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Basis for the quotient space of matrices under equivalence

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Dear editor,
The semi-tensor product (STP) of matrices, proposed by Cheng in 2001 [1], is a generalization of the conventional matrix product and well defined at every two finite-dimensional matrices. In 2016, Cheng [2] proposed a new concept of semi-tensor addition (STA), which is a generalization of the conventional matrix addition and well defined at every two finite-dimensional matrices with the same ratio between the numbers of rows and columns. In addition, Cheng [2] defined an identity equivalence relation on matrices; STP and STA are proved valid for the corresponding identity equivalence classes, and the corresponding quotient space is endowed with an algebraic structure, a manifold structure, and an analytical structure. In this follow-up study, we give a new basis for the quotient space, which also shows that the quotient space is of countably infinite dimension.

Matrices (all matrices considered in this study are finite-dimensional) can be seen as linear operators/transformations on sets (e.g., topological/linear spaces). Then, the products of matrices are compositions of operators on sets. The conventional matrix product works for two matrices A and B satisfying the condition where the number of columns of A and the number of rows of B are equal, and preserves many properties, such as the associative law, the distributive law, and several inverse-order laws (e.g., $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$, where A^{T} denotes the transpose of A). Is it possible to extend the concept of matrix product and simultaneously preserve the basic properties of the conventional matrix product? Taking the Kronecker product for example, the Kronecker product works for any two matrices, and preserves the associative law, but does not preserve inverse-order laws (e.g., one has $(A \otimes B)^{\mathrm{T}} = A^{\mathrm{T}} \otimes B^{\mathrm{T}}$ but does not necessarily have $(A \otimes B)^{\mathrm{T}} = B^{\mathrm{T}} \otimes A^{\mathrm{T}}$). Furthermore, the Kronecker product is not a generalization of the conventional matrix product, because when the number of columns of matrix A equals the number of rows of matrix B, one usually does not have $AB = A\otimes B$. In 2001, Cheng [1] proposed a generalization of the conventional matrix product, called the STP of matrices, which works for every two matrices, and preserves the associative law, the distributive law, and several inverse-order laws. During the subsequent years, STP has been applied to many fields, such as analysis and control of Boolean control networks [3], control-theoretic problems [1], symmetry of dynamical systems [4], differential geometry and Lie algebras [5], finite games [6–8], and practical engineering fields [9]. In particular, using STP, logic variables are represented as vectors, and then logic operations are represented as the STP of vectors and the so-called logical matrices (a special class of Boolean matrices).

Except for extending the conventional matrix product, Cheng also generalized the conventional matrix addition. In 2016, Cheng [2] proposed a

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new concept of STA that is a generalization of the conventional matrix addition and well defined at every two matrices with the same ratio between the numbers of rows and columns. In addition, Cheng defined an identity equivalence relation of matrices, and proved that STP and STA are both valid for the corresponding equivalence classes. Hence, the corresponding quotient space under the identity equivalence relation forms an associative algebra, which we might call semi-tensor algebra as well. In this study, we give a new basis for the quotient space, which also shows that the associative algebra is of countably infinite dimension.

Preliminaries. We briefly introduce preliminaries shown in [1, 2].

(1) Semi-tensor product. In this study, we use $\mathcal{M}_{m\times n}$ to denote the set of $m \times n$ real matrices, where m, n are two positive integers.

Definition 1. Let $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$. $t = \text{lcm}\{n, p\}$ denotes the least common multiple of n and p. The STP of A and B is defined as $A \ltimes B := \left(A \otimes I_{t/n}\right)\left(B \otimes I_{t/p}\right)$, where $\otimes$ is the Kronecker product.

Remark 1. For every $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$, one has $A \ltimes B \in \mathcal{M}_{(mt)/n\times(qt)/p}$, where $t = \text{lcm}(n, p)$.

