1 can be found in many applications. A typical example is a continuous-time plant controlled either by a physically implemented regulator or by a digitally implemented one together with a switching rule between them [24–26]. The operation mechanism of (1) and (2) is that: when $t \in (t_{2k}, t_{2k+1}]$, subsystem (1) evolves with initial condition $x(t_{2k})$ $(x(t_0) = \varphi(0))$, and then subsystem (2) works over $(t_{2k+1}, t_{2k+2}]$ with initial condition $x(t_{2k+1})$, where $t = 1, 2, \dots, m-1$ with $t_{2k+1} + mh = t_{2k+2}$, $m \in \mathbb{N}^+$. Note that the sampling period h is usually larger than the time-step in the simulation for continuous-time system.

Remark 2. Note that only the continuous-time subsystem (3) is controlled. Such an intermittent control scheme is reasonable because the signal transmitted from a discrete-time drive system is weak or even unavailable. In the previous literature, results in [24–26] require that both the continuous-time and discrete-time subsystems are stable, and results in [27, 28] require that both the continuous-time and discrete-time subsystems need to be controlled to achieve consensus. However, if the signal is not available in the discrete-time case, all the results in [24–28] are not applicable. Therefore, it is necessary to develop new analytical techniques to investigate intermittent synchronization.

As a standard assumption, the white noises $\mathrm{d}\omega_i(t)$ and $w_i(t)$ are independent of $\mathrm{d}\omega_j(t)$ and $w_j(t)$, respectively, for $i \neq j$. Moreover, the mathematical expectations of $\mathrm{d}\omega_i(t)$ and $w_i(t)$ have the following properties:

$$\begin{aligned}&\mathrm{E}(\mathrm{d}\omega_i(t))=0,\ \mathrm{E}((\mathrm{d}\omega_i(t))^2)=\mathrm{d}t,\ \mathrm{E}(\mathrm{d}\omega_i(t)\mathrm{d}\omega_j(t))=0,\ i\neq j,\\&\mathrm{E}(w_i(t))=0,\ \mathrm{E}(w_i^2(t))=1,\ \mathrm{E}(w_i(t)w_j(t))=0,\ \mathrm{E}(w_i(\tilde{t})w_i(t))=0,\ i\neq j,\ \tilde{t}\neq t.\end{aligned}$$

Remark 3. Stochastic perturbations on both the continuous-time and discrete-time subsystems are considered in this paper, while stochastic perturbations are not considered in [24–28]. Moreover, stochastic perturbations are usually considered for pure continuous-time systems [40–43, 48] or pure discrete-time systems [45, 46] separately in the literature. To the best of our knowledge, authors seldom consider both discrete-time and continuous-time stochastic perturbations in one system. Therefore, considering stochastic perturbations on both continuous-time and discrete-time subsystems for the synchronization makes the obtained results more practical than existing ones.

The following assumptions are required for our study.

Assumption 2. There exists a constant L such that $\|f(x)-f(y)\| \leqslant L\|x-y\|$ for all $x, y \in \mathbb{R}^n$, $x \neq y$.

Assumption 3. There exist constants α_1 and α_2 such that

$$\begin{aligned}&\mathrm{tr}(\sigma^{\mathrm{T}}(t,e(t))\sigma(t,e(t)))\leqslant\alpha_1\|e(t)\|^2,\\&\mathrm{tr}(\tilde{\sigma}^{\mathrm{T}}(t,e(t))\tilde{\sigma}(t,e(t)))\leqslant\alpha_2\|e(t)\|^2.\end{aligned}$$

Definition 1 (Finite-time mean-square synchronization)**.** The hybrid response system (3) and (4) is said to be synchronized in mean square with the hybrid drive system (1) and (2) in finite time if there exists a positive constant $T>0$ (T depends on the initial condition of subsystems (1) and (3)) such that $\lim_{t\to T}\mathrm{E}\{\|e(t)\|^2\}=0$ and $\mathrm{E}\{\|e(t)\|^2\}\equiv 0$ for $t>T$, where T is called the settling time.

Subtracting (3) and (4) from (1) and (2), respectively, leads to the error system:

$$\begin{cases}\mathrm{d}e(t)=[Ae(t)+Bg(e(t))+U(t)]\mathrm{d}t+\sigma(t,e(t))\mathrm{d}\omega(t),\\ e(t+1)=\tilde{A}e(t)+\tilde{B}g(e(t))+\tilde{\sigma}(t,e(t))w(t),\end{cases}\tag{5}$$

where $g(e(t))=f(y(t))-f(x(t))$.

Design the QIC as follows:

$$U(t)=-rq(e(t))-\eta_k\mathrm{sign}(q(e(t))),\quad t\in(t_{2k},t_{2k+1}],\tag{6}$$

where $k\in\mathbb{N}^+$, r and η_k are positive constants to be determined, $q(e(t))=(q(e_1(t)),\dots,q(e_n(t)))^{\mathrm{T}}$ with $q(\cdot):\mathbb{R}\to\Gamma$ being a quantizer, and $\Gamma=\{\pm\Gamma_i:\Gamma_i=\rho^i\Gamma_0, 0<\rho<1, i=0,\pm1,\pm2,\dots\}\cup\{0\}$ with a sufficiently large constant $\Gamma_0>0$. The quantizer $q(v)$, $\forall v\in\mathbb{R}$ is defined as follows:

$$q(v)=\begin{cases}\Gamma_i, & \text{if } \frac{1}{1+\delta}\Gamma_i<v\leqslant\frac{1}{1-\delta}\Gamma_i,\\ 0, & \text{if } v=0,\\ -q(-v), & \text{if } v<0,\end{cases}\tag{7}$$

where $\delta=\frac{1-\rho}{1+\rho}$. It is found from (7) that there exists $\Delta\in[-\delta,\delta]$ such that $q(v)=(1+\Delta)v$, $\forall v\in\mathbb{R}$. Let $\Lambda(t)=\mathrm{diag}(\Lambda_1(t),\dots,\Lambda_n(t))$ with $\Lambda_i(t)\in[-\delta,\delta], i=1,2,\dots,n$, and then

$$q(e(t))=(I_n+\Lambda(t))e(t).\tag{8}$$

3 Main results

According to Definition 1, finite time synchronization of the hybrid system (3) and (4) with the hybrid drive system (1) and (2) is equivalent to stabilizing the error system (5) in a finite time. With the aid of the QIC (6), three cases are considered. By developing new analytical techniques, three synchronization criteria are given through strict mathematical proofs. Note that existing finite-time stability theorems are not applicable to the hybrid systems.

Define $t_c(k) = t_{2k+1} - t_{2k}$, $t_d(k) = t_{2k+2} - t_{2k+1}$, $k \in \mathbb{N}^+$, $d_0 = (\mathrm{E}\{e^{\mathrm{T}}(0)e(0)\})^{1/2}$. Let $\lambda_1 = \lambda_{\max}(\mathrm{sym}(A))$, $\lambda_2 = \|B\|$, $\tilde{\lambda}_1 = \lambda_{\max}(\tilde{A}^{\mathrm{T}}\tilde{A})$, $\tilde{\lambda}_2 = \|\tilde{A}^{\mathrm{T}}\tilde{B}\|$, and $\tilde{\lambda}_3 = \lambda_{\max}(\tilde{B}^{\mathrm{T}}\tilde{B})$. The following is our first main result.

Theorem 1. Suppose that Assumptions 1–3 hold. For a given constant $\varepsilon > 0$, take a sequence $\{M_k : M_k \geqslant \max\{0, d_0 - k\varepsilon\}, k \in \mathbb{N}^+\}$. Then the hybrid system (3) and (4) can be synchronized in mean square with the hybrid system (1) and (2) in a finite settling time T if the following inequalities hold:

$$r \geqslant \left(\lambda_1 + \lambda_2 L + \frac{1}{2}\alpha_1\right)/(1-\delta), \tag{9}$$

$$\eta_k \geqslant (M_k(\gamma^{t_d(k+1)} - 1) + \varepsilon)/(\gamma^{t_d(k+1)} t_c(k)), \tag{10}$$

$$\gamma \triangleq (\tilde{\lambda}_1 + \tilde{\lambda}_2 L + \tilde{\lambda}_3 L^2 + \alpha_2)^{1/2} > 1. \tag{11}$$

Moreover, the settling time T is estimated as

$$T \leqslant t_{2N+1}, \tag{12}$$

where N is the minimum integer of the problem:

$$N = \left\{k \in \mathbb{N}^+ : \min_k \ k \ \ \text{s.t.}\ d_0 - (k-1)\varepsilon - \eta_k t_c(k) \leqslant 0\right\}. \tag{13}$$

Proof. Consider $V(t) = e^{\mathrm{T}}(t)e(t)$ and let $z(t) = \mathrm{E}\{V(t)\}^{1/2} = \mathrm{E}\{e^{\mathrm{T}}(t)e(t)\}^{1/2}$. By Itô's differential rule, the stochastic derivative of $V(t)$ along the error system (5) is

$$\mathrm{d}V(t) = \mathcal{L}V(t)\mathrm{d}t + 2e^{\mathrm{T}}(t)\sigma(t, e(t))\mathrm{d}w(t), \quad t \in (t_{2k}, t_{2k+1}], \tag{14}$$

where the weak infinitesimal operator $\mathcal{L}$ is given by

$$\begin{aligned}\mathcal{L}V(t) = {} & e^{\mathrm{T}}(t)(A + A^{\mathrm{T}})e(t) + 2e^{\mathrm{T}}(t)Bg(e(t)) - 2re^{\mathrm{T}}(t)q(e(t)) \\ & - 2\eta_k e^{\mathrm{T}}(t)\mathrm{sign}(q(e(t))) + \mathrm{tr}(\sigma^{\mathrm{T}}(t, e(t))\sigma(t, e(t))).\end{aligned} \tag{15}$$

Using Assumptions 2 and 3, one has

$$\begin{cases} e^{\mathrm{T}}(t)\mathrm{sym}(A)e(t) \leqslant \lambda_1 e^{\mathrm{T}}(t)e(t) = \lambda_1 V(t), \\ e^{\mathrm{T}}(t)Bg(e(t)) \leqslant \lambda_2 e(t)g(e(t)) \leqslant \lambda_2 LV(t), \\ \varrho^{\mathrm{T}}(t, \mathrm{tr}(\sigma^{\mathrm{T}}(t, e(t))\sigma(t, e(t))) \leqslant \alpha_1 V(t). \end{cases} \tag{16}$$

In view of (6) and (8), it is found that

$$\begin{aligned}-e^{\mathrm{T}}(t)q(e(t)) &= -e^{\mathrm{T}}(t)(I + \Delta(t))e(t) \\ &= -V(t) - e^{\mathrm{T}}(t)\Delta(t)e(t) \\ &\leqslant V(t)(-1 + \delta).\end{aligned} \tag{17}$$

It follows from (8) and $\delta < 1$ that $e(t) > 0$ (or < 0) if and only if $q(e(t)) > 0$ (or < 0). Hence,

$$e^{\mathrm{T}}(t)\mathrm{sign}(q(e(t))) = e^{\mathrm{T}}(t)\mathrm{sign}(e(t)) = \|e(t)\|_1 \geqslant V^{1/2}(t). \tag{18}$$

The inequalities (14)–(18) lead to

$$\begin{aligned}\mathrm{d}V(t) \leqslant {} & [2\lambda_1 V(t) + 2\lambda_2 LV(t) - 2rV(t) + 2\delta rV(t) + \alpha_1 V(t) - 2\eta_k V(t)^{1/2}]\mathrm{d}t \\ & + 2e^{\mathrm{T}}(t)\sigma(t, e(t))\mathrm{d}w(t), \quad t \in [t_{2k}, t_{2k+1}).\end{aligned} \tag{19}$$

Taking the expectation on both sides of (19) yields

$$\mathrm{dE}\{V(t)\}/\mathrm{d}t \leqslant (2\lambda_1 + 2\lambda_2 L + \alpha_1 + 2r\delta - 2r)\mathrm{E}\{V(t)\} - 2\eta_k \mathrm{E}\{V(t)\}^{1/2},$$

which can be simplified as

$$\mathrm{d}z(t)/\mathrm{d}t \leqslant \left(\lambda_1 + \lambda_2 L + \frac{\alpha_1}{2} + r\delta - r\right) z(t) - \eta_k, \quad t \in [t_{2k}, t_{2k+1}). \tag{20}$$