(2) Semi-tensor addition. $\mathcal{M}_\mu := \{M \in \mathcal{M}_{m\times n} | m, n \in \mathbb{Z}_+, m/n = \mu\}, \mu \in \mathbb{Q}_+$, where $\mathbb{Z}_+$ and $\mathbb{Q}_+$ denote the sets of positive integers and positive rational numbers, respectively. Let

$$\mathcal{M} = \bigcup_{\mu\in\mathbb{Q}_+} \mathcal{M}_\mu. \tag{1}$$

Note that the right-hand side of (1) is a partition of the left-hand side of (1). That is,

$$\mathcal{M}_{\mu_1} \cap \mathcal{M}_{\mu_2} = \emptyset, \quad \forall \mu_1 \neq \mu_2.$$

Definition 2. Let A, B be in $\mathcal{M}_\mu$, where $\mu \in \mathbb{Q}_+$. Precisely, $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$, and $m/n = p/q = \mu$. Set $t = \text{lcm}\{m, p\}$. The STA of A and B, denoted by $\pm$, is defined as $A \pm B := \left(A \otimes I_{t/m}\right) + \left(B \otimes I_{t/p}\right)$. We also define the left semi-tensor subtraction as $A \vdash B := A \pm (-B)$.

Remark 2. Let $\sigma \in \{\pm, \vdash\}$ be one of the two binary operations and μ_1 and μ_2 be two positive rational numbers.

(i) If $\mu_1 = \mu_2 =: \mu$, then for all A, $B \in \mathcal{M}_\mu$, $A\sigma B \in \mathcal{M}_\mu$.

(ii) For all $A \in \mathcal{M}_{\mu_1}$ and $B \in \mathcal{M}_{\mu_2}$, $A \ltimes B \in \mathcal{M}_{\mu_1\mu_2}$.

(iii) If A and B are as in Definition 2, then $A\sigma B \in \mathcal{M}_{t\times\frac{nt}{m}}$, where $t = \text{lcm}(m, n)$.

(3) Identity equivalence relation.

Definition 3. Let A, B be in $\mathcal{M}$. A and B are said to be identity equivalent, denoted by $A \sim B$, if there exist two identity matrices I_s, I_t, where $s, t \in \mathbb{Z}_+$, such that $A \otimes I_s = B \otimes I_t$.

Remark 3. The relation identity equivalence (IE) $\sim$ is an equivalence relation. That is, it is (i) reflexive ($A \sim A$), (ii) symmetric (if $A \sim B$ then $B \sim A$), and (iii) transitive (if $A \sim B$ and $B \sim C$ then $A \sim C$).

Definition 4. Let A be in $\mathcal{M}$.

(i) The equivalent class of A is denoted by

$$\langle A\rangle := \{B \in \mathcal{M} \mid B \sim A\}.$$

(ii) A is called reducible if there are an identity matrix I_s, where $s \geqslant 2$, and a matrix B, such that $A = B \otimes I_s$; and called irreducible otherwise.

Theorem 1. For every $\bar{A} \in \mathcal{M}$, in $\langle\bar{A}\rangle$ there exists a unique irreducible element.

By Theorem 1, for each $\langle A\rangle$ and all matrices $B \in \mathcal{M}_{m\times n}$ and $C \in \mathcal{M}_{p\times q}$ both in $\langle A\rangle$, one has $m/n = p/q$. Then, the matrix in $\langle A\rangle$ that has the minimal number of rows is the unique irreducible element of $\langle A\rangle$.

The following corollary reveals the structure of identity equivalence classes.

Corollary 1. For every given $A \in \mathcal{M}$, let A_0 be the only irreducible element of $\langle A\rangle$. Then,

$$\langle A\rangle = \{A_0 \otimes I_s \mid s = 1, 2, \dots\}. \tag{2}$$

For a binary operation $\sigma : S \times S \to S$, and an equivalence relation $\sim \subset S \times S$, $\sim$ is called a congruence with respect to σ if for all $A_1, A_2, B_1, B_2 \in S$, $(A_1, A_2) \in \sim$ and $(B_1, B_2) \in \sim$ imply $(A_1\sigma B_1, A_2\sigma B_2) \in \sim$. In this case, σ is called valid for the equivalence classes generated by $\sim$.