It follows from (9) that

$$\dot{z}(t) \leqslant -\eta_k, \quad t \in [t_{2k}, t_{2k+1}),\ k \in \mathbb{N}^+, \tag{21}$$

by which one has, for $k \in \mathbb{N}^+$,

$$z(t_{2k+1}) - z(t_{2k}) \leqslant -\eta_k (t_{2k+1} - t_{2k}). \tag{22}$$

On the other hand, when $t \in (t_{2k+1}, t_{2k+2}]$, one has from the second equation of (5) that

$$\begin{aligned}
&V(t+1)\\
&\quad= e^{\mathrm{T}}(t+1)e(t+1)\\
&\quad= [\tilde{A}e(t) + \tilde{B}g(e(t)) + \tilde{\sigma}(t,e(t))w(t)]^{\mathrm{T}}[\tilde{A}e(t) + \tilde{B}g(e(t)) + \tilde{\sigma}(t,e(t))w(t)]\\
&\quad= e^{\mathrm{T}}(t)\tilde{A}^{\mathrm{T}}\tilde{A}e(t) + 2e^{\mathrm{T}}(t)\tilde{A}^{\mathrm{T}}\tilde{B}g(e(t)) + 2e^{\mathrm{T}}(t)\tilde{A}^{\mathrm{T}}\rho(t,e(t))w(t)\\
&\qquad+ g^{\mathrm{T}}(e(t))\tilde{B}^{\mathrm{T}}\tilde{B}g(e(t)) + 2g^{\mathrm{T}}(e(t))\tilde{B}^{\mathrm{T}}\tilde{\sigma}(t,e(t))w(t)\\
&\qquad+ w^{\mathrm{T}}(t)\tilde{\sigma}^{\mathrm{T}}(t,e(t))\tilde{\sigma}(t,e(t))w(t).
\end{aligned} \tag{23}$$

Taking the expectation on both sides of (23) gives

$$\begin{aligned}
&\mathrm{E}\{V(t+1)\}\\
&\quad= \mathrm{E}\{e^{\mathrm{T}}(t+1)e(t+1)\}\\
&\quad= \mathrm{E}\{e^{\mathrm{T}}(t)\tilde{A}^{\mathrm{T}}\tilde{A}e(t)\} + \mathrm{E}\{2e^{\mathrm{T}}(t)\tilde{A}^{\mathrm{T}}\tilde{B}g(e(t))\} + \mathrm{E}\{g^{\mathrm{T}}(e(t))\tilde{B}^{\mathrm{T}}\tilde{B}g(e(t))\}\\
&\qquad+ \mathrm{E}\{w^{\mathrm{T}}(t)\tilde{\sigma}^{\mathrm{T}}(t,e(t))\tilde{\sigma}(t,e(t))w(t)\}\\
&\quad= \mathrm{E}\{e^{\mathrm{T}}(t)\tilde{A}^{\mathrm{T}}\tilde{A}e(t)\} + \mathrm{E}\{2e^{\mathrm{T}}(t)\tilde{A}^{\mathrm{T}}\tilde{B}g(e(t))\} + \mathrm{E}\{g^{\mathrm{T}}(e(t))\tilde{B}^{\mathrm{T}}\tilde{B}g(e(t))\}\\
&\qquad+ \mathrm{E}\{\mathrm{tr}(\tilde{\sigma}^{\mathrm{T}}(t,e(t))\tilde{\sigma}(t,e(t)))\}.
\end{aligned} \tag{24}$$

In view of Assumption 3, the following inequalities are true:

$$\begin{cases}
e^{\mathrm{T}}(t)\tilde{A}^{\mathrm{T}}\tilde{A}e(t) \leqslant \tilde{\lambda}_1 V(t),\\
e^{\mathrm{T}}(t)\tilde{A}^{\mathrm{T}}\tilde{B}g(e(t)) \leqslant \tilde{\lambda}_2 L V(t),\\
g^{\mathrm{T}}(e(t))\tilde{B}^{\mathrm{T}}\tilde{B}g(e(t)) \leqslant \tilde{\lambda}_3^2 L^2 V(t),\\
\mathrm{tr}(\tilde{\sigma}^{\mathrm{T}}(t,e(t))\tilde{\sigma}(t,e(t))) \leqslant \alpha_2 V(t).
\end{cases} \tag{25}$$

Substituting (25) into (24) and considering (11) produce

$$\mathrm{E}\{V(t+1)\} \leqslant (\tilde{\lambda}_1 + \tilde{\lambda}_2 L + \tilde{\lambda}_3 L^2 + \alpha_2)\mathrm{E}\{V(t)\} = \gamma^2 \mathrm{E}\{V(t)\},$$

which is equivalent to

$$z(t+1) \leqslant \gamma z(t),\ t \in [t_{2k+1}, t_{2k+2}),\ k \in \mathbb{N}^+, \tag{26}$$

and further means

$$z(t_{2k+2}) \leqslant \gamma^{t_{2k+2} - t_{2k+1}} z(t_{2k+1}),\ k \in \mathbb{N}^+. \tag{27}$$

Next, mathematical induction method is adopted to prove that

$$z(t_{2k+2}) \leqslant \max\{0, z(0) - (k+1)\varepsilon\}, \quad k \in \mathbb{N}^+. \tag{28}$$

Step 1. When $k = 0$, it can be derived from (22) that $z(t_1) \leqslant z(0) - \eta_0 t_c(0)$. If $z(0) - \eta_0 t_c(0) \leqslant 0$, then $z(t_1) = 0$ owing to $z(t) \geqslant 0$; hence $z(t_2) \leqslant \gamma^{t_d(1)} z(t_1) = 0 \leqslant \max\{0, z(0) - \varepsilon\}$. So Eq. (28) holds. If $z(0) - \eta_0 t_c(0) > 0$, then it follows from (10) and (27) that

$$\begin{aligned} z(t_2) &\leqslant \gamma^{t_d(1)} z(t_1) \\ &\leqslant \gamma^{t_d(1)} \left(z(0) - \frac{M_0(\gamma^{t_d(1)} - 1) + \varepsilon}{\gamma^{t_d(1)} t_c(0)} t_c(0) \right) \\ &\leqslant \gamma^{t_d(1)} z(0) - M_0(\gamma^{t_d(1)} - 1) - \varepsilon \\ &\leqslant z(0) + z(0)(\gamma^{t_d(1)} - 1) - M_0(\gamma^{t_d(1)} - 1) - \varepsilon \\ &\leqslant z(0) - \varepsilon \\ &\leqslant \max\{0, z(0) - \varepsilon\}, \end{aligned} \tag{29}$$

where $\max\{0, d_0\} \leqslant M_0$ has been used. Therefore, Eq. (28) holds for $k = 0$.

Step 2. Suppose that Eq. (28) holds for $k > 0$. Then it is required to prove Eq. (28) holds for $k + 1$.

It is found from (22) that

$$z(t_{2k+1}) \leqslant z(t_{2k}) - \eta_k t_c(k). \tag{30}$$

If $z(t_{2k}) - \eta_k t_c(k) \leqslant 0$, then $z(t_{2k+1}) = 0$. Thus $z(t_{2k+2}) \leqslant \gamma^{t_d(k+1)} z(t_{2k+1}) = 0 \leqslant \max\{0, z(t_{2k}) - \varepsilon\}$, which implies Eq. (28) holds for $k + 1$.

If $z(t_{2k}) - \eta_k t_c(k) > 0$, one has from (10) and (27) that

$$\begin{aligned} z(t_{2k+2}) &\leqslant \gamma^{t_d(k+1)} z(t_{2k+1}) \\ &\leqslant \gamma^{t_d(k+1)} \left(z(t_{2k}) - \frac{M_k(\gamma^{t_d(k+1)} - 1) + \varepsilon}{\gamma^{t_d(k+1)} t_c(k)} t_c(k) \right) \\ &\leqslant \gamma^{t_d(k+1)} z(t_{2k}) - M_k(\gamma^{t_d(k+1)} - 1) - \varepsilon \\ &\leqslant z(t_{2k}) + z(t_{2k})(\gamma^{t_d(k+1)} - 1) - M_k(\gamma^{t_d(k+1)} - 1) - \varepsilon \\ &\leqslant z(t_{2k}) - \varepsilon \\ &\leqslant \max\{0, z(t_{2k}) - \varepsilon\}. \end{aligned}$$

Therefore,

$$\begin{aligned} z(t_{2k+2}) &\leqslant \max\{0, z(t_{2k}) - \varepsilon\} \\ &\leqslant \max\{0, \max\{0, z(0) - k\varepsilon\} - \varepsilon\} \\ &\leqslant \max\{0, z(0) - (k+1)\varepsilon\}. \end{aligned}$$

Hence, Eq. (28) is true for $k + 1$.

Combining Steps 1 and 2, Eq. (28) holds for all $k \in \mathbb{N}^+$.

In the following, it is necessary to prove that there exists $N < \infty$ such that

$$z(t_{2N+1}) = 0 \text{ and } z(t) = 0 \text{ for } t \geqslant t_{2N+1}. \tag{31}$$

Suppose $z(t_{2N+1}) > 0$ when $N \to \infty$, it follows from (22) and (28) that

$$0 \leqslant z(t_{2N+1}) \leqslant z(t_{2N}) - \eta_N t_c(N) \leqslant z(0) - N\varepsilon - \eta_N t_c(N),$$

which contradicts $z(0) - N\varepsilon - \eta_N t_c(N) < 0$ when $N \to \infty$. Therefore, there exists $N < \infty$ such that $z(t_{2N+1}) = 0$.

Now we prove $z(t) = 0$ for $t \geqslant t_{2N+1}$. Suppose that there exists $\tilde{t} > t_{2N+1}$ such that $z(\tilde{t}) > 0$. Then there exists $\tilde{k} > N$ such that $\tilde{t} \in (t_{2\tilde{k}}, t_{2\tilde{k}+1}]$ or $\tilde{t}h \in (t_{2\tilde{k}+1}, t_{2\tilde{k}+2}]$. Let $t_s = \sup\{t \in (t_{2N+1}, \tilde{t}] : z(t) = 0\}$. So $t_s < \tilde{t}$, $z(t_s) = 0$ and $z(t) > 0$ for all $t \in (t_s, \tilde{t}]$.

In the following, two cases are discussed.

Case 1. If $\tilde{t} \in (t_{2\tilde{k}}, t_{2\tilde{k}+1}]$, there exists $t_* \in (t_{2\tilde{k}}, t_{2\tilde{k}+1}] \subset (t_s, \tilde{t}]$ such that $z(t)$ is monotonously increasing on the interval $[t_*, \tilde{t}]$, i.e., $\dot{z}(t) > 0$ for $t \in [t_*, \tilde{t}]$, which contradicts (21). Therefore, $\tilde{t} \in (t_{2\tilde{k}}, t_{2\tilde{k}+1}]$ does not hold, which further implies $z(t) = 0$ for $t \in (t_{2k}, t_{2k+1}]$, $k > N$. From the continuity of $z(t)$ on $(t_{2k}, t_{2k+1}]$, one has

$$z(t_{2k+1}) = 0, \text{ for } \ k > N. \tag{32}$$

Case 2. If $\tilde{t} \in (t_{2\tilde{k}+1}, t_{2\tilde{k}+2}]$ such that $z(\tilde{t}) > 0$, then it is found from (26) that $z(\tilde{t}) \leqslant \gamma^{\tilde{t}-t_{2\tilde{k}+1}} z(t_{2\tilde{k}+1})$. Noticing $z(t_{2\tilde{k}+1}) = 0$, known from (32), we have $z(\tilde{t}) = 0$, which contradicts $z(\tilde{t}) > 0$.

Therefore, Eq. (31) is true. Based on Definition 1, the system achieves synchronization in finite time.

Furthermore, because $\gamma > 1$ and $z(t)$ decreases in the interval $[t_{2k}, t_{2k+1})$, one has that, when $z(t) > 0$,

$$z(t_{2k+1}) < z(t_{2k}), \quad z(t_{2k+1}) \leqslant z(t_{2k+2}).$$

Hence, $z(t_{2k+1})$ is the minimum value when $t \in (t_{2k}, t_{2k+2}]$. If the finite synchronization time exists, it must be in $(t_{2k}, t_{2k+1}]$ for some integer k.