Theorem 2. Consider the algebraic system $(\mathcal{M}_\mu, \pm)$, where $\mu \in \mathbb{Q}_+$. The relation IE $\sim$ is a congruence with respect to $\pm$.

Given $\mu \in \mathbb{Q}_+$, define the quotient space Σ_μ as

$$\Sigma_\mu := \mathcal{M}_\mu / \sim = \{\langle A\rangle \,|A \in \mathcal{M}_\mu\}. \tag{3}$$

According to Theorem 2, the operation $\pm$ can be extended to Σ_μ as

$$\langle A\rangle \pm \langle B\rangle := \langle A \pm B\rangle, \quad \forall \langle A\rangle, \langle B\rangle \in \Sigma_\mu. \tag{4}$$

Theorem 3. With the definition in (4), the quotient space $(\Sigma_\mu, \pm)$ is a vector space.

Basis for the quotient space. In this study, we show that for each positive rational number μ, the quotient space $(\Sigma_\mu, \pm)$ is of countably infinite dimension; we give a generator for the space and then use the generator to construct a basis for the space.

Proposition 1. For every positive rational number $\mu = p/q$, where $p, q \in \mathbb{Z}_+$ are co-prime, the vector space $(\Sigma_\mu, \vdash\!\vdash)$ is of countably infinite dimension, and has a generator

$$\bigcup_{k\in\mathbb{Z}_+} \bigcup_{\substack{1\leqslant i_{kp}\leqslant kp \\ 1\leqslant j_{kq}\leqslant kq}} \left\{\left\langle E^{kp\times kq}_{i_{kp}j_{kq}}\right\rangle\right\}, \tag{5}$$

where and hereinafter $E^{kp\times kq}_{i_{kp}j_{kq}}$ denotes the $kp \times kq$ matrix with the (i_{kp}, j_{kq})-th entry equal to 1 and all other entries 0.

Theorem 4. For every positive rational number $\mu = p/q$, where $p, q \in \mathbb{Z}_+$ are co-prime, the following countably infinite set of equivalence classes

$$\mathsf{D}_\mu \cup \mathsf{N}_\mu \tag{6}$$

is a basis for the vector space $(\Sigma_\mu, \vdash\!\vdash)$, where

$$\begin{aligned} \mathsf{D}_\mu &= \bigcup_{i=1}^{\infty} \bigcup_{\substack{1\leqslant j\leqslant i \\ \gcd(i,j)=1}} \bigcup_{\substack{1\leqslant k\leqslant p \\ 1\leqslant l\leqslant q}} \left\{\left\langle E^{p\times q}_{kl} \otimes E^{i\times i}_{jj}\right\rangle\right\}, \\ \mathsf{N}_\mu &= \bigcup_{i=2}^{\infty} \bigcup_{\substack{1\leqslant j_1,j_2\leqslant i \\ j_1\neq j_2 \\ \gcd(i,j_1,j_2)=1}} \bigcup_{\substack{1\leqslant k\leqslant p \\ 1\leqslant l\leqslant q}} \left\{\left\langle E^{p\times q}_{kl} \otimes E^{i\times i}_{j_1j_2}\right\rangle\right\}, \end{aligned} \tag{7}$$

where gcd() denote the greatest common divisor of numbers in (). Note that in (7), $E^{p\times q}_{kl} \otimes E^{i\times i}_{jj}$ is the unique irreducible element of $\langle E^{p\times q}_{kl} \otimes E^{i\times i}_{jj}\rangle$, and $E^{p\times q}_{kl} \otimes E^{i\times i}_{j_1j_2}$ is the unique irreducible element of $\langle E^{p\times q}_{kl} \otimes E^{i\times i}_{j_1j_2}\rangle$.

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