By (28) and (30), it can be obtained that

$$\begin{aligned} z(t_{2N+1}) &\leqslant \max\{0, z(t_{2N}) - \eta_N t_c(N)\} \\ &\leqslant \max\{0, \max\{0, z(0) - (N-1)\varepsilon\} - \eta_N t_c(N)\} \\ &= \max\{0, z(0) - (N-1)\varepsilon - \eta_N t_c(N)\}. \end{aligned} \tag{33}$$

Considering the condition (13), one has

$$\begin{cases} z(0) - (N-1)\varepsilon - \eta_N t_c(N) \leqslant 0, \\ z(0) - (N-2)\varepsilon - \eta_{N-1} t_c(N-1) \geqslant 0. \end{cases}$$

By solving the inequalities above, one has $T \leqslant t_{2N+1}$, which is the condition (12).

Remark 4. It is shown that the settling time is dependent of the initial value d_0, the positive constant ε, the control gain η_k, and control width $t_c(k)$. Naturally, when any one of the parameters ε, η_k, or $t_c(k)$ increases, the settling time will decrease. On the other hand, given control gain η_k satisfying the condition (10), it is known from (13) that the settling time increases with increasing $t_d(k+1)$, which is consistent with practical applications. It should be noted that, although finite-time synchronization has been extensively investigated by designing various continuous-time controllers, few authors consider finite-time synchronization via intermittent control, let alone the same issue for switched systems with continuous-time and discrete-time subsystems. The main difficulty comes from the increment of the Lyapunov function on the uncontrolled time-intervals. Recently, the authors in [35] tried to investigate finite-time synchronization of networks by using a kind of quantized intermittent control. Unfortunately, a special controller including the term $\mathrm{sign}(q(e(t)))$ has to work all the time, which means the control scheme in [35] is, in a strict sense, not a classical intermittent control. On the other hand, although finite-time consensus of hybrid continuous-time and discrete-time subsystems is considered in [28], both the continuous-time and discrete-time subsystems have to be controlled. Obviously, the results in [28] cannot be extended to intermittent control. Hence, results in Theorem 1 are new and essentially extend the results in [28, 35].

Note that the control gain η_k depends on the next uncontrolled width $t_d(k+1)$, which in practice is difficult to be known in advance. Therefore, when the next uncontrolled width $t_d(k+1)$ is unknown, Theorem 1 is not applicable anymore. For the convenience of practical applications, the following criterion is developed.

Theorem 2. Suppose that Assumptions 1–3 hold. For given constants η_0 and $\varepsilon > 0$, take a sequence $\{M_k : M_k \geqslant \max\{0, d_0 - \eta_0 t_c(0) - (k-1)\varepsilon\}, k \in \mathbb{N}^+\}$. Then the hybrid system (3) and (4) can be synchronized in mean square with hybrid system (1) and (2) in a finite settling time T if the inequalities (9) and (11) hold, and

$$\eta_k \geqslant (M_k(\gamma^{t_d(k)} - 1) + \varepsilon)/t_c(k). \tag{34}$$

Moreover, the settling time T is estimated as $T \leqslant t_{2N+1}$, where $N = [(d_0 - \eta_0 t_c(0))/\varepsilon]$.

Proof. Consider the same $V(t)$ and $z(t)$ as those given in the proof of Theorem 1. It is easy to know that Eqs. (21), (22), (26), and (27) still hold. Then, it is required to prove that, for $k \in \mathbb{N}^+$,

$$z(t_{2k+1}) \leqslant \max\{0, z(0) - \eta_0 t_c(0) - k\varepsilon\}. \tag{35}$$

In the following, the mathematical induction method is utilized to prove (35).

When $k = 0$, it follows from (22) that

$$z(t_1) \leqslant z(0) - \eta_0 t_c(0). \tag{36}$$

Considering the condition on the sequence $\{M_k\}$, the above inequality implies that

$$z(t_{2\times 1-1}) = z(t_1) \leqslant \max\{0, z(0) - \eta_0 t_c(0)\} \leqslant M_1. \tag{37}$$

It is found from (22) and (27) that

$$\begin{aligned} z(t_3) &= z(t_{2\times 1+1}) \\ &\leqslant z(t_{2\times 1}) - \eta_1(t_{2\times 1+1} - t_{2\times 1}) \\ &\leqslant \gamma^{t_d(1)} z(t_1) - \eta_1 t_c(1). \end{aligned}$$

Substituting (34) and (37) into the above inequality gives

$$\begin{aligned} z(t_3) &\leqslant \gamma^{t_d(1)} z(t_1) - (M_1(\gamma^{t_d(1)} - 1) + \varepsilon)/t_c(1) \cdot t_c(1) \\ &\leqslant z(t_1) + z(t_1)(\gamma^{t_d(1)} - 1) - M_1(\gamma^{t_d(1)} - 1) - \varepsilon \\ &\leqslant z(t_1) - \varepsilon. \end{aligned} \tag{38}$$

By combining (36) and (38) and noticing $z(t) \geqslant 0$, it is found that

$$z(t_3) \leqslant \max\{0, z(t_1) - \varepsilon\} \leqslant \max\{0, z(0) - \eta_0 t_c(0) - \varepsilon\},$$

which implies that Eq. (35) holds for $k = 1$.

Suppose Eq. (35) holds for some $k > 1$. Now it is necessary to prove Eq. (35) holds for $k + 1$.

In view of (22) and (27), one has

$$\begin{aligned} z(t_{2(k+1)+1}) &\leqslant z(t_{2(k+1)}) - \eta_{k+1}(t_{2(k+1)+1} - t_{2(k+1)}) \\ &\leqslant \gamma^{t_d(k+1)} z(t_{2k+1}) - \eta_{k+1} t_c(k+1). \end{aligned} \tag{39}$$

By the definition of $\{M_k\}$, it holds that

$$z(t_{2(k+1)-1}) = z(t_{2k+1}) \leqslant \max\{0, z(0) - \eta_0 t_c(0) - k\varepsilon\} \leqslant M_{k+1}. \tag{40}$$

Recalling (34), (39), and (40), one has

$$\begin{aligned} &z(t_{2(k+1)+1}) \\ &\leqslant \gamma^{t_d(k+1)} z(t_{2k+1}) - (M_{k+1}(\gamma^{t_d(k+1)} - 1) + \varepsilon) \\ &\leqslant z(t_{2k+1}) + z(t_{2k+1})(\gamma^{t_d(k+1)} - 1) - M_{k+1}(\gamma^{t_d(k+1)} - 1) - \varepsilon \end{aligned}$$

$$\leqslant z(t_{2k+1}) - \varepsilon.$$

Hence,

$$\begin{aligned} & z(t_{2(k+1)+1}) \\ & \quad \leqslant \max\{0, z(t_{2k+1}) - \varepsilon\} \\ & \quad \leqslant \max\{0, \max\{0, z(0) - \eta_0 t_c(0) - k\varepsilon\} - \varepsilon\} \\ & \quad \leqslant \max\{0, \max\{-\varepsilon, z(0) - \eta_0 t_c(0) - (k+1)\varepsilon\}\} \\ & \quad \leqslant \max\{0, z(0) - \eta_0 t_c(0) - (k+1)\varepsilon\}, \end{aligned}$$

which implies Eq. (35) holds for $k+1$. Thus Eq. (35) is true for all $k \in \mathbb{N}^+$.

Referring to the same analytical method as that in the proof of Theorem 1, there exists $N < \infty$ such that

$$z(t_{2N+1}) = 0 \text{ and } z(t) = 0 \text{ for } t \geqslant t_{2N+1}. \tag{41}$$

On the other hand, it can be seen from (26), (35), and (41) that

$$\begin{cases} z(0) - \eta_0 t_c(0) - (N-1)\varepsilon > 0, \\ z(0) - \eta_0 t_c(0) - N\varepsilon \leqslant 0. \end{cases} \tag{42}$$

By solving (42), it can be obtained that $N = \lceil (d_0 - \eta_0 t_c(0))/\varepsilon \rceil$ and $T \leqslant t_{2N+1}$.

Remark 5. Note that the control gain η_k in Theorem 2 is dependent on both the control width and non-controlled width. Specially, one can choose $\{M_k : M_k = \max\{0, d_0 - \eta_0 t_c(0) - (k-1)\varepsilon\}, k \in \mathbb{N}^+\}$ to minimize the value of η_k such that the control cost is reduced. Given ε, if $t_c(k)$ and $t_d(k)$ are fixed, the control gain η_k is monotonously decreasing when $\{M_k : M_k = \max\{0, d_0 - \eta_0 t_c(0) - (k-1)\varepsilon\}, k \in \mathbb{N}^+\}$. It is found from (34) that, when the synchronization has been realized, the control gain η_k can be taken as $\eta_k = \varepsilon / t_c(k)$. On the other hand, the settling time T decreases as the value of ε increases, and hence one can adjust the settling time by tuning the value of ε.

Both Theorems 1 and 2 are applicable to finite-time synchronization with the condition $\gamma > 1$, which means that the second equation of the error system (5) is unstable. Obviously, when $\gamma \leqslant 1$, Theorems 1 and 2 can still guarantee the finite-time synchronization. However, the estimated settling time is conservative. To improve the accuracy of the estimation, the following theorem is given.

Theorem 3. Suppose that Assumptions 1–3 hold. Then the hybrid system (3) and (4) can be synchronized in mean square with the hybrid system (1) and (2) in finite time T if the condition (9) is satisfied, $\gamma \leqslant 1$, and

$$\sum_{k=0}^{\infty} \eta_k t_c(k) > d_0. \tag{43}$$

Moreover, the settling time T is estimated as

$$T \leqslant t_{2N+1} - \left(\sum_{k=0}^{N} \eta_k t_c(k) - d_0 \right) \Big/ \eta_N, \tag{44}$$

where N is an integer satisfying

$$\sum_{k=0}^{N-1} \eta_k t_c(k) < d_0, \quad \sum_{k=0}^{N} \eta_k t_c(k) \geqslant d_0. \tag{45}$$

Proof. Considering $V(t)$, $z(t)$, and $z(0) = d_0$ as those given in Theorem 1, one has (21), (22), (26), and (27). Owing to (27) and $\gamma \leqslant 1$, $z(t)$ will not increase in the interval $(t_{2k+1}, t_{2k+2}]$. Moreover, the inequality (22) means that $z(t)$ decreases in the interval $t \in (t_{2k}, t_{2k+2}]$, $k \in \mathbb{N}^+$. Meanwhile, the condition (43) implies that $\sum_{k=0}^{\infty} \eta_k t_c(k) = \lim_{N\to\infty} \sum_{k=0}^{N} \eta_k t_c(k) > d_0$. So there exists $N < \infty$ such that Eq. (45) is satisfied.

Let $(\sum_{k=0}^{N} \eta_k t_c(k) - d_0)/\eta_N = \theta$. One derives from (45) that $t_{2N+1} - \theta \in (t_{2N}, t_{2N+1}]$. It can be known from (21) that

$$\begin{aligned} & z(t_{2N+1} - \theta) \\ & \quad \leqslant z(t_{2N}) - \eta_N \left(t_c(N) - \theta\right) \\ & \quad \leqslant d_0 - \sum_{k=0}^{N-1} \eta_k t_c(k) - \eta_N \left(t_c(N) - \theta\right) \\ & \quad \leqslant d_0 - \sum_{k=0}^{N} \eta_k t_c(k) + \eta_N \left(\sum_{k=0}^{N} \eta_k t_c(k) - d_0\right) \Big/ \eta_N \\ & \quad \leqslant d_0 - d_0 = 0. \end{aligned}$$

Therefore, $z(t_{2N+1} - \theta) \leqslant z(t_{2N}) - \eta_N \left(t_c(N) - \theta\right) = 0$ because $z(t)$ is always greater than zero. Considering (21) and (26), it is found that $z(t) \equiv 0$ when $t > t_{2N} - \eta_N \left(t_c(N) - \theta\right)$. Thus, Eq. (44) holds and the proof is completed.

4 Numerical examples

In this section, two numerical simulations are given to illustrate the effectiveness of Theorems 1 and 2.

Example 1. Models in this example come from the Chua's circuits [50] as

$$\begin{cases} \mathrm{d}x(t) = [Ax(t) + Bf(x(t))]\mathrm{d}t, \\ x(t+1) = \tilde{A}x(t) + \tilde{B}f(x(t)), \end{cases} \tag{46}$$

and

$$\begin{cases} \mathrm{d}y(t) = [Ay(t) + Bf(y(t)) + U(t)]\mathrm{d}t + \sigma(t, e(t))\mathrm{d}\omega(t), \\ y(t+1) = \tilde{A}y(t) + \tilde{B}f(y(t)) + \tilde{\sigma}(t, e(t))w(t), \end{cases} \tag{47}$$

where $h = 0.01$, $t_{2k} = 4kh, t_{2k+1} = (4k+1)h$, $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$, $f(x) = (|x_1+1| - |x_1-1|, 0, 0)^{\mathrm{T}}$, $\sigma(t, e(t)) = 3\mathrm{diag}(e_1(t), e_2(t), e_3(t))^{\mathrm{T}}$, $\tilde{\sigma}(t, e(t)) = 0.3\mathrm{diag}(e_1(t), e_2(t), e_3(t))^{\mathrm{T}}$,

$$A = \begin{pmatrix} -19/7 & 9 & 0 \\ 1 & -1 & 1 \\ 0 & -14.28 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 27/7 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to derive that Assumptions 1 and 2 are satisfied with $L = 2, \alpha_1 = 9, \alpha_2 = 0.09$. By simple computation, one has $\lambda_1 = 7.4146, \lambda_2 = 3.8571, \tilde{\lambda}_1 = 1.1618, \tilde{\lambda}_2 = 0.0377, \tilde{\lambda}_3 = 0.0015$. It follows by (11) that $\gamma = 1.1546 > 1$. Take $\rho = 0.8$, $\varepsilon = 0.1$, $M_k = \max\{0, 2.8178 - 0.1k\}$, $k \in \mathbb{N}^+$. According to Theorem 1, the system (47) with (6) can be synchronized with (46) in the settling time $T = t_{2\times N+1} = (4N+1)h = 1.13$ with $N = 28$ when the control gains satisfy the conditions: $r \geqslant 22.0824$, $\eta_1 \geqslant 104.1773, \eta_2 \geqslant 100.7085, \eta_3 \geqslant 97.2396, \eta_4 \geqslant 93.7708, \ldots$.

In the simulations, the time step on the continuous intervals $[t_{2k}, t_{2k+1})$, $k \in \mathbb{N}^+$ is 0.00001, and the control gains are taken as $\eta_0 = 0$, $\eta_1 = 104.1773, \eta_2 = 100.7085, \eta_3 = 97.2396, \eta_4 = 93.7708, \ldots, r = 22.0824$. The trajectories of the synchronization error between systems (47) and (46) are presented in Figure 1, which converges to zero before $0.09 < 1.13$. Figure 1 verifies Theorem 1 perfectly. The signals $q(e(t))$ used in the QIC are given in Figure 2.

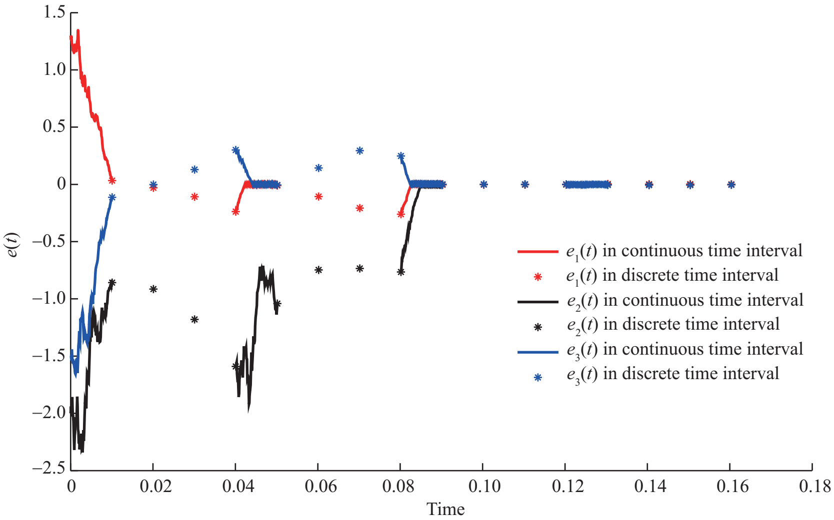

Figure 1 (Color online) Trajectory of the error system $e(t)$ with initial condition $e(0) = y(0) - x(0)$.

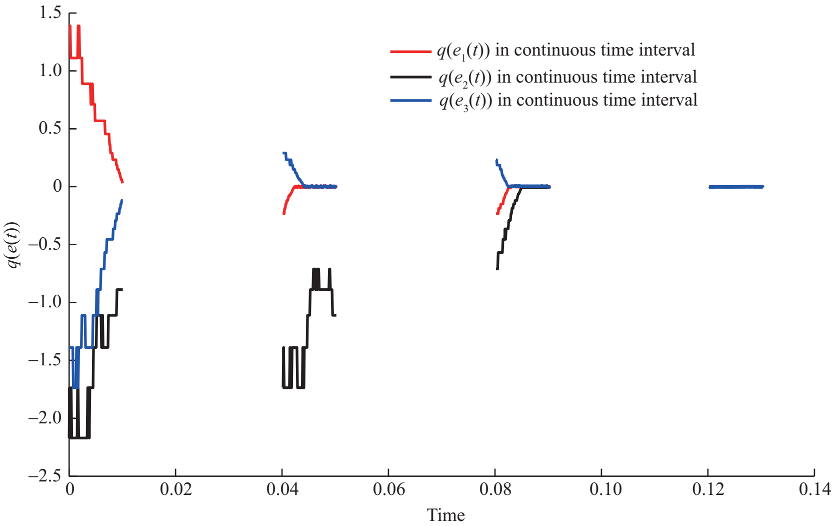

Figure 2 (Color online) Trajectory of $q(e(t))$ in continuous time interval under quantizer (7) with initial condition $q(e(0))$.

Example 2. The drive system is given by

$$\begin{cases} \mathrm{d}x(t) = [Ax(t) + Bf(x(t))]\mathrm{d}t, \\ x(t+1) = \tilde{A}x(t) + \tilde{B}f(x(t)), \end{cases} \tag{48}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ is the state vector, $f(x(t)) = (0, -x_1(t)x_3(t), x_1(t)x_2(t))^{\mathrm{T}}$, $h = 0.002$, and

$$A = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The time sequence $\{t_k, k \in \mathbb{N}^+\}$ is assumed to satisfy $t_{2k} = 4kh, t_{2k+1} = (4k+1)h$.

The response system is given by

$$\begin{cases} \mathrm{d}y(t) = [Ay(t) + Bf(y(t)) + U(t)]\mathrm{d}t + \sigma(t, e(t))\mathrm{d}\omega(t), \\ y(t+1) = \tilde{A}y(t) + \tilde{B}f(y(t)) + \tilde{\sigma}(t, e(t))w(t), \end{cases} \tag{49}$$

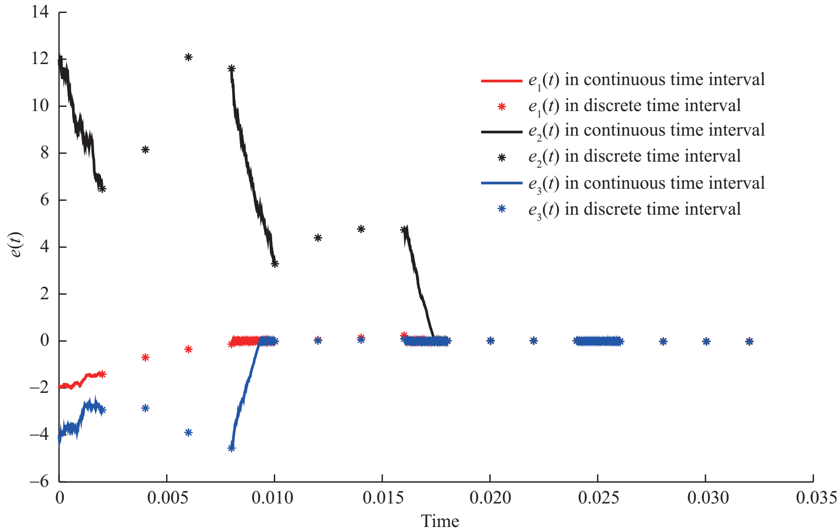

Figure 3 (Color online) Trajectory of the error system with initial condition $e(0) = y(0) - x(0)$.

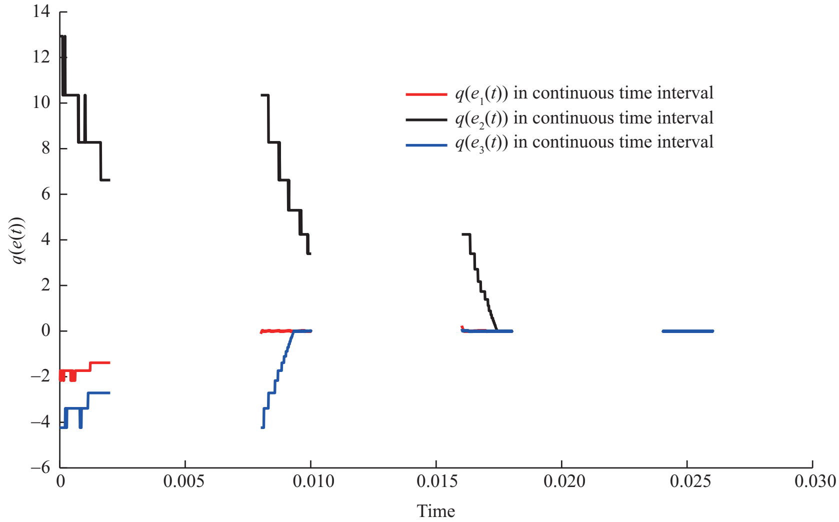

Figure 4 (Color online) Trajectory of $q(e(t))$ in continuous time interval under quantizer (7) with initial condition $q(e(0))$.

where $\sigma(t, e(t)) = 4\mathrm{diag}(e_1(t), e_2(t), e_3(t))^{\mathrm{T}}, \tilde{\sigma}(t, e(t)) = 0.2\mathrm{diag}(e_1(t), e_2(t), e_3(t))^{\mathrm{T}}$.

Note that both the drive (48) and response system (49) come from the Lorenz system. In this example, the initial values are chosen as $y_0 = (2, 17, 2)$, $x_0 = (4, 5, 6)$.

The Lipschitz coefficient is estimated as $L = 80$ with small initial state and Assumptions 1 and 2 are satisfied with $\alpha_1 = 16, \alpha_2 = 0.04$. By simple computation, one has $\lambda_1 = 14.0256, \lambda_2 = 1, \tilde{\lambda}_1 = 1.0572, \tilde{\lambda}_2 = 0.0021, \tilde{\lambda}_3 = 4 \times 10^{-6}$. Take $\rho = 0.8$ in quantizer (7), $\varepsilon = 0.1$, $M_0 = ||x_0 - y_0|| = 12.8062$, $M_k = \max\{0, 12.8062 - 0.1(k-1)\}$, $k \in \mathbb{N}$, $\gamma = 1.1346 > 1$. By Theorem 2, the system (49) with the controller (6) can be synchronized with (48) in the settling time $T = t_{2\times N+1} = (4N+1)h = 1.026$ with $N = 128$ if $\eta_0 = 0$, $\eta_1 \geqslant 2984.5, \eta_2 \geqslant 2961.6, \eta_3 \geqslant 2938.7, \ldots, r \geqslant 114.7788$.

In the simulations, the time step on the continuous intervals $[t_{2k}, t_{2k+1})$, $k \in \mathbb{N}^+$ is 0.00001, and the control gains are taken as $\eta_0 = 0$, $\eta_1 = 2984.5, \eta_2 = 2961.6, \eta_3 = 2938.7, \ldots, r = 114.7788$. The trajectory of the synchronization error between systems (49) and (48) is shown in Figure 3, from which one can see that the synchronization is achieved before $0.018 < 1.026$ and Theorem 2 is matched perfectly. The signals $q(e(t))$ used in the QIC are given in Figure 4.

5 Conclusion

This paper has addressed the finite-time drive-response synchronization in mean square of nonlinear systems with stochastic perturbations which consist of continuous-time and discrete-time dynamics. QICs are designed for the synchronization goal. Sufficient conditions have been given by developing novel analytical methods. Our results show that the settling time is dependent on control gains, systems' initial conditions, as well as the widths of both continuous time and discrete time. The effectiveness of the theoretical results is illustrated by numerical simulations.

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